ESTIMATION OF STEADY-STATE CENTRAL MOMENTS BY THE REGENERATIVE METHOD OF SIMULATION

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Let X be a positive recurrent regenerative process on state space S with steady-state distribution π. Given a function $f: S \to \mathbf{R}$, we consider the problem of estimating the steady-state central moments $\mu_k(f) = \int_S (f(x) - r)^k \pi(\mathrm{d}x)$ where r is the steady-state mean of $f(X(\cdot))$. We obtain strong laws, central limit theorems, and confidence intervals for our estimators, and present numerical results.

statistical analysis of simulation output * regenerative processes

1. Introduction

Suppose we are simulating a stochastic process $X = \{X(t): t \geq 0\}$ with the intent of estimating the steady-state mean of the process. This problem has been extensively studied in the simulation literature and a number of methods have been developed to do the job. However, suppose that we are also interested in estimating the steady-state varience of X. To the best of our knowledge, no technique is available for estimating this variance. Our goal in this paper is to provide such a technique in the context of the regenerative method.

To be specific, let X be a (possibly) delayed regenerative process taking values in state space S, with regeneration times $T(-1) = 0 \leq T(0) < T(1) < \cdots$ (to incorporate regenerative sequences $\{X_n: n \geq 0\}$, we pass to the continuous time process $X(\cdot)$ defined by $X(t) = X_{[t]}$, where $[t]$ is the greatest integer less than or equal to t). Under quite general conditions (see, for example, Heyman and Sobel [4, p. 185], there exists a probability distribution π on S such that

$$r_t(f) = \frac{1}{t}\int_0^t f(X(s))\,\mathrm{d}s \to \int_S f(y)\pi(\mathrm{d}y) \equiv r(f), \tag{1.1}$$

a.s. as $t \to \infty$, for a broad class of functions $f: S \to \mathbf{R}$.

As is clear from (1.1), $r(f)$ has an interpretation as the steady-state mean of $f(X(\cdot))$. In certain applications, however, it may also be of interest to estimate the fluctuations of $f(X(\cdot))$ around its steady-state limit. To be precise, set $f_c(\cdot) = f(\cdot) - r(f)$ and put $v(f) = r(f_c^2)$; for $g: S \to \mathbf{R}$, define $g^m: S \to \mathbf{R}$ via $g^m(x) = g(x) \cdot g(x) \cdots g(x)$ (m times). Letting f_c^2 play the role of f in (1.1), we observe that $v(f)$ may be regarded as the steady-state variance of $f(X(\cdot))$. In the same spirit, the simulator may be interested in estimating the skewness or kurtosis of the steady-state distribution of $f(X(\cdot))$. Then estimating $r(f_c^3)$ and $r(f_c^4)$ would be required.

More generally, let $\mu_m(f) = r(f_c^m)$; then $\mu_m(f)$

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is the mth steady-state central moment of $f(\cdot)$. Clearly, $\mu_1(f)=0$ and $\mu_2(f)=v(f)$. Note that estimation of $\mu_m(f)=r(f_c^m)$ is not a special case of the standard regenerative method (see, for example, Iglehart [5]), since f_c depends on the unknown parameter $r(f)$, which itself must be estimated. Our goal, in this paper, is to develop an estimation methodology for the central moments, $\mu_m(f)$. In Section 2, we prove the required limit theorems upon which our methods are based. Section 3 develops confidence intervals for central moments, and numerical results are presented in Section 4 only for the case of steady-state variances ($m=2$). The reader who is only interested in estimating the steady-state variance, $\mu_2(f)$, should consult the confidence intervals given in eqs. (3.4) and (3.5) and the examples in Section 4.

2. Estimators and limit theorems for central moments

We assume throughout the remainder of this paper that

$$E\left(Y_1(|f|^{2m})+\tau_1^{2m}\right)<\infty, \tag{2.1}$$

where

$$\tau_n=T(n)-T(n-1) \quad \text{and}$$
$$Y_n(g)=\int_{T(n-1)}^{T(n)} g(X(s))\,ds$$

for functions g: $S\to\mathbf{R}$. Our goal is to estimate the central moments $\mu_k(f)$, $1\le k\le$ m.

