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EQUITABLE ASSIGNMENT RULES

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ABSTRACT

This paper investigates the formalization of an important class of management decision problems. The problems considered are those of making equitable workload assignments to personnel. The paper proposes a series of intuitively appealing assignment rules, including random assignment, fixed assignment, block rotation and rules that reverse inequities caused by the last period's assignments. It is shown that in the two-person case none of these rules satisfies the simple criterion that cumulative differences of workload assignments among personnel become and remain small. Differences in the properties of these rules are investigated under three additional but less strenuous criteria. It is shown that a new assignment rule called the "counter-current" rule does satisfy the criterion stated above; further, it is shown that it is an optimal rule under a fairly weak set of requirements. The extension of the results from the two-person case to the n-person case to the n-person case is discussed briefly and some initial results are presented.

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## SIGNIFICANCE AND EXPLANATION

Consider the following assignment problem. A car rental company wishes to assign cars in such a way so as to equalize wear. The assignment rule must reflect the fact that the wear introduced by a customer is a random quantity. A similar situation is faced by a manager who must assign random workloads to employees in an equitable manner. In this paper, we consider decision rules for problems of the above type. We show that while a number of commonly used decision rules have undesirable asymptotic properties, there exists a rule, which we call countercurrent, that has good long-run characteristics and is optimal for a reasonable criterion function. Furthermore, the countercurrent decision rule is easy to implement.

## 1. Introduction

Consider a facility with two employees that is required to process two tasks per day. Assume that each task has an associated index of effort and that the task effort indices form stochastic sequences. The goal of the facility manager is to assign tasks to employees in as "equitable" a manner as possible.

The concept of equitable treatment of employees is common in the organizational behavior literature. In particular, the perceived equity of pay received for work delivered has received attention. According to Shapiro and Wahba (1978), "Dissatisfaction (with pay) results in many dysfunctional reactions such as turnovers, absenteeism, alcohol and drug problems, union formations, strikes, slowdowns, decreased performance, grievances, increased training costs, high accident rates and in turn increased unemployment and accident insurance costs." Also of interest are definitions of perceived equity based on perceptions of the ratio of rewards to the individual from the organization to the input effort provided by the individual and the comparison of these reward ratios among individuals in the organization. (Adams (1963 and 1965))

Shapiro and Wahba (1978) in their empirical study, show that social comparison of pay received for work done is the single most important variable in explaining pay dissatisfaction. Andrews and Henry (1963), who surveyed managers who were dissatisfied with their pay, report that 87% of those managers felt that their subordinates had a better outcome/input ratio than

they did. Lawler (1971) reports similar results. In the area of pay satisfaction, perceived equity clearly is an important, and possibly the premier issue.

Communications with a number of directors of nursing in acute care hospitals indicate that perceived inequities of work assignments among staff nurses can profoundly disrupt a nursing unit, resulting in absenteeism, turnover and other problems cited by Shapiro and Wahba. In this case, the equity issue surfaces around work assignment alone, aside from any differences in pay.

In a related setting, Adams (1963) has developed a theory of motivation related to perceived equity of rewards. He explores the implications of equity for prediction of individual behavior in business organizations.

In our case, we assume that the manager is only free to adjust work assignments. Thus, the equity issue refers to equalization of the work effort assigned to the employee pool.

(1.1) Example. A director of nursing manages two nurses and two nursing units. Assume that the difficulty of each assignment is given by the combined patient care acuity measure associated with the patients on a unit for the shift of duty being considered.

Nursing care acuity measures are systems that survey the patient's need for clinical care, assistance with daily living (e.g. feeding, bathing, etc.), education, emotional support and other aspects of nursing care. They are not designed to assess the severity of illness of the patient. They are instead used primarily to assess the amount and level of nursing care required.

Clearly, as patients come and go, the sum of the acuity levels of patients on a unit may fluctuate unpredictably. In some cases, the acuity assessments made on a unit at the beginning of a shift will remain accurate over the entire shift. In this case we know in advance the level of effort and skill required to staff that unit on that shift. In other cases, such as psychiatric and geriatric units, the assessments made at the beginning of the shift may be grossly inaccurate at the end of the tour. Older patients may become confused and psychiatric patients may experience an exacerbation of their condition, thus requiring higher levels of attention. In the first case, we assume that we know the acuity requirement in advance of the work assignment event though that level varies stochastically from assignment period to assignment period. In the second case, we cannot predict the level of acuity associated with an assignment in advance, although we may know the average value of acuity associated with the assignment.

The problem of equalizing effort also arises outside of the nursing context described above.

(1.2) Example. The manager of a typing pool receives two typing assignments per day. The goal of the manager is to assign the typing tasks among the two available typists so as to equalize the amount of work assigned to the employees over a given time horizon.

(1.3) Example. Consider a factory production unit with two machines. On each day, the unit is required to process two tasks. The goal of the production manager is to assign tasks to the available machines so as to equalize wear.

(1.4) Example. The manager of a car rental company wishes to assign the company's two cars to customers so as to equalize mileage.

In this paper, we will analyze decision rules for the above task assignment problems. Section 2 develops the mathematical framework for the problem and discusses basic properties of several common, intuitively appealing decision rules. In Section 3, we examine a rule, which we call counter-current, which is optimal with respect to a number of different criteria. Section 4 is devoted to the finer stochastic properties of the counter-current decision rule. Finally, in Section 5, we briefly discuss the n-task, n-person case, and offer some concluding remarks.

