not been extensively studied in numerical analysis. With much computational experience and the use of internal accuracy checks, one may program the algorithms so as to have great confidence in their numerical results. A rigorous error analysis of our matrix methods and of the solution of various related integral equations appears to be possible and mathematically challenging. We hope that it will be undertaken in the near future.

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SIMULATION OUTPUT ANALYSIS FOR GENERAL STATE SPACE MARKOV CHAINS

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1. Introduction

The statistical analysis of simulation output has been the primary focus of recent research in simulation methodology. Methods have been developed which permit the simulator to construct confidence intervals for steady-state characteristics of the system being simulated. The principal methods in current use are autoregressive modeling, batch means, regenerative, and replication. With the exception of the replication method, all methods are based on just one simulation run. These methods for constructing confidence intervals are all based on central limit theorems for the underlying stochastic processes being simulated. Thus all methods are only valid asymptotically for long simulation runs.

In this paper we discuss three new methods for analyzing simulation output. All three methods are aimed at analyzing the simulation output of general state space Markov chains. This class of processes encompasses the embedded jump chain generated by a generalized semi-Markov process (GSMP). GSMP's are important for simulation since they may be used to model a general discrete event simulation. GSMP's have been discussed in recent papers by FOSSETT (1979), HORDIJK and SCHASSBERGER (1981), and WHITT (1980).

The first method, called the extended regenerative method (ERM), is

based on some recent work on general state space Markov chains by ATHREYA and NEY (1978) and NUMMELIN (1978). This method involves a construction which creates regeneration points for Markov chains which do not hit a single point infinitely often. While this idea is very attractive in principle, there are a number of practical considerations which limit its application. However, the method can be used to increase the rate of regeneration points when using the standard regenerative method. For more details on this method and some related results see GLYNN (1981).

The second method, called the random blocking method (RBM), is based on blocks of the process which begin when the process enters a given set in the state space. This method is reminiscent of the regenerative method except the blocks created here are not independent and identically distributed. Details on this second method can be found in a forthcoming paper by GLYNN (1981).

The last method is a variation of the method of autoregressive modeling. This method, the multivariate autoregressive method (MARM), fits a multivariate autoregressive model to the simulation output data. The model fitting is done automatically based on Akaike's AIC-criterion; see AKAIKE (1976) for a full discussion of these criterion. A forthcoming paper by JOW (1981) will develop this method and other related methods for simulation applications.

This paper is organized as follows. Section 2 is devoted to a discussion of GSMP's and their relation to simulation. The ERM is covered in Section 3 and RBM in Section 4. Section 5 is devoted to the MARM and Section 6 to an illustrative example.

2. Generalized Semi-Markov Processes

In a discrete-event simulation a finite number of events occurring

at random times cause changes in the state of the system being simulated. The number of events active at any given time is a function of the state of the system. This type of simulation is well modeled by the generalized semi-Markov process (GSMP) which we now describe.

Let S be the finite (or countable) set of states which describes the GSMP at the successive transition epochs and G the finite number of events which can cause a transition. If $G = \{e_1, e_2, \ldots, e_m\}$, then we let $G(s) = \{e_1(s), \ldots, e_{n_s}(s)\}$ denote the subset of events active when the GSMP is in state s. For each event active in state s, associate a clock which records the time until that event would trigger a state change. If in state s, the clock (associated with an event in G(s)) with the minimal reading triggers the next state change when it runs down to zero. Let C(s) denote the possible clock readings when the GSMP is in state s:

$$C(s) = \{c \in \mathbb{R}_+^m : c_i > 0 \text{ iff } e_i \in G(s);\ c_i \neq c_j \text{ for } i \neq j\},$$

where $\mathbb{R}_+^m$ is the Cartesian product of m copies of $[0,\infty)$. Next define the space Σ by

$$\Sigma = \bigcup_{s \in S} (\{s\} \times C(s))$$

and the process $(\underset{\sim}{S}, \underset{\sim}{C}) = \{(S_n, C_n) : n \geq 0\}$ which lives on Σ and represents the state values and clock readings at the successive transition epochs. The process $(\underset{\sim}{S}, \underset{\sim}{C})$ is a general state space Markov chain whose transition kernel will be defined in Section 3. Finally, the GSMP is a piece-wise constant process, $\{X_t : t \geq 0\}$, constructed from the embedded jump process $(\underset{\sim}{S}, \underset{\sim}{C})$ in the usual manner.