The following binomial representation of the central moments is crucial to our development:

$$\mu_k(f)=r\left((f-r(f))^k\right)=\sum_{j=0}^{k}\binom{k}{j}r(f^j)(-1)^{k-j}r(f)^{k-j}. \tag{2.2}$$

For g: $S\to\mathbf{R}$, set $r(n,g)=\bar{Y}_n(g)/\bar{\tau}_n$, where

$$\bar{Y}_n(g)=\frac{1}{n}\sum_{k=1}^{n}Y_k(g)$$

and

$$\bar{\tau}_n=\frac{1}{n}\sum_{k=1}^{n}\tau_k.$$

Relation (2.2) suggests that

$$u(n,k)=\sum_{j=0}^{k}\binom{k}{j}r(n,f^j)(-1)^{k-j}r(n,f)^{k-j} \tag{2.3}$$

should be a reasonable estimator for $\mu_k(f)$.

Proposition 1.

$$u(n,k)\to\mu_k(f) \quad a.s.\ as \quad n\to\infty,$$
$$for \quad 1\le k\le 2m. \tag{2.4}$$

Proof. From (2.2) and (2.3), it is clear that we need only show that $r(n,f^k)\to r(f^k)$ a.s. as $n\to\infty$, for $k\le 2m$. The strong law of large numbers applies to both the numerator $\bar{Y}_n(f^k)$ and denominator $\bar{\tau}_n$, yielding the required convergence, provided that $E|Y_1(f^k)|<\infty$ and $E\tau_1<\infty$. Clearly, (2.1) implies that $E\tau_1<\infty$, whereas the inequality

$$E|Y_1(f^k)|\le E\left(\int_{T(0)}^{T(1)}|f(X(s))|^k ds\right)$$
$$\le E\left(\int_{T(0)}^{T(1)}(1+|f(X(s))|^{2m})ds\right)$$
$$=E\tau_1+EY_1(|f|^{2m})<\infty$$

provides the finiteness of the other moment. □

By Proposition 1, $u(n,k)$ is strongly consistent for $\mu_k(f)$. To state our next result, we shall use that notation $o_p(n^{-\frac{1}{2}})$ to denote any sequence of random variables (r.v.'s) δ_n such that $n^{\frac{1}{2}}\delta_n\Rightarrow 0$ as n $\to\infty$, where $\Rightarrow$ means weak convergence. The following properties follow easily from our definition and standard results about weak convergence (see Chung [2, p. 92]):

(i) if $X_n=o_p(n^{-\frac{1}{2}})$ and $Y_n=o_p(n^{-\frac{1}{2}})$, then $Z_n=X_n+Y_n=o_p(n^{-\frac{1}{2}})$,

(ii) if $X_n\Rightarrow X$ and $Y_n=o_p(n^{-\frac{1}{2}})$, then $Z_n=X_n\cdot Y_n=o_p(n^{-\frac{1}{2}})$. (2.5)

For our next limit result, we will need the following central limit theorem (CLT).

Proposition 2. *For* $1 \le k \le m$,

$$n^{\frac{1}{2}}\left(r(n, f^k) - r(f^k)\right) \Rightarrow \sigma(f^k)\mathrm{N}(0, 1)$$

as $n \to \infty$, *where*

$$\sigma^2(g) = \mathrm{E}Z_1^2(g)/(\mathrm{E}\tau_1)^2, \qquad Z_n(g) \equiv Y_n(g) - r(g)\tau_n, \tag{2.6}$$

and $\mathrm{N}(0, 1)$ *is a mean zero normal r.v. with unit variance.*

Proposition 2 is well known in the regenerative simulation literature (see, for example, Iglehart [5]); it is a CLT for estimators of the uncentered steady-state moments of $f(X(\cdot))$.

Proposition 3.

$$u(n, k) = r(n, f_k) + o_p(n^{-\frac{1}{2}}) \quad \textit{for} \quad 1 \le k \le m,$$

where

$$f_k(\cdot) = f_c^k(\cdot) - k\mu_{k-1}(f)f_c(\cdot). \tag{2.7}$$

Proof. Observe that

$$\begin{aligned} u(n, k) = {} & \sum_{j=0}^{k} \binom{k}{j} r(n, f^j)(-1)^{k-j} r(f)^{k-j} \\ & + \sum_{j=0}^{k-1} \binom{k}{j} r(n, f^j)(-1)^{k-j} \\ & \times \left(r(n, f)^{k-j} - r(f)^{k-j}\right). \end{aligned} \tag{2.8}$$