## 2. The Class of Task Assignment Rules

In order to develop a mathematical framework for the task assignment problem, we consider Example 1.1. Let the acuity measures on day n for the two nursing units be given by  $V_n$  and  $W_n$ . We assume, throughout this paper, that:

- A1.  $\{(V_n, W_n): n \geq 1\}$  is a sequence of independent, identically distributed (i.i.d.) random vectors (r.v.'s)
- A2. there exists  $K < \infty$  such that  $P\{|D_n| \leq K\} = 1$
- A3.  $P\{D_n \leq x\}$  has a density which is positive on  $[-K, K]$   
where  $D_n = V_n - W_n$
- A4.  $ED_n > 0$ .

We also suppose that there exists an independent sequence  $\{U_n : n \geq 1\}$  of i.i.d. uniform r.v.'s.

A decision rule is a sequence  $\{\alpha_n : n \geq 1\}$  of r.v.'s taking values in  $\{0,1\}$ .

A decision rule is said to be a randomized non-anticipating decision rule

if

$$\alpha_n = I_{\{U_n \leq p_n\}} f_n(V_1, W_1, \alpha_1, \dots, V_{n-1}, W_{n-1}, \alpha_{n-1}, V_n, W_n)$$

for some Borel-measurable function  $f_n$  and real number  $p_n$  ( $I_A$  denotes a r.v.

which is 1 or 0 depending on whether or not A has occurred). A rule is said

to be randomized and strictly non-anticipating if it can be represent-

ed as

$$\alpha_n = I_{\{U_n \leq p_n\}} f_n(V_1, W_1, \alpha_1, \dots, V_{n-1}, W_{n-1}, \alpha_{n-1}).$$

Set

$$X_n = \sum_{i=1}^n \alpha_i V_i + (1 - \alpha_i) W_i$$

$$Y_n = \sum_{i=1}^n (1 - \alpha_i) V_i + \alpha_i W_i.$$

If we follow the nursing unit example, we interpret  $X_n$  and  $Y_n$  as the cumulative amount of task effort assigned to nurses 1 and 2, respectively, by time  $n$ .

(2.1) Example. Set  $\alpha_n \equiv 1$ . Since  $ED_n > 0$ ,  $Z_n \rightarrow \infty$  a.s. where

$$Z_n = X_n - Y_n.$$

Such a decision rule is clearly inequitable, due to an obvious lack of symmetry. Any "good" rule should be symmetric in the sense that for all  $n$ ,

$$P\{\alpha_n = 0 | \vec{V}_n, \vec{W}_n\} = 1/2 \text{ a.s.}$$

where  $\vec{V}_n = (V_1, \dots, V_n)$ ,  $\vec{W}_n = (W_1, \dots, W_n)$ . Other desirable properties of a decision rule are:

(P1). the r.v.  $|Z_n|$  should grow as slowly as possible

(P2).  $E\{T_n | X_n \geq Y_n\}$  should be as small as possible, where

$$T_n = \inf\{m \geq n : X_{m+n} < Y_{m+n}\}$$

(P3).  $N_n/n$  should be as close to 1/2 as possible, where

$$N_n = \sum_{k=1}^n I_{\{X_k \geq Y_k\}}$$

Before proceeding to a discussion of optimal decision rules in Section 3, we first examine five decision rules that either have been used in practice for scheduling or are intuitively appealing as assignment rules.



$$(R1). \alpha_n = I_A, \text{ where } A = \{U_1 \leq 1/2\}$$

$$(R2). \alpha_{2n} = I_A, \alpha_{2n+1} = 1 - I_A, \text{ where } A = \{U_1 \leq 1/2\}$$

$$(R3). \alpha_n = I_{A_n}, \text{ where } A_n = \{U_n \leq 1/2\}$$

$$(R4). \alpha_n = I_{B_n}, \text{ where } B_1 = \{U_1 \leq 1/2\}, B_n = \{Z_{n-1} \leq Z_{n-2}\}, \text{ for } n \geq 2.$$

$$(R5). \alpha_n = I_{B_n}, \text{ where } B_1 = \{U_1 \leq 1/2\}, B_n = \{Z_{n-1} \leq Z_{n-2}, D_n > 0\} \cup \\ \{Z_{n-1} \geq Z_{n-2}, D_n < 0\} \text{ for } n \geq 2.$$

Rule R1 is equivalent to making the first assignment of personnel to duties

by flipping a fair coin. However, once these assignments are made they are assumed to remain there indefinitely. In other words, if nurse "A" draws Unit 2 on the first assignment it is assumed that she/he remains on assignment there in the future. This rule is "fair" in the sense that both nurses have equal probability of being assigned to a particular unit.

Rule R2 makes the first assignment as in Rule R1 but thereafter strictly rotates the nurses between assignments. This rule is a trivial example of the "Block Rotation" systems that have been used in personnel scheduling for some time. The rule is "fair" in the sense that both nurses are assigned to a given duty station the same fraction of the time.

Rule R3 assigns personnel to duties by extending the randomization employed at the beginning of Rule R1 to every period. This "purely random" rule could be realized in practice by repeated application of the "coin tossing" mechanism and may be approximated by haphazard processes where the assignment mechanism employed no previous memory of previous assignments or their outcomes.

Rules R4 and R5 look to the previous period and examine the outcome of the assignments of that period. The assignment made in this period attempts to reverse the inequities of the previous period. In R5, the rule is "clairvoyant"; it assumes knowledge of the true acuity measures for the shift before the assignments are made. Rule R4 is the non-clairvoyant version of R5, in which the outcome of the assignment is unpredictable. Both are "fair"--they try to continually reverse any inequities that arose from the previous

period's assignment. While it may be difficult to point to instances in which these rules are employed explicitly, they represent a natural tendency of management to compensate employees for previous inequities.

