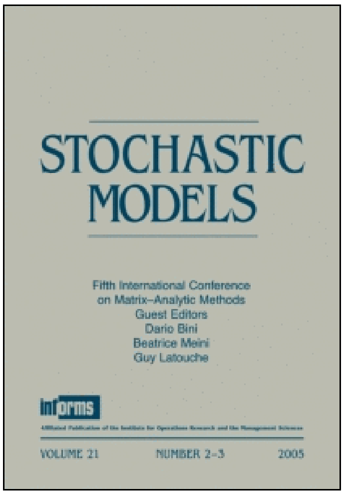

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LIMIT THEOREMS FOR THE METHOD OF REPLICATION

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ABSTRACT

The method of replication is frequently used by simulators to estimate steady-state quantities. In this paper, we obtain conditions under which this method yields asymptotically valid confidence intervals for steady-state means. In particular, we show that the length of a replication must be large relative to the number of replications, in order that the method work efficiently. We also contrast variants of the method in which the number of replications stay fixed with run length with those in which the number tends to infinity. Procedures in which the number of replications increase to infinity with the sample size produce confidence intervals which have smaller expected half-width and are less variable.

1. INTRODUCTION

Let $Y = \{Y(t) : t \geq 0\}$ be a real-valued stochastic process such that

$$\overline{Y}(t) \Longrightarrow r \tag{1}$$

as $t \to \infty$, where $\overline{Y}(t) = t^{-1} \int_0^t Y(s)ds$ and $\Longrightarrow$ denotes weak convergence. The parameter r is called the steady-state mean of Y. Simulators are frequently interested in estimating steady-state means. For example, when $Y(t)$ corresponds to the rate at which cost is incurred at time t, r is the long-run average cost of running the system; this parameter is frequently difficult to calculate analytically and must then be calculated numerically.

To estimate r, observe that (1) suggests that

$$E\overline{Y}(T) \approx r$$

for T large; Monte Carlo simulation can then be used to estimate $E\overline{Y}(T)$. Specifically, given a computational budget of size t, one can generate $k(t)$ independent replicates of $\overline{Y}(t/k(t))$. The resulting sample mean $r(t)$ of the $k(t)$ replicates is then used to estimate r. If $k(t) \equiv 1$ for all t, the resulting procedure is called a single-run method; otherwise, it is known as a multiple replication procedure.

The method of multiple replicates has two principal advantages for the simulator over single run methods:

i) It frequently leads to a (slight) reduction in mean square error (MSE) of the point estimator for r.

ii) It simplifies the construction of confidence intervals for the parameter r.

The first point is discussed in KELTON [9]. It is shown there that for simulation of stationary processes Y with positive correlation, multiple replication methods are preferable to single-run procedures in which r is estimated from one long run of Y.

In this paper, we will focus on the confidence interval generation aspect of the method of multiple replications. Confidence intervals play a central role in the theory and practice of simulation output analysis, since they are the primary tool used to assess error in the simulation context. As we shall see in Section 2, the construction of confidence intervals in the multiple replication setting is relatively straightforward. In particular, confidence intervals for the method of multiple replications do not require specification of additional user-supplied parameters (beyond specification of $k(t)$) and need little additional computation overhead. These factors contrast strongly with the demands imposed by most of the competing steady-state confidence interval methods. For example, spectral algorithms (see Chapter 3 of BRATLEY, FOX, and SCHRAGE [2]) require a user-supplied "bandwidth" for the spectral window, and involve significant additional computation.

This paper is organized as follows. Section 2 describes the main results of this paper; proofs of the more difficult results are deferred to the Appendix.