Evidently, for $0 \le p \le m$ and $1 \le l \le m$,

$$\begin{aligned} & \left(r(n, f)^l - r(f)^l\right) r(n, f^p) \\ & = \left(r(n, f) - r(f)\right) \cdot \left(\sum_{j=0}^{l-1} r(n, f)^j r(f)^{l-1-j}\right) \\ & \quad \times r(n, f^p). \end{aligned} \tag{2.9}$$

From the proof of Proposition 1,

$$\left(\sum_{j=0}^{l-1} r(n, f)^j r(f)^{l-1-j}\right) \cdot r(n, f^p) \to l r(f)^{l-1} \cdot r(f^p) \tag{2.10}$$

a.s. as $n \to \infty$, so that (2.5), (2.9) and (2.10), together with Proposition 2, imply that

$$\begin{aligned} & \left(r(n, f)^l - r(f)^l\right) \cdot r(n, f^p) \\ & = \left(r(n, f) - r(f)\right) \cdot l r(f)^{l-1} \cdot r(f^p) \\ & \quad + o_p(n^{-\frac{1}{2}}). \end{aligned} \tag{2.11}$$

By noting that the first sum on the right-hand side of (2.8) is $r(n, f_c^k)$, we can combine (2.8) and (2.11) to obtain

$$\begin{aligned} u(n, k) & = r(n, f_c^k) + \left(r(n, f) - r(f)\right) \\ & \quad \times \sum_{j=0}^{k-1} \binom{k}{j} r(f^j)(-1)^{k-j}(k-j) \\ & \quad \times r(f)^{k-j-1} + o_p(n^{-\frac{1}{2}}) \\ & = r(n, f_c^k) - r(n, f_c) \cdot k \\ & \quad \times \sum_{j=0}^{k-1} \binom{k-1}{j} r(f^j)(-1)^{k-1-j} \\ & \quad \times r(f)^{k-j-1} + o_p(n^{-\frac{1}{2}}) \\ & = r(n, f_k) + o_p(n^{-\frac{1}{2}}). \quad \square \end{aligned}$$

Let $U(n) = (u(n, 1), \ldots, u(n, m))^{\mathrm{T}}$ and $\boldsymbol{\mu} = (\mu_1(f), \ldots, \mu_m(f))^{\mathrm{T}}$. We take all vectors as column vectors and T denotes transpose.

Theorem 1. $n^{\frac{1}{2}}(U(n) - \boldsymbol{\mu}) \Rightarrow \mathrm{N}(\boldsymbol{0}, \boldsymbol{C}(f))$ *where* $\mathrm{N}(\boldsymbol{0}, \boldsymbol{C}(f))$ *is a multivariate normal r.v. with mean vector* $\boldsymbol{0}$. *and covariance matrix* $\boldsymbol{C}(f)$ *with elements given by*

$$C_{ij}(f) = \mathrm{E}Z_1(f_i)Z_1(f_j)/(\mathrm{E}\tau_1)^2.$$

Proof. We shall use the Cramér–Wold device (Billingsley [1, p. 48]) to prove this result. Let $\boldsymbol{a}$ be an arbitrary column vector in $\mathbf{R}^m$, and note that Proposition 3 implies that

$$\begin{aligned} & \boldsymbol{a}^{\mathrm{T}}(U(n) - \boldsymbol{\mu}) \\ & = \sum_{i=1}^{m} a_i\left(r(n, f_i) - \mu_i(f)\right) + o_p(n^{-\frac{1}{2}}) \\ & = \sum_{i=1}^{m} a_i \bar{Z}_n(f_i)/\bar{\tau}_n + o_p(n^{-\frac{1}{2}}), \end{aligned}$$

where

$$\bar{Z}_n(g) \equiv n^{-1} \sum_{k=1}^{n} Z_k(g) \quad \text{for} \quad g: S \to \mathbf{R}.$$