The following table summarizes the behavior of the five rules under criteria P1 - P3 (all limits are limits in weak convergence).

		R1	R2	R3	R4	R5
P1	$\lim_{n \rightarrow \infty} n^{-1/2}  Z_n $	$\infty$	$\sigma_2 C$	$\sigma_3 C$	$\sigma_4 C$	$\sigma_5 C$
P2	$E\{T_n   X_n > Y_n\}$	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$
P3	$\lim_{n \rightarrow \infty} N_n/n$	$L_1$	$L_2$	$L_2$	$L_2$	$L_2$

In the above table, the limit r.v.'s  $C$ ,  $L_1$ , and  $L_2$  have distributions given by

$$P\{C \leq x\} = (2/\pi)^{1/2} \int_0^x \exp(-t^2/2) dt; \quad x \geq 0$$

$$P\{L_1 \leq x\} = \begin{cases} 0; & x < 0 \\ 1/2; & 0 \leq x < 1 \\ 1; & x \geq 1 \end{cases}$$

$$P\{L_2 \leq x\} = 2(\arcsin(x^{1/2}))/\pi, \quad 0 < x < 1$$

and

$$\sigma_2 = (\text{var}(D_1))^{1/2}$$

$$\sigma_3 = (ED_1^2)^{1/2}$$

$$\sigma_4 = (ED_1^2 - (ED_1 \cdot E|D_1|) / P\{D_1 \geq 0\})^{1/2}$$

$$\sigma_5 = (ED_1^2 - (E|D_1|)^2)^{1/2}$$

While the rules R1, R2, R3, and R4 have some intuitive appeal as "fair" assignment rules, we see from the table above the none of these simple rules satisfies performance criteria necessary for a truly equitable assignment rule. First, all the rules have the property that the cumulative difference in acuity measures ( $Z_n$ ) grows as the square root of n or faster (in the case of R1). In the case of property P2 we find that if, on a given trial, one of the nurses is ahead of the other nurse in cumulated acuity measure, then in the case of R1, R2 and R3 the expected time required to equalize the cumulated acuity measures is infinite. In the case of the third property, we wish to know the fraction of the time that one nurse finds her/himself with a higher cumulative acuity measure than the second nurse. Ideally, this random variable would converge to a distribution with a single atom at 1/2. We see that none of the rules has this property. In the case of R1, the distribution converges to a single atom at 0 or 1; in the other cases, the convergence is to the arcsin law, which guarantees that although the limiting distribution is symmetric, 1/2 is the least likely region for the limit random variable.

With the exception of the R4 entry for criterion P1, all entries of the table may be verified by routine application of the central limit theorem (CLT) for i.i.d. r.v.'s, classical random walk results (Theorem 8.4.4 of Chung, (1974)) and the functional form of the arcsin law (Billingsley, (1968) p. 80).

One merely uses the following representations for  $Z_n$ :

$$(R1). \quad Z_n = (2I_A - 1) \left( \sum_{i=1}^n D_i \right)$$

$$(R2). \quad Z_{2n} = (2I_A - 1) \left( \sum_{i=1}^n D_{2i} - D_{2i-1} \right)$$

$$(R3). \quad Z_n = \sum_{i=1}^n (2I_{A_i} - 1) D_i$$

$$(R5). \quad Z_{2n} = (2I_\Lambda - 1) \left( \sum_{i=1}^n |D_{2i-1}| - |D_{2i}| \right)$$

where  $\Lambda = \{D_1 > 0, \alpha_1 = 1\} \cup \{D_1 < 0, \alpha_1 = 0\}$ . The R4 entry uses the fact that  $Z_n$  then has the form

$$Z_n = (2I_\Lambda - 1) \sum_{i=1}^n (-1)^{\ell(i)} D_i$$

where  $\Lambda$  has been defined above and  $\ell(i) = \max\{k: S_k \leq i\}$ , where  $S_1 = 1$  and

$$S_{k+1} = \inf\{n > S_k : D_n \geq 0\}$$

To obtain a CLT for  $Z_n$ , note that

$$Z_{S_n} = (2I_\Lambda - 1) \left( |D_1| + \sum_{k=1}^{n-1} (-1)^k \sum_{j=S_k+1}^{S_{k+1}} D_j \right)$$

$$\triangleq (2I_\Lambda - 1) \left( |D_1| + \sum_{k=1}^{n-1} (-1)^k \beta_k \right)$$

where  $\{\beta_k : k \geq 1\}$  is i.i.d. Now,  $\sigma^2(Z_{S_n}) = n\sigma^2(\beta_1)$  and

$$\sigma^2(\beta_1) = \sigma^2(D_1)E(S_2 - S_1) + \sigma^2(S_2 - S_1)(ED_1)^2 + 2ED_1(E(S_2 - S_1)\hat{\beta}_1)$$

where  $\hat{\beta}_k = \beta_k - (S_{k+1} - S_k)ED_1$ ; the above equality is the second moment version of Wald's identity. To simplify the above expression, observe that  $S_2 - S_1$  is geometric so

$$E(S_2 - S_1) = 1/P\{D_1 \geq 0\}$$

$$\sigma^2(S_2 - S_1) = P\{D_1 < 0\}/P\{D_1 \geq 0\}^2.$$

Also,

$$E\{(S_2 - S_1)\beta_1 | S_2 - S_1 = k\} = k(E\{D_1 | D_1 \geq 0\} + (k-1)E\{D_1 | D_1 < 0\})$$