3. The Extended Regenerative Method for GSSMC's

We start by formalizing the notion of a general state space Markov chain (GSSMC). Let E be a complete, separable metric space with $\mathcal{E}$ its

associated Borel field. A function $P: E \times \mathcal{E} \to [0,1]$ is called a probability transition kernel if:

i) $P(x,\cdot)$ is a probability on $(E,\mathcal{E})$ for each x in E,

ii) $P(\cdot,B)$ is $\mathcal{E}$-measurable for all $B \in \mathcal{E}$.

One should think of $P(x,B)$ as representing the one-step transition probability of the chain passing from x into B. The analogous n-step transition probabilities are then given through the Chapman-Kolmogorov equations, namely

$$P^{n+1}(x,B) = \int_E P^n(y,B)\, P(x,dy)$$

where, of course, $P^0(x,B) = \delta_x(B)$ ($\delta_x(B)$ is 1 or 0 depending on whether or not $x \in B$). Given a kernel P and an initial probability μ on $(E,\mathcal{E})$, one can construct a measure P on $(\Omega,\mathcal{F}) = (E \times E \times \ldots, \mathcal{E} \times \mathcal{E} \times \ldots)$ such that

$$P_\mu\{X_0 \in B_0,\ X_1 \in B_1,\ldots,X_n \in B_n\}$$
$$= \int_{B_0} \mu(dx_0) \int_{B_1} P(x_0,\ dx_1) \cdots \int_{B_n} P(x_{n-1},\ dx_n)$$

where $X_i(\omega) = \omega_i$ and $(\omega_0,\omega_1,\ldots)$ is a typical element of Ω. The above construction yields a process $\{X_n\colon n \geq 1\}$ that is endowed with the Markov property

$$P_\mu\{X_{n+1} \in B \mid X_0,\ldots,X_n\} = P_\mu\{X_{n+1} \in B \mid X_n\}$$

and is referred to as the GSSMC associated with kernel P and initial distribution μ.

Example 3.1. Let $E = \{0,1,\ldots\}$ and put $P(i,B) = \sum_{j \in B} p_{ij}$, where P is a stochastic matrix on E. This gives the classical countable state Markov chain.

Example 3.2. Let $E = \{W_n\colon n \geq 0\}$ be the waiting time process of a GI/G/1 queue. Suppose that u and v are the interarrival and service r.v.'s respectively, and that $v-u$ has a density $f(x)$. Then $E = [0,\infty)$ and

$$P(x,B) = \int_B f(y-x)dy + \delta_0(B) \int_0^x f(y)dy.$$

Example 3.3. Consider a model that imitates the dynamics of a lake. Let S_i, Z_i and X_i represent the volume of water stored in the lake, the inflow of water, and outflow of water respectively, at time i. Then, the mass-balance equation

$$S_i = S_{i-1} + Z_i - X_i \tag{3.4}$$

holds. If one now assumes that output increases linearly with storage through the relation $X_i = \alpha S_i$ $(0 < \alpha < 1)$, then (3.4) takes the form

$$X_i = \rho X_{i-1} + \varepsilon_i$$

where $\rho = 1/(1+\alpha)$ and $\varepsilon_i = (\alpha Z_i)/(1+\alpha)$. Finally, the further assumption that $\{\varepsilon_i\}$ is i.i.d. with

$$P\{\varepsilon_i \in B\} = p\delta_0(B) + (1-p) \int_B f(y)dy$$

yields a Markov chain model for the outflows $\{X_n\}$, where $E = [0,\infty)$ and

$$P(x,B) = p\delta_{\rho x}(B) + (1-p) \int_B f(y-\rho x)dy\ .$$

Example 3.4. The embedded jump process $(\underset{\sim}{S},\underset{\sim}{C})$ introduced in Section 2 is a GSSMC on state space Σ and with kernel

$$P((s,c),\ B)$$
$$= p(s';\ s,i^*) \prod_{i \in N_{s'}} \int_{B_i} f(y;\ s',i,s,i^*)dy \prod_{j \in O_{s'}} \delta_{c_i^*}(B_j)$$

where B is that subset of Σ corresponding to the GSMP entering state s' with the i'th clock of s' set to a value in B_i.

The numbers $p(s';\ s,i^*)$ govern state transitions of the GSMP and represent the probability of a jump to s' from s, given that clock i^* initiated the jump. The rest of the kernel governs clock readings. Those clocks $i \in N_{s'}$ (new clocks) are set stochastically according to a density $f(y;\ s',i,s,i^*)$, whereas those clocks $j \in O_{s'}$ (old clocks) are inherited from the previous state and so must be set deterministically

at the previous value c_j^*.