In the second equality we have used the fact that $\mu_i(f) = r(f_i)$. Standard arguments then show that

$$n^{\frac{1}{2}}\left(\boldsymbol{a}^{\mathrm{T}}(U(n) - \boldsymbol{\mu})\right) \Rightarrow \left(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{C}(f)\boldsymbol{a}\right)^{\frac{1}{2}}\mathrm{N}(0, 1)$$

as $n \to \infty$; the Cramér–Wold device completes the proof. $\square$

Theorem 1 shows that our estimators have an asymptotically normal distribution; it is, of course, a limit theorem expressed in terms of an index n corresponding to the number of regenerative cycles simulated. However, in certain settings, it is more natural to express limit theorems in terms of t, the amount of time that the process X has been simulated. Then, $N(t) = \max\{n: T(n) \le t\}$ is the number of regenerative cycles completed by time t. Set

$$u_t(k) = \begin{cases} u(N(t), k), & N(t) \ge 1 \\ 0, & N(t) < 1, \end{cases}$$

and

$$U_t = (u_t(1), \ldots, u_t(m)).$$

Proposition 4. *$u_t(k) \to \mu_k(f)$ a.s. as $t \to \infty$, for $1 \le k \le 2m$*

This follows immediately from Proposition 1 and the fact that $N(t) \to \infty$ a.s. as $t \to \infty$.

Theorem 2. *$t^{\frac{1}{2}}(U_t - \mu) \Rightarrow N(0, C^*(f))$ as $t \to \infty$, where $C^*(f)$ is given by*

$$C^*_{ij}(f) = EZ_1(f_i)Z_1(f_j)/E\tau_1.$$

The proof of Theorem 2 is based on making a 'random time change' by substituting the process $N(t)$ for the parameter n in Theorem 1, giving

$$N(t)^{\frac{1}{2}}(U(N(t)) - \mu) \approx N(0, C(f));$$

since $N(t) \approx t/E\tau_1$, we obtain the result. For a rigorous proof of a more general result, see Glynn and Iglehart [3, Theorem 3.10]. □

Specializing our results to the steady-state variance, we observe that the estimator $u(n, 2)$, respectively $u_t(2)$, is asymptotically normal with limiting variance given by $EZ_1(f_2)^2/(E\tau_1)^2$, respectively $EZ_1(f_2)^2/E\tau_1$. Since $EZ_1(f_2)^2 = EZ_1(f_c^2)^2$, it follows that the asymptotic variability of our steady-state variance estimator is unaffected by having to estimate $r(f)$. (Note that $EZ_1(f_c^2)^2/(E\tau_1)^2$ is the variance of the limiting normal r.v. which approaches $r(n, f_c^2)$.) For higher-order central moments, however, the variances will generally differ.

3. Confidence interval generation

In this section, we use our limit theorems of Section 2 to construct confidence intervals for the kth central moment. To accomplish this, we need consistent estimators for the covariance matrices $C(f)$ and $C^*(f)$. Set

$$\begin{aligned} A_k(n, i, j) &= \int_{T(k-1)}^{T(k)} (f(X(s)) - r(n, f))^i \mathrm{d}s \\ &\quad \times \int_{T(k-1)}^{T(k)} (f(X(u)) - r(n, f))^j \mathrm{d}u, \end{aligned}$$

and

$$\bar{A}_n(i, j) = \frac{1}{n} \sum_{k=1}^{n} A_k(n, i, j).$$

Proposition 5. *For $1 \le i, j \le m$,*

$$\bar{A}_n(i, j) \to EY_1(f_c^i)Y_1(f_c^j)$$

a.s. as $n \to \infty$.

Proof. Note that

$$\begin{aligned} \bar{A}_n(i, j) &= \sum_{k=0}^{i} \sum_{l=0}^{j} \tbinom{i}{k}\tbinom{j}{l}(-1)^{k+l} r(n, f)^{k+l} \\ &\quad \times \frac{1}{n} \sum_{p=1}^{n} Y_p(f^{i-k}) Y_p(f^{j-l}) \tau_p^{k+l} \\ &\to \sum_{k=0}^{i} \sum_{l=0}^{j} \tbinom{i}{k}\tbinom{j}{l}(-1)^{k+l} r(f)^{k+l} \\ &\quad \times EY_1(f^{i-k}) Y_1(f^{j-l}) \tau_1^{k+l} \\ &= EY_1(f_c^i) Y_1(f_c^j) \end{aligned}$$

a.s. as $n \to \infty$, by the strong law of large numbers. □

Let

$$\begin{aligned} C(n, i, j) = \frac{1}{\bar{\tau}_n^2} \{ & \bar{A}_n(i, j) - ju(n, j-1) \\ & \times A_n(i, 1) - u(n, j)\bar{A}_n(i, 0) \\ & - iu(n, i-1)\bar{A}_n(1, j) \\ & + iju(n, i-1)u(n, j-1)\bar{A}_n(1, 1) \\ & + iu(n, i-1)u(n, j)\bar{A}_n(1, 0) \\ & - u(n, i)\bar{A}_n(0, j) \\ & + ju(n, i)u(n, j-1)\bar{A}_n(0, 1) \\ & + u(n, i)u(n, j)\bar{A}_n(0, 0) \}. \end{aligned}$$

Propositions 1 and 5 together yield the following result.