2. THE MAIN RESULTS

To precisely define a multiple replication estimator, we need to consider a sequence $\{Y_i : i \geq 1\}$ of i.i.d. replicates of the continuous-time process Y. (In order to incorporate discrete-time sequences $U = \{U_n : n \geq 0\}$ into our framework, we set $Y(t) = U_{[t]}$, where $[t]$ denotes the greatest integer less than or equal to t.) Given a computational budget t, suppose $k(t)$ is the number of replicates to be simulated; each replicate involves independently simulating Y up to time $m(t) = t/k(t)$. The resulting point estimate $r(t)$ then takes the form

$$r(t) = \frac{1}{k(t)} \sum_{i=1}^{k(t)} \overline{Y}_i(m(t))$$

where $\overline{Y}_i(t) = t^{-1} \int_0^t Y_i(s)ds$.

In order that $r(t)$ consistently estimate r, it should be clear that $m(t) \to \infty$ as $t \to \infty$ is generally necessary. (To see this, observe that if $m(\cdot) = c$ and $E|\overline{Y}(c)| < \infty$, then $r(t) \to E\overline{Y}(c)$ a.s. as $t \to \infty$.) One obvious way to arrange that $m(t) \to \infty$ is to let $k(\cdot) = k(\geq 1)$, in which case $m(t) = t/k$. To analyze the rate of convergence of $r(t)$ to r, we need to make the following assumption:

H1. There exist finite constants r and σ such that

$$X(t) \equiv t^{1/2}(\overline{Y}(t)-r) \Longrightarrow \sigma N(0,1)$$

as $t \to \infty$, where $N(0,1)$ is a standard normal r.v.

Assumption H1 is a mild requirement on the simulation output process Y. In particular, mixing processes, regenerative processes, and associated sequences all satisfy H1 (see ETHIER and KURTZ [6], p. 350-353, SMITH [12], and NEWMAN and WRIGHT [10] for proofs).

From H1, relation (1) follows, and the constant r appearing in H1 can be identified as the steady-state mean of Y. Noting that r(t) can be expressed as the sum of k(t) independent copies of $\overline{Y}(m(t))$, the continuous mapping lemma (Theorem 5.1 of BILLINGSLEY [1]) yields the following result.

THEOREM 1. Assume H1 and suppose that $k(\cdot) = k > 1$. Then, $t^{1/2}(r(t)-r) \Longrightarrow \sigma N(0,1)$ as $t \to \infty$.

Theorem 1 shows that the rate of convergence of r(t) to r is independent of the choice of the number of replicates k. This implies that the class of multiple replication procedures described by Theorem 1 enjoys the same rate of convergence as a single run method. However, as noted earlier, more precise asymptotics suggest that for positively correlated processes, multiple replication procedures are preferable.

We now turn to the question of constructing confidence intervals for the estimators considered in Theorem 1. Let

$$\Gamma(t) = \frac{1}{k(t)-1} \sum_{i=1}^{k(t)} (\overline{Y}_i(m(t)) - r(t))^2 .$$

The following result is easily established, again by appealing to the continuous mapping lemma.

THEOREM 2. Assume H1, $\sigma^2 > 0$, and suppose that $k(\cdot) = k > 1$. Then, as $t \to \infty$

$$t\,\Gamma(t) \Longrightarrow \sigma^2 \chi^2_{k-1}$$

$$k^{1/2}(r(t)-r)/\Gamma^{1/2}(t) \Longrightarrow t_{k-1}$$

where χ^2_{k-1} and t_{k-1} are, respectively, a chi-square r.v. and Student-t r.v. with $k-1$ degrees of freedom.

Confidence intervals for r can easily be constructed from Theorem 2; such an interval is clearly based on the Student-t limit distribution. It is desirable to employ a Student-t limit distribution with k as large as possible, since the asymptotic expected half-width and variability are both decreasing functions of k (see SCHMEISER [11]). This suggests that it is desirable to consider multiple replication procedures in which $k(t)$ increases with the computation effort t. To analyze the resulting estimator $r(t)$, we need to add two additional hypotheses:

H2. The process $\{X^2(t) : t \geq 0\}$ is uniformly integrable.