Note that by setting $\tau_n = c_0(i^*(C_0)) + \cdots + c_{n-1}(i^*(C_{n-1}))$ for $n \geq 1$ and $\tau_0 = 0$ ($c_j(k)$ is the value of the k'th clock just after the j'th jump of X_t), we can retrieve $\{X_t: t \geq 0\}$, the GSMP, from $\{(S_n,C_n): n \geq 0\}$ via

$$X_t = \sum_{k=0}^{\infty} \delta_t([\tau_k,\tau_{k+1}))S_k.$$

Let us turn now to a recurrence condition for GSSMC's first formulated by ATHREYA and NEY (1978) and NUMMELIN (1978). We say that $\{X_n: n \geq 0\}$ is (A,B,λ,φ,k) recurrent if there exist A, B $\in \mathcal{E}$, a positive number λ, an integer k, and a probability φ on B, such that:

i) $P\{\sum_{n=0}^{\infty} \delta_{X_n}(A) = +\infty \,|\, X_0 = x\}$ for all x in E,

ii) $P^k(x,\Gamma) \geq \lambda\varphi(\Gamma)$ for each x in A and measurable subset Γ of B.

In our setting, this recurrence notion is in fact equivalent to one first proposed by HARRIS (1956).

To gain some appreciation for the significance of these conditions, we observe that in the $(A,B,\lambda,\varphi,1)$ case, we can decompose P over A as

$$P(x,\Gamma) = \lambda\varphi(\Gamma) + (1-\lambda)\, Q(x,\Gamma)$$

where $Q(x,\cdot)$ is a probability on $(E,\mathcal{E})$. The key idea of Athreya, Ney and Nummelin was to exploit this decomposition via the following embedding.

Let $E' = E \times \{0,1\}$ and $\mathcal{E}'$ be the associated product σ-field. We extend P to a kernel P' on $(E',\mathcal{E}')$ by setting

$$P'((x,\delta), \Gamma \times \{0\}) = \begin{cases} (1-\lambda)\, P(x,\Gamma), & x \notin A \\ (1-\lambda)\, Q(x,\Gamma), & x \in A \end{cases}$$

$$P'((x,\delta), \Gamma \times \{1\}) = \begin{cases} \lambda P(x,\Gamma), & x \notin A \\ \lambda\varphi(\Gamma \cap B), & x \in A \end{cases}$$

For a probability μ on $(E,\mathcal{E})$, define μ' on $(E',\mathcal{E}')$ by

$$\mu'(\Gamma \times \{i\}) = \delta_0(i)\,(1-\lambda)\,\mu(\Gamma) + \delta_1(i)\lambda\,\mu(\Gamma).$$

Then, it can be readily verified that the Markov chain $X_n' = (X_n,\delta_n)$ on $(E',\mathcal{E}')$ associated with P', μ' has the property that the coordinate process $\{X_n: n \geq 0\}$ is the original GSSMC on $(E,\mathcal{E})$ associated with P, μ.

The importance of this embedding is that it furnishes one with a sequence of regeneration times T_k defined in terms of X_n' via

$$T_1 = \inf\{n \geq 1: X_{n-1} \in A,\ \delta_n = 1\},$$

$$T_k = \inf\{n > T_{k-1}: X_{n-1} \in A,\ \delta_n = 1\}, \quad k > 1.$$

The regenerative character of X_n' guarantees "nice" ergodic behavior for X_n. For example, if we assume that $E'(T_2-T_1) < +\infty$, then the classical regenerative theory shows that (e.g., SMITH (1955))

$$\frac{1}{n}\sum_{k=1}^{n} f(X_k) \to r = E_\pi f(X) \qquad P'_{\mu'} \text{ a.s.}$$

where

$$\pi(B) = E'\Big(\sum_{j=T_1}^{T_2-1} \delta_{X_j}(B)\Big)/E'(T_2-T_1).$$

Estimation of r is a common goal of simulators. The above discussion suggests an "extended regenerative method" (ERM) for producing confidence intervals for r based on the T_k sequence.