so

$$E(S_2 - S_1)\beta_1 = (ED_1 + E(D_1 I_{\{D_1 < 0\}}))/P\{D_1 \geq 0\}^2.$$

Substituting the above relations in our expression for  $\sigma^2(\beta_1)$  and simplifying, we find that  $\sigma^2(Z_{S_n}) = n\sigma^2/P\{D_k \geq 0\}$ . Application of the classical CLT therefore proves that

$$n^{-1/2}Z_{S_n} \Rightarrow (\sigma^2/P\{D_k \geq 0\})^{1/2} N(0,1)$$

as  $n \rightarrow \infty$ , where  $\Rightarrow$  denotes weak convergence, and  $N(0,1)$  is a normal r.v. with zero mean and unit variance. Since  $\ell(n)/n \Rightarrow P\{D_1 \geq 0\}$  as  $n \rightarrow \infty$ , we may apply Theorem 7.3.2 of Chung (1974) to conclude that

$$\ell(n)^{-1/2}Z_{S_{\ell(n)}} \Rightarrow (\sigma^2/P\{D_k \geq 0\})^{1/2} N(0,1)$$

as  $n \rightarrow \infty$ . Noting that  $n^{-1/2} (Z_{S_{\ell(n)}} - Z_n) \Rightarrow 0$ , a standard argument then yields

$$n^{-1/2} Z_n \Rightarrow \sigma_4 \quad N(0,1)$$

as  $n \rightarrow \infty$ , which is P1; for P3, we apply the continuous mapping theorem to an invariance principle version of the above CLT.

The following inequalities are easily proved  $\sigma_2 \leq \sigma_3$ ,  $\sigma_4 \leq \sigma_2$ ,  $\sigma_5 \leq \sigma_2$ .

Thus, in terms of criterion P1, R1, R3, and R2 are the worst, second worst, and third worst decision rules, respectively. Somewhat surprisingly, it can be shown by example that both  $\sigma_4 \leq \sigma_5$  and  $\sigma_5 \leq \sigma_4$  are possible -- the direction of the inequality (and thus the P1 performance) depends on the joint distribution of  $(V_i, W_i)$ .

### 3. Counter-current Decision Rules: Definition and Basic Properties

Consider the deterministic situation where  $D_n \equiv \mu > 0$ . We then have

(3.1) Lemma: The rule  $\{\alpha_i : 1 \leq i \leq n\}$  which minimizes  $E\{\sum_{k=1}^n |Z_k|\}$  is given

by

$$\alpha_1 = I\{U_1 \leq 1/2\}, \quad \alpha_k = I\{Z_{k-1} < 0\}, \quad k > 1.$$

Proof: Observe that

$$\sum_{k=1}^n |z_k| = \mu \sum_{k=1}^n \left| \sum_{j=1}^k (2\alpha_j - 1) \right|$$

We claim that for each  $k \geq 1$ ,

$$\left| \sum_{j=1}^k 2\alpha_j - 1 \right| + \left| \sum_{j=1}^{k+1} 2\alpha_j - 1 \right| \geq 1$$

for if either term vanishes, then the other term must equal 1. Hence,

$$(3.2) \quad \sum_{k=1}^n |z_k| \geq j\mu$$

if  $n = 2j$  or  $2j - 1$ . Now, it is trivially verified that the rule given in the lemma attains the lower bound. ||

The above rule behaves nicely for deterministic sequences. For example,  $|z_k|$  remains bounded (see P1),  $E\{T_n | X_n - Y_n\} = 1$  (see P2), and  $N_n/n \rightarrow 1/2$  (see P3). This behavior suggests that one should try to generalize the rule to the stochastic case.

(3.3) Definition: We call the rule defined by

$$\alpha_1 = I\{U_1 \leq 1/2\}, \quad \alpha_n = I\{z_{n-1} < 0\}$$

the strict counter-current decision rule, and the rule

$$\alpha_1 = I\{U_1 \leq 1/2\}$$

$$\alpha_n = I\{z_{n-1} < 0, D_n > 0\} + I\{z_{n-1} > 0, D_n < 0\}$$



the counter-current decision rule.

The strict counter-current decision rule is strictly non-anticipating and the counter-current decision rule is anticipating. We now proceed to derive certain asymptotic properties of  $\{Z_n : n \geq 1\}$  under the two decision rules.

First, we examine the counter-current decision rule. Note that for  $n \geq 2$ ,

$$(3.4) \quad Z_{n+1} = Z_n - (\text{sign } Z_n) |D_{n+1}|$$

and hence  $Z_n$  is a Markov chain (M.C.) on the real line  $R$ ; its transition kernel is given by

$$P_1(x, z) \triangleq P\{Z_{n+1} \leq x | Z_n = z\}$$

$$= \begin{cases} 1 - G(z-x); & z \geq 0 \\ G(x-z); & z < 0 \end{cases}$$

where  $G(x) = P\{|D_n| \leq x\}$ . It is easily verified that

$$(3.5) \quad h_1(y) = \frac{1}{2E|D_n|} (1 - G(|y|))$$

is a stationary density for  $\{Z_n : n \geq 1\}$  (see Feller, (1971) p. 208) for

calculation of a closely related density). For further analysis of this M.C., it is convenient to introduce the following notion.

(3.6) Definition. A M.C.  $\{Z_n : n \geq 1\}$  on  $R$  is said to be  $\lambda$ -irreducible if there exists a probability measure  $\lambda(\cdot)$  such that if  $\lambda(E) > 0$  ( $E$  a Borel set), then

$$\sum_{n=1}^{\infty} 2^{-n} P\{Z_n \in E | Z_1 = z\} > 0$$

for all  $z$ .