H3. $E\overline{Y}(t) = r + O(1/t)$ as $t \to \infty$.

Assumption H2 is a mild technical strengthening of H1; precise conditions, guaranteeing H2 for regenerative processes, may be found in GLYNN and IGLEHART [8]. Assumption H3 is also valid for a large class of regenerative processes, as the following proposition shows.

PROPOSITION 1. Let Y be a non-delayed regenerative process with regeneration times $0 = T(0) < T(1) < \ldots$ and assume that $E(\int_0^{T(1)}(1+|Y(s)|)ds)^3 < \infty$. Then, H3 holds.

For many Markov processes, it can be shown that the rate of convergence to the steady-state is exponentially fast (see p. 221 of DOOB [5]), so that there exists $\alpha > 0$ for which

$$EY(t) = r + O(e^{-\alpha t}) \quad . \tag{2}$$

If (2) holds, then

$$E(\int_0^t Y(s)ds) = rt + b - \int_t^\infty (EY(s)-r)ds$$

$$= rt + b + O(e^{-\alpha t})$$

where $b = \int_0^\infty (EY(s)-r)ds$. For such processes, the following strengthened version of H3 is then in force.

H4. $E\bar{Y}(t) = r + b/t + o(1/t)$ as $t \to \infty$.

We now return to the analysis of $r(t)$ when $k(t)$ is allowed to increase with t. Theorem 3 shows that the behavior of the estimator $r(t)$ depends critically on the manner in which $m(t)$ and $k(t)$ jointly converge to infinity.

THEOREM 3.

i) Assume H1 - H3. If $m(t)/k(t) \to \infty$, then,

$$t^{1/2}(r(t)-r) \Rightarrow \sigma N(0,1) .$$

ii) Assume H1, H2 and H4 with $b \neq 0$. If $m(t)/k(t) \to 0$ and $m(t) \to \infty$, then

$$t^{1/2}|r(t) - r| \Rightarrow \infty .$$

iii) Assume H1, H2 and H4 with $b \neq 0$. If $m(t)/k(t) \to c(0 < c < \infty)$, then

$$t^{1/2}(r(t)-r) \Rightarrow \sigma N(0,1) + b/c^{1/2} .$$

Theorem 3 asserts that the length of a replicate should be large relative to the number of replicates, in order that a reasonable convergence rate be achieved. In particular, we require that $k(t) = O(t^{1/2})$ and $m(t)^{-1} = O(t^{-1/2})$. Note that $m(t) \sim (ct)^{1/2}$ is a "critical rate" for this result, in the sense that any slower rate of increase of $m(t)$ results in a suboptimal rate of convergence for $r(t)$.

To produce confidence intervals in the current setting in which $k(t) \to \infty$, we need to consistently estimate σ^2. A natural candidate for estimating σ^2 is, of course, $m(t)\Gamma(t)$. (Note that $E\Gamma(t) \approx \text{var}\,\bar{Y}(m(t)) \approx \sigma^2/m(t)$.)

THEOREM 4. Assume that the process $\{X^4(t) : t > 0\}$ is uniformly integrable and that H1 - H2 are in force. If $m(t)/k(t) \to \infty$ and $k(t) \to \infty$, then

$$m(t)\,\Gamma(t) \Rightarrow \sigma^2$$

as $t \to \infty$.

See GLYNN and IGLEHART [8] for conditions guaranteeing uniform integrability of $\{X^4(t) : t > 0\}$ when Y is regenerative.

Combining Theorems 3 and 4, we obtain the following corollary, from which confidence intervals are easily produced.

COROLLARY 1. Under the assumptions of Theorem 4,

$$t^{1/2}(r(t)-r)/(m(t)\Gamma(t))^{1/2} \Longrightarrow N(0,1)$$

as $t \to \infty$, provided $\sigma^2 > 0$.

Corollary 1 and Theorem 2 together confirm the claim, made in the Introduction, that confidence intervals can be easily constructed when the method of replication is employed; in particular, no additional user-supplied parameters need be specified to generate confidence intervals.