Proposition 6. *For* $1 \le i, \ j \le m, \ C(n, i, j) \to C_{ij}(f)$ *a.s. as* $n \to \infty$.

Application of the converging-together lemma (Billingsley [1, p. 5]) to Theorem 1 and Proposition 6 shows that if $C_{kk}(f) > 0$ $(1 \le k \le m)$, then

$$\left[u(n, k) - z(\delta)\frac{C_{kk}(f)^{\frac{1}{2}}}{n^{\frac{1}{2}}},\right.$$
$$\left. u(n, k) + z(\delta)\frac{C_{kk}(f)^{\frac{1}{2}}}{n^{\frac{1}{2}}}\right] \tag{3.1}$$

is an asymptotic $100(1-\delta)\%$ confidence interval for $\mu_k(f)$, where $z(\delta)$ solves $P\{N(0,1) \le z(\delta)\} = 1 - \delta/2$.

A similar confidence interval can be based on simulation of X to time t. Set

$$C_t(i, j) = \begin{cases} C(N(t), i, j)\bar{\tau}_{N(t)}, & N(t) \le 1 \\ 0, & N(t) < 1. \end{cases}$$

Proposition 7. *For* $1 \le i, \ j \le m, \ C_t(i, j) \to C_{ij}^*(f)$ *a.s. as* $t \to \infty$.

This result is an immediate consequence of Proposition 6, and leads to the following asymptotic $100(1-\delta)\%$ confidence interval for $\mu_k(f)$ (assuming $C_{kk}(f) > 0$, $1 \le k \le m$):

$$\left[u_t(k) - z(\delta)\frac{C_{kk}^*(f)^{\frac{1}{2}}}{t^{\frac{1}{2}}},\right.$$
$$\left. u_t(k) + z(\delta)\frac{C_{kk}^*(f)^{\frac{1}{2}}}{t^{\frac{1}{2}}}\right]. \tag{3.2}$$

For the steady-state variance $v(f) = \sigma_2(f)$, we can use our knowledge that $\mu_1(f) = 0$ to obtain a simpler family of estimates for $C_{22}(f)$ and $C_{22}^*(f)$. Note that

$$\begin{aligned} EZ_1(f_2)^2 &= EZ_1(f_c^2)^2 \\ &= EY_1(f_c^2)^2 - 2v(f)EY_1(f_c^2)\tau_1 \\ &\quad + v(f)^2 E\tau_1^2. \end{aligned} \tag{3.3}$$

Set

$$\begin{aligned} C^0(n) = \frac{1}{\bar{\tau}_n^2}\{&\bar{A}_n(2,2) - 2u(n,2)\bar{A}_n(2,0) \\ &+ u(n,2)^2\bar{A}_n(0,0)\}, \end{aligned}$$

$$C_t^0 = \begin{cases} C^0(N(t))\bar{\tau}_{N(t)}, & N(t) \ge 1 \\ 0, & N(t) < 1. \end{cases}$$

Propositions 1 and 5 show that $C^0(n) \to C_{ij}(f)$ a.s. as $n \to \infty$ and $C_t^0 \to C_{ij}(f)$ a.s. as $t \to \infty$, provided $m \ge 2$. Thus, if $C_{22}(f) > 0$, the following intervals are $100(1-\delta)\%$ asymptotic confidence intervals for $v(f)$:

$$\left[u(2, n) - z(\delta)\frac{C^0(n)^{\frac{1}{2}}}{n^{\frac{1}{2}}},\right.$$
$$\left. u(2, n) + z(\delta)\frac{C^0(n)^{\frac{1}{2}}}{n^{\frac{1}{2}}}\right], \tag{3.4}$$

$$\left[u_t(2) - z(\delta)\frac{(C_t^0)^{\frac{1}{2}}}{t^{\frac{1}{2}}},\ u_t(2) + z(\delta)\frac{(C_t^0)^{\frac{1}{2}}}{t^{\frac{1}{2}}}\right]. \tag{3.5}$$

4. Numerical results

To illustrate the results obtained in the previous section we have simulated three models: the waiting time process in the $M/M/1$ queue ($\rho = 0.5$), and (s, S) inventory model, and the classical repairman model. For all three models we selected f to be the identity function ($f(x) = x$) and $k = 2$, so that our goal was to estimate the variance of the steady-state distribution.