(3.7) Lemma: Under  $A1 - A4$ , then M.C.  $\{Z_n : n \geq 1\}$  defined by (3.4) is  $\lambda$ -irreducible.

Proof. Taking  $\lambda$  to be normalized Lebesgue measure on  $(-K/2, K/2)$  we observe that by choosing  $n = [z/K] + 2$ , ( $[ ]$  denotes greatest integer) one obtains

$$P\{Z_n \in E | Z_1 = z\} > 0$$

for any  $E$  for which  $\lambda(E) > 0$ . ||

We can now prove the following ergodic theorem.

(3.8) Theorem. Let  $\{Z_n : n \geq 1\}$  be defined by the counter-current decision rule, and suppose that  $A1 - A4$  hold. Then, for any function  $k(\cdot)$  satisfying

$$\int_{-\infty}^{\infty} |k(y)| \cdot h_1(y) dy < \infty,$$

it follows that

$$(3.9) \quad \frac{1}{n} \sum_{j=1}^n k(Z_j) \rightarrow \int_{-\infty}^{\infty} k(y) h_1(y) dy \text{ a.s.}$$

Proof. By Lemma 3.7, we have that  $Z_n$  is  $\lambda$ -irreducible; thus  $h_1(\cdot)$  is the unique stationary probability of  $\{Z_n : n \geq 1\}$  (Revuz (1975)). It follows that if  $Z_1$  has the stationary distribution, then  $\{Z_n : n \geq 1\}$  is a stationary ergodic sequence (Ash (1972)), so one may apply Birkhoff's ergodic theorem (Lamperti (1977), p. 92) to conclude that

$$\int_{-\infty}^{\infty} h(z) P\left\{ \frac{1}{n} \sum_{j=1}^n k(Z_j) \rightarrow \int_{-\infty}^{\infty} k(y) h_1(y) dy \mid Z_1 = z \right\} dz = 1$$

$$(3.10) \text{ i.e. } P\left\{ \frac{1}{n} \sum_{j=1}^n k(Z_j) \rightarrow \int_{-\infty}^{\infty} k(y) h_1(y) dy \mid Z_1 = z \right\} = 1$$

for a.e.  $z \in [-K, K]$  (observe that by A3,  $h(\cdot)$  vanishes for  $|z| > K$ ). But if  $Z_n$  evolves according to the counter-current decision rule, then  $|Z_1| = |D_1|$ , and so  $Z_1$  is concentrated on  $[-K, K]$  and has a density there. Thus, (3.9) follows immediately from (3.10). ||

In particular, setting  $k(x) = I_{[0, \infty]}(x)$ , it is immediate that

$$N_n/n \rightarrow 1/2 \text{ a.s.}$$

Thus, the counter-current rule leads to a  $Z_n$  sequence with a "good" P3

property. Also, (3.5) implies that  $Z_n$  remains "bounded" in some sense, and thus improves on decision rules R1 - R5 (in which  $|Z_n|$  grows at rate  $n^{1/2}$ ).

As for criterion P2, observe that on  $\{Z_n > 0\}$ ,

$$T_n = \min\{k: |D_{n+1}| + \dots + |D_{n+k}| > Z_n\}$$

from which it follows that  $E\{T_n | Z_n\}$  on  $\{Z_n > 0\}$  is given by  $M(Z_n) + 1$ , where  $M(x)$  is the renewal function given by

$$M(x) = \max\{k: |D_1| + \dots + |D_k| < x\}.$$

Thus, by the elementary renewal theorem,  $E\{T_n | Z_n\}$  is asymptotic to  $Z_n / E|D_n|$  for  $Z_n$  large. Hence,  $\{Z_n: n \geq 1\}$  performs well under criterion P2 when a counter-current policy is followed.

We turn now to the asymptotic behavior of  $Z_n$  under the strict counter-current policy. Once again,  $\{Z_n: n \geq 1\}$  is a Markov chain, this time defined recursively by

$$(3.11) \quad Z_{n+1} = Z_n - \text{sign}(Z_n) \cdot D_{n+1}$$

Our first order of business is to show that  $Z_n$  possesses a stationary distribution under the strict counter-current rule. Let  $\Gamma_n = |Z_n|$ ; observe that  $\Gamma_n$  satisfies, for  $n \geq 1$ ,

$$(3.12) \quad \Gamma_{n+1} = |\Gamma_n - D_{n+1}|$$

We shall need the following result.

(3.13) Lemma. Let  $\xi_n$  be defined by

$$\xi_{n+1} = [\xi_n - D_{n+1}]^+, \text{ for } n \geq 1, \text{ with } \xi_1 = |D_1|. \text{ Then,}$$

$$\Gamma_k \leq \xi_k + K$$

Proof. For  $n = 1$ , the result is trivial. Proceeding by induction,

assume the inequality holds for  $n = k$ , and consider:

$$\begin{aligned} \Gamma_{k+1} &= \max\{\Gamma_k - D_{k+1}, D_{k+1} - \Gamma_k\} \\ &\leq \max\{\Gamma_k - D_{k+1}, K\} \\ &\leq \max\{\xi_k + K - D_{k+1}, K\} \\ &\leq \max\{K + [\xi_k - D_{k+1}]^+, K\} \\ &= K + [\xi_k - D_{k+1}]^+ \\ &= K + \xi_{k+1} \end{aligned}$$

The next result follows easily from Lemma 3.13.

(3.14) Proposition. The M.C.  $\{Z_n : n \geq 1\}$  defined by (3.11) possesses a stationary distribution.