The replication schemes described above, in which the number of replicates $k(t) \to \infty$, permit estimation of the parameter σ^2 (see Theorem 4); for schemes in which $k(t) \equiv k < \infty$, no such estimation is possible (see Theorem 2). In addition to its importance in the confidence interval setting, the parameter σ^2 is itself of some interest. The parameter σ^2 measures the asymptotic variability of $\bar{Y}(t)$; if $Y(t)$ measures the cost of running a stochastic system at time t, then r is the long-run average cost, and σ^2 measures the extent to which the average cost $\bar{Y}(t)$ may deviate from r over the interval $[0,t]$. Thus, σ^2 may itself be important in determining the suitability of a policy to be evaluated over a planning horizon of t time units duration.

We conclude that schemes in which $k(t) \to \infty$ offer two main advantages over those in which $k(t)$ is fixed; improved confidence interval asymptotics (shorter expected length and variability) and the opportunity to estimate the parameter σ^2.

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REFERENCES

[1]. P. BILLINGSLEY, Convergence of Probability Measures. John Wiley, New York (1968).

[2]. P. BRATLEY, B. L. FOX, and L. SCHRAGE, A Guide to Simulation, Springer-Verlag, New York (1983).

[3]. K. L. CHUNG, A Course in Probability Theory. Academic Press, New York (1974).

[4]. E. ÇINLAR, Introduction to Stochastic Processes. Prentice-Hall, Englewood Cliffs, NJ. (1975).

[5]. J. L. DOOB, Stochastic Processes, John Wiley, New York (1953).

[6]. S. N. ETHIER and T. G. KURTZ, Markov Processes: Characterization and Convergence, John Wiley, New York (1986).

[7]. W. FELLER, An Introduction to Probability Theory and its Applications, Volume II, John Wiley, New York (1971).

[8]. P. W. GLYNN and D. L. IGLEHART, Consequences of uniform integrability for simulation. Technical Report, Department of Operations Research, Stanford University, Stanford, CA (1986).

[9]. W. D. KELTON, Replication splitting and variance for simulating discrete-parameter stochastic processes, Operations Research Letters 4, 275-279 (1986).

[10]. C. M. NEWMAN and A. L. WRIGHT, An invariance principle for certain dependent sequences, Ann. Probability 9, 671-675 (1981).

[11]. B. SCHMEISER, Batch size effects in the analysis of simulation output, Operations Research 30, 556-568 (1982).

[12]. W. L. SMITH, Regenerative stochastic processes. Proc. Roy. Soc. London Ser. A 232, 6-31 (1955).

APPENDIX:

Proof of Proposition 1: It is easy to see that

$$\int_0^t Y(s)ds - rt = \sum_{k=1}^{N(t)+1} Z_k - \int_t^{T(N(t)+1)} (Y(s) - r)ds \tag{A1}$$

where $Z_k = \int_{T(k-1)}^{T(k)} (Y(s)-r)ds$ and $N(t) = \max\{k \geq 0 : T(k) \leq t\}$.

From the moment condition, it is obvious that $E(|Z_1| + \tau_1) < \infty$ and by Wald's identity

$$E\left(\sum_{k=1}^{N(t)+1} Z_k\right) = E(N(t)+1)EZ_k \quad .$$

(see p. 137 of CHUNG [3]). Since $r = E(\int_0^{T(1)} Y(s)ds)/ET(1)$, $EZ_k = 0$ and thus the first term in the right-hand side of (A1) vanishes. For the second term, set

$$a(t) = E\left(\int_t^{T(N(t)+1)} (Y(s) - r)ds\right) \quad . \tag{A2}$$

A simple renewal argument shows that a satisfies the renewal equation

$$a(t) = b(t) + (a * F)(t) \quad ,$$

with

$$b(t) = E\{\int_t^{T(1)} (Y(s) - r)ds \; ; \; T(1) > t\}$$

$$F(t) = P\{T(1) \leq t\} \quad .$$

Since

$$a(t) \leq E\left(\sum_{k=1}^{N(t)+1} \int_{T(k-1)}^{T(k)} |Y(s) - r|ds\right)$$