1. Generate the sequence $\{(X_n,\delta_n): n \geq 0\}$.

2. Let $Y_k = \sum_{j=T_k}^{T_{k+1}} f(X_j)$, $\alpha_k = T_{k+1} - T_k$ and form

$$\hat{r}(n) = \Big(\sum_{k=1}^{n} Y_k\Big)/\Big(\sum_{k=1}^{n} \alpha_k\Big)$$

$$s_{11}(n) = \frac{1}{n-1}\sum_{k=1}^{n} Y_k^2 - \frac{1}{n(n-1)}\Big(\sum_{k=1}^{n} Y_k\Big)^2$$

$$s_{12}(n) = \frac{1}{n-1}\sum_{k=1}^{n} Y_k\,\alpha_k - \frac{1}{n(n-1)}\Big(\sum_{k=1}^{n} Y_k\Big)\Big(\sum_{k=1}^{n} \alpha_k\Big)$$

$$s_{22}(n) = \frac{1}{n-1}\sum_{k=1}^{n}\alpha_k^2 - \frac{1}{n(n-1)}\left(\sum_{k=1}^{n}\alpha_k\right)^2$$

$$s^2(n) = s_{11}(n) - 2\hat{r}(n)\, s_{12}(n) + \hat{r}^2(n)\, s_{22}(n).$$

3. To form a $100(1-\delta)\%$ confidence interval for r choose z_δ so that $\Phi(z_\delta) = 1 - \delta/2$, where Φ is the standard normal distribution function. Then

$$\hat{I} = \left[r(n) - \frac{z_\delta\, s(n)\, n^{1/2}}{\sum_{k=1}^{n}\alpha_k},\ r(n) + \frac{z_\delta\, s(n)\, n^{1/2}}{\sum_{k=1}^{n}\alpha_k} \right]$$

is the desired confidence interval.

The above steps can all be justified under the assumption that $0 < E(Y_2 - r\alpha_2)^2 < +\infty$. Incidentally, although Step 1 appears to necessitate the ability to generate deviates distributed according to $Q(x,\cdot)$, this difficulty can be avoided by using an acceptance-rejection technique ([8], p. 57).

Example 3.1 (continued). Suppose that $X = \{X_n: n \geq 0\}$ is an irreducible, recurrent Markov chain with transition matrix $P = \{p_{ij}\}$. Then, X is $(\{0\}, E, 1, p_{0\cdot}, 1)$ recurrent and $T_k = N_k+1$, where N_k is the k'th hitting time of 0. The ERM reduces here to the classical regenerative technique (modulo a shift to the right).

Example 3.2 (continued). Assuming that $Ev \leq Eu$, $\{W_n: n \geq 0\}$ is $(\{0\}, E, 1, P(0,\cdot), 1)$ recurrent, and again $T_k = N_k+1$, when N_k is the k'th hitting time of 0, reducing our estimation technique to the usual one (again, modulo a shift).

Example 3.3 (continued). Assuming that $E\varepsilon_i < +\infty$ and that f is positive and continuous over $[0, +\infty)$, $\{X_n: n \geq 0\}$ is $([0,b], E, \lambda, \varphi, 1)$ recurrent where

$$\psi(y) = (1-p) \min_{0 \leq x \leq b} f(y-\rho x)$$

$$\lambda = \int_0^\infty \psi(y)dy;\ \varphi(y) = \psi(y)/\lambda\ .$$

Here the T_k's form a subsequence of the hitting times of $[0,b]$ -- this reflects the fact that X_n returns to no point infinitely often. The ERM is applicable here, whereas the classical regenerative method is not.

Example 3.4 (continued). We say that the GSMP has a single set if there exists a state s' with which is associated only one clock. Then, given that the GSMP hits $\{s'\}$ infinitely often, $(\underset{\sim}{S},\underset{\sim}{C})$ is $(A, \Sigma, 1, P((s',c), \cdot), 1)$ recurrent where $A = \{(s,c) \in \Sigma: s = s'\}$. Here, $T_k = N_k+1$ where the N_k's are consecutive hitting times of A.

It should be remarked that in the context of a GSMP, a simulator is often interested in the continuous-time quantity

$$\underset{\sim}{r} = \lim_{t\to\infty} \int_0^t f(X_s)ds/t\ .$$

This can be estimated by the ERM, provided that the definitions of Y_k and α_k are modified to

$$\underset{\sim}{Y}_k = \sum_{j=T_k}^{T_{k+1}-1} f(S_j)\, c_j(i^*(C_j))$$

$$\underset{\sim}{\alpha}_k = \sum_{j=T_k}^{T_{k+1}-1} c_j(i^*(C_j))\ .$$

The above discussion focussed on $(A, B, \lambda, \varphi, 1)$ GSSMC's. A fundamental difficulty arises in the $(A, B, \lambda, \varphi, k)$ case, a difficulty that can be partially circumvented via an embedding that endows the chain with an environment that is "loosely regenerative" in the sense of SMITH (1955). An estimation procedure can then be developed that bears close resemblance to the "extended regenerative method" outlined above (see [8]).