Proof. First, observe that since  $ED_k > 0$ , the process  $\xi_n$  possesses a limiting distribution (Kiefer and Wolfowitz (1956)). As a consequence, for  $\epsilon > 0$ , there exists  $K_\epsilon$  such that

$$\inf_k P\{\xi_k \leq K_\epsilon\} > 1 - \epsilon$$

from which it follows that

$$\inf_k P\{\Gamma_k \leq K_\epsilon + K\} > 1 - \epsilon$$

Thus, the probabilities  $P\{\Gamma_k \leq K_\epsilon\}$  are tight (see Billingsley (1968), p. 37), from which it follows that the probabilities  $P\{Z_k \leq K_\epsilon\}$  are tight. A glance at (3.11) shows that the recursion is continuous in  $Z_n$ , and hence a well-known theorem on weakly continuous kernels may be applied (see, for example, Karr, (1975)) to conclude that  $Z_n$  possesses an invariant probability.

Given Lemma 3.13 and Proposition 3.14, the proof of the following theorem follows the same pattern as that of Theorem 3.8.

(3.15) Theorem. Let  $\{Z_n: n \geq 1\}$  be defined by the strict counter-current decision rule, and suppose that A1 - A4 hold. Then,  $Z_n$  possesses a unique stationary density  $h_2$  and for any  $k(\cdot)$  satisfying

$$\int_{-\infty}^{\infty} |k(y)| h_2(y) dy < \infty$$

it follows that

$$(3.16) \quad \frac{1}{n} \sum_{j=1}^n k(Z_j) \rightarrow \int_{-\infty}^{\infty} k(y) h_2(y) dy \text{ a.s.}$$

Observe that if  $h_2(z)$  is stationary for  $Z_n$ , then so is  $h_2(-z)$ . Thus, by uniqueness of the stationary distribution,  $h_2(z) = h_2(-z)$  and therefore

$$\int_0^{\infty} h_2(z) dz = 1/2.$$

So we have, by applying (3.16), that  $N_n/n \rightarrow 1/2$  a.s.

and therefore the strict counter-current rule behaves well in terms of both criterion P1 and P3 ((3.16) says that  $|Z_n|$  remains "bounded").

As in the case of the counter-current rule, the analysis of property P2 requires the representation

$$T_n = \min\{k: D_{n+1} + \dots + D_{n+k} > Z_n\}$$

which is valid on  $\{Z_n > 0\}$ . Although the  $D$ 's are not positive r.v.s, it is still true that the renewal-type result

$$E\{T_n | Z_n\} \sim Z_n / ED_n$$

holds for  $Z_n$  large. If  $U(n,m) = D_{n+1} + \dots + D_{n+m}$ , Wald's equality implies that  $E\{U(n, T_n) | Z_n\} = E\{T_n | Z_n\} ED_n$ . But the boundedness of the  $D_k$ 's implies that  $Z_n \leq U(n, T_n) \leq Z_n + K$ , from which the asymptotic relation follows. Thus, the strict counter-current policy behaves well under P2.

Finally, we shall show that introduction of "noise" into the problem always leads to a degradation in the behavior of counter-current-type rules. From (3.5) we have that

$$\int_{-\infty}^{\infty} |y|^r h_1(y) dy = \frac{E|D_n|^{r+1}}{(r+1)E|D_n|}$$

for  $r > 0$ . In particular, taking  $r = 1$  and applying (3.9), we obtain

$$(3.17) \quad \frac{1}{n} \sum_{k=1}^n |Z_k| + \frac{ED_n^2}{2E|D_n|} = \frac{ED_n}{2} + \frac{\text{var}(D_n)}{E|D_n|}$$

a.s. From Lemma 3.1, it follows that  $ED_n/2$  is the deterministic lower bound;

introduction of stochastic noise leads to the presence of an additional positive term given by  $\text{var}(D_n)/ED_n$ .

This discussion also carries over to the strict counter-current decision rule. The argument in this case centers around (3.12). If  $\Gamma$  has the distribution of  $|Z|$ , where  $Z$  has stationary distribution  $h_2$ , then from (3.12),

$$(3.18) \quad \Gamma \stackrel{D}{=} |\Gamma - D|$$

( $\stackrel{D}{=}$  denotes equality in distribution), where  $\Gamma$  and  $D$  are independent. It is evident from Lemma 3.13 that

$$P\{\Gamma > x\} \leq P\{\xi + K > x\}$$

where  $\xi$  is the limiting distribution of  $\xi_k$ . It is well-known that under A3,  $E\xi^k < \infty$  for all  $k$ , (Kiefer and Wolfowitz (1956)), and thus  $E\Gamma^k < \infty$  for all  $k$ . So, we square both sides of (3.17) and cancel common terms to obtain

$$E\Gamma = \int_{-\infty}^{\infty} |y| h_2(y) dy = \frac{ED_{n+1}^2}{2ED_{n+1}}$$

Thus, we obtain that under the strict counter-current decision rule,

$$(3.19) \quad \frac{1}{n} \sum_{k=1}^n |Z_k| \rightarrow \frac{ED_{n+1}^2}{2ED_{n+1}} \quad \text{a.s.}$$

It is worth observing that since  $ED_n < E|D_n|$ , the strict counter-current rule behaves worse than the counter-current rule under the "sum of the absolute value" criterion (compare (3.17) and (3.19)).



#### 4. Optimality of Counter-current Decision Rules

In Section 3, we saw that counter-current decision rules enjoy a number of desirable properties -- in this section, we shall show that counter-current rules possess certain optimality characteristics.