$$\leq E(N(t) + 1) \cdot E\left(\int_0^{T(1)} |Y(s) - r|ds\right) < \infty \quad ,$$

by Wald's equality and the moment hypothesis, it is clear that $a(\cdot)$ is bounded over finite intervals. Hence, by Theorem 2.3 of ÇINLAR [4],

$$a(t) = (b * U)(t) \quad , \tag{A3}$$

where $U(t) = \sum_{k=0}^{\infty} P\{T(k) \leq t\} = EN(t) + 1$. But

$$|b(t)| \leq E\{\int_t^{T(1)} (|Y(s)| + r)ds \quad ; \quad T(1) > t\}$$

$$\leq \frac{1}{t^2} E\{T(1)^2 \int_t^{T(1)} (|Y(s)| + r)ds \quad ; \quad T(1) > t\}$$

$$\leq \frac{1}{t^2} E\{T(1)^2 \int_0^{T(1)} |Y(s)|ds + rT(1)^3\}$$

and thus $|b(t)| \leq c_1 \min\{1, t^{-2}\}$ for some c_1. So, there exists α and c_2 such that

$$|a(t)| = |\int_{[0,t]} b(t-s)U(ds)|$$

$$\leq \int_{(t-\alpha,t]} c_2 \, U(ds) + \int_{[0,t-\alpha]} c_2 \, t^{-1}U(ds) \qquad \text{(A4)}$$

$$\leq c_2(U(t) - U(t-\alpha)) + c_2 \, U(t)/t \quad ;$$

the first term above is bounded in t by relation (1.7), p. 360, of FELLER [7], whereas the second is bounded by the elementary renewal theorem (see Theorem 5.5.2 of CHUNG [3]). From (A2) and (A3), it is evident from (A1) that

$$E(\int_0^t Y(s)ds - rt)$$

is a bounded function of t, proving the proposition.

Proof of Theorem 3: Set $v(t) = EX^2(t)$ and note that H2 implies that $v(t) \to \sigma^2$ as $t \to \infty$.

To prove i), suppose first that $\sigma^2 > 0$, and note that

$$t^{1/2}(r(t) - r) = \alpha(t) \sum_{i=1}^{k(t)} Z_i(t) + \beta(t) \qquad \text{(A5)}$$

where

so that H2 and H3 imply that $\{V^2(t)\}$ is uniformly integrable; thus, (A8) goes to zero as $t \to \infty$, verifying (A7). Combining (A6) and (A7) yields i) when $\sigma^2 > 0$.

If $\sigma^2 = 0$, we can reduce this case to the one preceding by applying the above results to $\tilde{Y}$, where

$$\tilde{Y}(t) = Y(t) + 2^{-1}N(0,1)t^{-1/2}$$

and $N(0,1)$ is independent of Y; we leave the details to the reader.

The proofs of ii) and iii) are similar to that of i); only $\beta(t)$'s behavior is different. For ii),

$$\beta(t) = t^{1/2}(E\overline{Y}(m(t))-r) \sim t^{1/2} b/m(t) = b\,\frac{k(t)^{1/2}}{m(t)^{1/2}} \to \infty$$

whereas for iii),

$$\beta(t) \sim b\,\frac{k(t)^{1/2}}{m(t)^{1/2}} \to b/c^{1/2} \quad .$$

Proof of Theorem 4: Note that

$$\frac{k(t)-1}{k(t)}\, m(t)\Gamma(t) = \frac{1}{k(t)} \sum_{i=1}^{k(t)} m(t)(\overline{Y}_i(m(t)) - r(t))^2 \tag{A9}$$

$$= \frac{1}{k(t)} \sum_{i=1}^{k(t)} m(t)(\overline{Y}_i(m(t)) - r)^2 - m(t)(r(t) - r)^2 \quad .$$