Example 1: $M/M/1$ queue. This model is the single server queue with Poisson arrivals and exponential service times. We simulated the waiting time process $W = \{W_n: n \ge 0\}$, where W_n is the waiting time (exclusive of service time) of the nth customer. Our simulations were carried out for arrival rate $\lambda = 5$ and service rate $\mu = 10$, and hence the traffic intensity $\rho = 0.5$. This guarantees that $W_n \Rightarrow W$ as $n \to \infty$. Regenerative cycles begin at those values of n for which $W_n = 0$. The quantity being estimated here is $\sigma^2\{W\} = 3.0$. We did 50 replications of 5000 cycles each. The sample mean of the 50 point estimates, $u(2{,}5000)$, was

3.0417, and the sample mean of the 50 point estimates of $C_{22}(f)^{\frac{1}{2}}$, $C^0(5000)^{\frac{1}{2}}$, was 13.2782. As a result, the sample mean of the 90% confidence intervals was [2.7328, 3.3506]. The coverage fraction was 58%.

Example 2: (s, S) inventory model. This model is a periodic review inventory model with a stationary (s, S) ordering policy. An (s, S) policy is characterized by two positive integers: s and S with $s < S$. If the amount of inventory on hand plus on order is less than s, order to bring the sum up to S. If the inventory is greater than or equal to s, do not order. Let X_n denote the level of inventory on hand plus on order in period n immediately after the ordering decision. If d_n denotes the demand in period n, then

$$X_{n+1} = \begin{cases} X_n - d_n, & d_n \le X_n - s \\ S, & \text{otherwise.} \end{cases}$$

We assume that $s \le X_0 \le S$. The state space of $\{X_n: n \ge 0\}$ is $\{s, s+1, \ldots, S\}$. For this example we have selected $s = 6$, $S = 10$, and

$$P\{d_n = j\} = \begin{cases} \frac{3}{8}, & j = 0 \\ \frac{1}{4}, & j = 1 \\ \frac{3}{16}, & j = 2 \\ \frac{1}{8}, & j = 3 \\ \frac{1}{16}, & j = 4. \end{cases}$$

Again we simulate to estimate $\sigma^2\{X\} = 2.3333$. Using $i = 10$ as the regenerative state we ran 50 replications of 1000 cycles each. The sample mean of the 50 point estimates, $u(2,1000)$, was 2.3352 and the sample mean of the 50 point estimates for $C_{22}(f)^{\frac{1}{2}}$, $C^0(1000)^{\frac{1}{2}}$, was 1.2396. The sample mean of the resulting 50 90% confidence intervals was [2.2708, 2.3997]. The coverage fraction was 94%.

Example 3: classical repairman model. This model is a continuous time Markov chain with $X(t)$ denoting the number of failed units undergoing or waiting for service at the repair facility at time t. We have $m + n$ identical machines each with an exponential failure time with failure rate λ. At most n of the units operate at one time, the other m being thought of as spares. When a unit fails, it is sent to a repair facility consisting of s repairmen (servers) having exponential repair (service) times with repair rate μ. With these assumptions $\{X(t): t \ge 0\}$ is a birth–death process with state space $\{0, 1, \ldots, m+n\}$, birth parameters $\lambda_i = (n - [i - m]^+)\lambda$, and death parameters $\mu_i = \mu \cdot \min(i, s)$. For this example we have used $n = 10$, $m = 4$, $\lambda = 1$, $\mu = 4$ and $s = 3$. Again we are interested in estimating the steady-state variance, $\sigma^2\{X\} = 5.231$. We ran 50 replications of 1000 cycles each with the regenerative state taken to be $i = 2$. The sample mean of our 50 point estimates, $u(2,1000)$, was 5.1916, and the sample mean of the 50 point estimates of $C_{22}(f)^{\frac{1}{2}}$, $C^0(5000)^{\frac{1}{2}}$, was 11.5562. The sample mean of the 50 90% confidence intervals was [4.5905, 5.7927]. The coverage fraction was 74%.

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