(4.1) Theorem. Let  $\{Z_n : n \geq 1\}$  be constructed according to the counter-current decision rule. Then, under A1 - A4, the counter-current decision rule minimizes a.s. both

$$i.) \quad \sum_{k=n}^{n+T_n-1} q(Z_k) \text{ on } \{Z_n > 0\}, \text{ where } q(\cdot) \text{ is any increasing function}$$

$$ii.) \quad \overline{\lim}_n |Z_n| \geq K$$

over the class of non-anticipating decision rules.

Proof. For i.), suppose that the counter-current rule is first violated at time  $k$ , where  $n \leq k < n + T_n$ . Then, for  $k \leq m < n + T_n$ , we have

$$(4.2) \quad Z_m \geq Z_{k-1} + |D_k| - |D_{k+1}| - \dots - |D_m| = Z'_m + |D_k|$$

where  $Z'$  is constructed from the counter-current rule. Inequality (4.2) immediately implies (4.1) i.).

For ii.), it is clearly sufficient to prove that for any non-anticipating decision rule,

$$(4.3) \quad \overline{\lim}_n |Z_n| \geq K \text{ a.s.};$$

the result then follows since the counter-current decision rule clearly attains the lower bound of (4.3).

Note that for fixed  $\epsilon > 0$ , and any non-anticipating decision rule,

$$\begin{aligned}
 (4.4) \quad & P\{|D_{n+1} - \hat{K}| < \epsilon, |D_{n+2} - \hat{K} - Z_n| < \epsilon, \\
 & |D_{n+3} - \hat{K}| < \epsilon | Z_1, \dots, Z_n\} \\
 & = (F(\hat{K} + \epsilon) - F(\hat{K} - \epsilon))^2 (F(Z_n + \hat{K} + \epsilon) - F(Z_n + \hat{K} - \epsilon)) \\
 & \geq \left( \min_{1 \leq j \leq \hat{K}/\epsilon} F(\epsilon j/2) - F(\epsilon(j-1)/2) \right)^3 > 0
 \end{aligned}$$

where  $\hat{K} = [K/\epsilon]$  ( $[ ] =$  greatest integer function); Because of the uniformity of (4.4) over  $\vec{Z}_n$ , one can apply the conditional Borel-Cantelli Lemma (Doob (1953), p. 323) to infer that

$$\begin{aligned}
 (4.5) \quad & P\{|D_{n+1} - K| < \epsilon, |D_{n+2} - K - Z_n| < \epsilon, \\
 & |D_{n+3} - K| < \epsilon \text{ infinitely often}\} = 1
 \end{aligned}$$

Letting  $T$  be a generic time at which the inequalities (4.5) are satisfied, one can easily check (just go through the eight different cases for

$$(\alpha_{T+1}, \alpha_{T+2}, \alpha_{T+3}) \text{ such that } |Z_{T+3}| \geq K - 3\epsilon, \text{ if } |Z_T| \leq K/2.$$

A similar argument to that used above shows that for any decision rule,

$|z_n| \leq K/2$  must occur infinitely often, from which it follows that

$$\overline{\lim} |z_n| \geq K - 3\epsilon, \text{ proving ii.)} \quad ||$$

Recall that in Section 3, we proved that counter-current policies suffer a degradation in performance when stochastic "noise" is introduced. We shall now show that counter-current policies are optimal for the  $L^1$  criterion, thus proving that the optimal policies suffer in the presence of "noise." To accomplish this goal, we will invoke the theory of Markov decision chains on  $R$ .

We will first deal with the strictly non-anticipating situation. The decision chain involves two actions ( $\alpha = 0$  or  $\alpha = 1$ ), and consequently two transition kernels. A cost equal to the absolute value of the state occupied is charged for each transition; costs are independent of action. With this framework, the optimality equation for the average cost decision process is given by

$$(4.6) \quad \gamma + m(z) = |z| + \min\{E_m(z + D_1), E_m(z - D_1)\}$$

where  $m(z)$  is the optimal return function.

(4.7) Proposition. Under A1 - A4, a solution pair  $(\gamma, m)$  exists to

(4.6) and is given by

$$m(z) = z^2/2ED_1$$

$$\gamma = ED_1^2/2ED_1$$

Proof. Observe that

$$\begin{aligned} \min\{E_m(z + D_1), E_m(z - D_1)\} &= (z^2 + ED_1^2)/2ED_1 + \min\{-z, z\} \\ &= m(z) + \gamma - |z|. \end{aligned}$$

To give a concise proof of the next theorem, we shall consider a restricted class of decision rules.

$$\Delta = \{(\alpha_1, \alpha_2, \dots) : \lim_n EZ_n^2/n = 0\}$$

Note that if a decision rule is not in  $\Delta$ , then  $EZ_n^2$  is of order  $n$ , which indicates that  $|Z_n|$  is growing unboundedly in expectation; clearly this is undesirable from a practical viewpoint. We henceforth restrict ourselves to decision rules in  $\Delta$ .

(4.8) Theorem. Under A1 - A4,

$$(4.9) \quad \lim_n \frac{1}{n} \sum_{k=1}^n E|Z_k| \geq \frac{ED_1^2}{2ED_1}$$

for any randomized strictly non-anticipating decision rule in  $\Delta$ ; the minimum in (4.9) is attained by the strict counter-current decision rule.

Proof. We apply a theorem due to Ross [1968]. Our class of randomized strictly non-anticipating decision rules corresponds to the set of policies enunciated there. Following Ross's proof, we see that for any randomized strictly non-anticipating decision rule (whether in or not in  $\Delta$ ),  $Z_n$  satisfies

$$(4.10) \quad \frac{ED^2}{2ED_1} \leq EZ_n^2 - EZ_1^2 + \sum_{k=1}^n E|Z_k|;$$

thus, if the rule is in  $\Delta$ , one obtains (4.9) (since  $EZ_1^2 \leq K^2$ ). As usual, equality in (4.10) occurs if one uses the policy which consistently minimizes the right-hand side of the optimality equation (4.6); this minimizing policy is easily seen to be the strict counter-current rule. However, because of our  $\Delta$  restriction, we still need to show that the strict counter-current rule is in  $\Delta$ .