By Theorem 3, $t(r(t) - r)^2 \Longrightarrow \sigma^2 N(0,1)^2$ as $t \to \infty$ so that

$$m(t)(r(t) - r)^2 = \frac{m(t)}{t} \cdot t(r(t) - r)^2 \Longrightarrow 0 \quad , \tag{A10}$$

as $t \to \infty$, since $k(t) \to \infty$. For the other term in (A8), we note that

$$m(t)E(\overline{Y}_i(m(t)) - r)^2 = v(t) \quad ,$$

and

$$\operatorname{var}\Big(\frac{1}{k(t)} \sum_{i=1}^{k(t)} m(t)(\overline{Y}_i(m(t)) - r)^2\Big) \tag{A11}$$

$$= \frac{1}{k(t)} \{E|X(t)|^4 - v^2(t)\} \quad .$$

$$Z_i(t) = \gamma(t)\ (\overline{Y}_i(m(t)) - Er(t))$$

$$\gamma(t) = m(t)^{1/2}/(k(t)v(m(t)))^{1/2}$$

$$\alpha(t) = v(m(t))^{1/2}$$

$$\beta(t) = t^{1/2}(Er(t) - r) \quad .$$

Since $Er(t) = E\overline{Y}(m(t))$, H3 implies that

$$\beta(t) = t^{1/2}\,O(1/m(t)) = k(t)^{1/2}\,O(1/m(t)^{1/2}) \to 0 \qquad (A6)$$

as $t \to \infty$. Also, $\alpha(t) \to \sigma$ as $t \to \infty$. To treat the sum in (A5), we view the family of r.v.'s $\{Z_i(t) : 1 \le i \le k(t),\ t > 0\}$ as a triangular array. Note that $EZ_i(t) = 0$ and

$$\sum_{i=1}^{k(t)} EZ_i^2(t) = 1 \quad .$$

Furthermore, by Chebyshev's inequality,

$$\max_{1 \le i \le k(t)} P\{Z_i^2(t) > \varepsilon\} \le \frac{EZ_i^2(t)}{\varepsilon} = \frac{1}{k(t)\varepsilon} \to 0$$

as $n \to \infty$, so we conclude that $\{Z_i(t)\}$ is holospoudic (see p. 196-206 of CHUNG [3] for results and definitions); holospoudicity requires that each of the summands make a vanishingly small contribution to the total sum. To show that as $t \to \infty$,

$$\sum_{i=1}^{k(t)} Z_i(t) \Longrightarrow N(0,1) \quad , \qquad (A7)$$

we need to verify Lindeberg's condition. Observe that for $\eta > 0$,

$$\sum_{i=1}^{k(t)} E\{Z_i^2(t) \; ; \; Z_i^2(t) > \eta\} = k(t)E\{Z_i^2(t) \; ; \; Z_i^2(t) > \eta\} \qquad (A8)$$

$$= E\{V^2(t) \; ; \; V^2(t) > k(t)\eta\}$$

where $V^2(t) = m(t)(\overline{Y}(m(t)) - Er(t))^2/v(m(t))$. But

$$V^2(t) \le 2v(m(t))^{-1}\{X^2(m(t)) + (EX(m(t)))^2\}$$

Since $\{X^4(t) : t \ge 0\}$ is uniformly integrable, it follows that $EX^4(\cdot)$ is a bounded function (see (5.1) of [1]) so that the variance term (A9) tends to zero as $t \to \infty$. Thus,

$$\frac{1}{k(t)} \sum_{i=1}^{k(t)} m(t)(\overline{Y}_i(m(t)) - r)^2 - v(t) \Longrightarrow 0$$

as $t \to \infty$. Combining H2, (A9), (A10), and (A11) yields the result.

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