From Lemma 3.13, it follows that

$$(4.11) \quad EZ_n^2 \leq 2(K^2 + E\xi_n^2).$$

Now, it is well-known that  $\xi_n$  is stochastically increasing to its steady-state  $\xi$  and that  $E\xi^2 < \infty$  since  $ED_1^3 < \infty$  (see Kiefer and Wolfowitz (1956)).

Hence,  $EZ_n^2/n \rightarrow 0$  for the strict counter-current rule. ||

We conclude this section with a statement and short proof of the corresponding result for the counter-current decision rule.

(4.12) Theorem. Under A1 - A4,

$$(4.13) \quad \liminf_n \frac{1}{n} \sum_{k=1}^n E|Z_k| \geq \frac{ED_1^2}{2E|D_1|}$$

for any randomized non-anticipating decision rule in  $\Delta$ ; the minimum in (4.13)

is attained by the counter-current decision rule.

The key idea here is to define an appropriate state space for the decision chain. We choose to use states of the form  $(z,d)$ , where the  $z$  component corresponds to the current value of the sum of absolute values of  $Z_i$ 's;  $d$  corresponds to the current value of the r.v.  $D_k$ . Letting  $m(z,d)$  be the optimal return function for the decision chain, it is easily seen that the optimality equation now takes on the form

$$\gamma + m(z,d) = \min\{|z + d| + E m(z + d, D_1), |z - d| + E m(z - d, D_1)\}.$$

The pair  $(\gamma, m)$  which solves the above equation is given by

$$\gamma = E D_1^2 / 2E |D_1|$$

$$m(z,d) = (|z| - |d|)^2 / 2E |D_1|,$$

again, the counter-current rule is the minimizing rule for the optimality equation. Since  $|Z_1| \leq K$  under the counter-current rule, it lies in  $\Delta$ .

Note that the optimality results given above show that the optimality of counter-current decision rules does not depend on the detailed form of the distribution of the  $D_k$ 's. Hence, one expects these results to be quite robust in practice.

#### SUMMARY AND CONCLUSIONS

In the previous sections we have formalized an important class of management decision problems, investigated the properties of a number of intuitively

appealing assignment rules and developed a new class of assignment rules, the counter-current policies. We have argued, by citing a number of examples, that the decision problem under investigation finds application in a wide variety of circumstances.

We have shown that ordinary intuition is often of little assistance in developing equitable assignment policies. It is remarkable that none of the easily conceived assignment rules will assure reasonable local or asymptotic properties. (This raises a number of interesting psychological and philosophical questions about our ability to make informed judgments about the properties of stochastic processes from the properties of the "generating" mechanism.) We have shown that among the rules considered, rule R1 is worst, followed by R3 and R2. Rules R4 and R5 are better than the others, but one cannot know which of these is "better" without examining the distribution of the D 's in some detail. "Better" refers to the rate of growth of  $|Z_n|$ , with faster growth being "worse." To restate the results in less formal terms, we see that "Fixed Assignment" is worse than "Random Assignment," which in turn is worse than "Fixed Alternation" (or "Block Rotation"). Both of the rules which attempt to reverse the last period's inequities (i.e. R4 and R5) are better than the other rules. An interesting new result emerges: under certain circumstances the "clairvoyant" rule, R5, is inferior to R4, a rule that does not assume knowledge of the severity of the assignments in advance.

Despite the disappointing performance of the more obvious rules we have proposed a class of rules (counter-current), which not only have the desired asymptotic and local properties but also are optimal in the sense described in

Sections 3 and 4.

An important question arises at this point. Is it possible to extend these results to the n-person assignment problem? The negative results of Section 2 almost certainly will carry over into the multi-person problem. More importantly, is there a multi-person analogy for the counter-current rule? Does it have the same properties as the two-person case? In the n-person analogue to the counter-current rule, one assigns duties in any period in the following way: the individual with the largest cumulative workload measure to date receives the assignment with the smallest value or smallest expected value. That individual is removed from the list of individuals to be considered and the rule is repeated with n-1 remaining assignments. It can be shown relatively easily from the results of Section 4 that this n-person generalization of the counter-current rule has at least the following desirable property.

Let  $(X_1^t, \dots, X_n^t)$  be the duty assignments on units 1, ..., n in period t and we assume that the joint density of  $(X_1^t, \dots, X_n^t)$  is everywhere positive on  $|X_i^t| \leq K$  and  $|X_j^t| \leq K$  for every  $i = 1, \dots, n$ , and every  $t = 1, 2, \dots$

If the X-vectors are i.i.d., then for any assignment rule

$$\overline{\lim} \max_{j, k \leq n} |U_k^t - U_j^t| \geq K$$

where  $U_i^t$  is the sum of all workload measures for individual i through time t. Further, the lower bound of this inequality is obtained by using the n-person counter-current rule discussed above. This follows from the fact that if  $U_i^t \geq U_j^t$ , then the assignment of individual j will be greater than



that to individual  $i$  at time  $t + 1$ .

A number of related mathematical and psychological issues deserve additional investigation. The  $n$ -person case presents interesting challenges. Consider areas of application such as equitable distribution of merit pay among university faculty or public school teachers. These issues require more complex models since they presuppose inherent inequalities in "true" performance, though they do share characteristics with examples we have discussed.

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