Before leaving this topic, it should be mentioned that a number of difficulties remain in terms of implementation of the ERM. The most fundamental problem is that the decomposition of P over A requires an

explicit form for the kernel over that set. The "event-scheduling" approach normally used by simulators to generate sample paths does not require an explicit representation for the kernel, and so a simulator is left with the burdensome task of calculating such a form. Additional difficulties arise in determining appropriate λ, φ, and A, although here simple numerical techniques would be applicable ([8], p. 52).

4. Random Blocking Method for GSSMC's

Our major incentive in studying GSSMC's here has been in terms of application to GSMP's. It turns out that GSMP's possessing no single set cannot be (A, B, λ, φ, 1) recurrent ([8], p. 60), and hence the ERM discussed in the previous section does not apply. This factor together with the ERM-related difficulties already mentioned, motivates development of other methods.

Suppose that $\{X_n: n \geq 0\}$ is an (A, B, λ, φ, k) GSSMC with invariant probability π. This will in fact be the case for the embedded jump process $(\underset{\sim}{S},\underset{\sim}{C})$, corresponding to a GSMP $\{X_t: t \geq 0\}$, provided that (see [9])

i) the state space S is finite

ii) the "road map" ;(s', s, i*) satisfies a natural irreducibility condition (see [7], p. 16)

iii) the densities f(·; s', i, s, i*) are positive and continuous on $[0,\infty)$, with finite mean.

For such an (A, B, λ, φ, k) recurrent $\{X_n: n \geq 0\}$ one can show that if $E_\pi|f(X)| < +\infty$, then

$$\frac{1}{n+1}\sum_{k=0}^{n} f(X_k) \to r = E_\pi f(X) \qquad P_\mu \quad \text{a.s.}$$

for any μ on $(E,\mathcal{E})$. A simulator commonly wishes to obtain an estimate for r, together with an associated confidence interval.

Let $T_1, T_2, \ldots$ be the consecutive hitting times of the set A. The "random blocking method" (RBM) hinges on the observation, due essentially to OREY (1959), that

$$V_k = (X_{T_k+1}, X_{T_k+2}, \ldots, X_{T_{k+1}})$$

is a Doeblin recurrent Markov chain with a single ergodic set (see DOOB (1953) for definitions and results). Under the assumption that V_k is aperiodic (this will generally be the case for GSMP's), functions of V_k will enjoy a central limit theorem with a variance constant of the form

$$\sigma^2 = \sigma^2(Z_0) + 2\sum_{k=1}^{\infty} \operatorname{cov}(Z_0, Z_k). \tag{4.1}$$

By truncating the infinite sum and estimating the finite number of remaining terms in the series by the standard sample moments, we obtain the RBM.

1. Choose a truncation number m (the number of covariance terms of (4.1) to be retained).

2. Put

$$Y_k = \sum_{j=T_{k+1}}^{T_{k+1}} f(X_j), \quad \alpha_k = T_{k+1} - T_k \quad \text{and form}$$

$$\hat{r}(n) = \Big(\sum_{k=1}^{n} Y_k\Big)\Big/\Big(\sum_{k=1}^{n} \alpha_k\Big)$$

$$c_{o\ell}(n) = \frac{1}{n-\ell}\sum_{k=1}^{n-\ell} Y_k Y_{k+\ell} - \frac{\hat{r}(n)}{n-\ell}\sum_{k=1}^{n-\ell} \alpha_k Y_{k+1}$$

$$- \frac{\hat{r}(n)}{n-\ell}\sum_{k=1}^{n-\ell} Y_k \alpha_{k+\ell} + \frac{\hat{r}^2(n)}{n-\ell}\sum_{k=1}^{n-\ell} \alpha_k \alpha_{k+\ell}$$

for $\ell = 0,1,\ldots,m$

$$s^2(n) = c_{oo}(n) + 2\sum_{\ell=1}^{m} c_{o\ell}(n).$$

3. A $100(1-\delta)\%$ confidence interval for r is

$$\hat{I} = \left[\hat{r}(n) - \frac{z_\delta \, s(n) \, n^{1/2}}{\sum_{k=1}^{n} \alpha_k} \, , \; \hat{r}(n) + \frac{z_\delta \, s(n) \, n^{1/2}}{\sum_{k=1}^{n} \alpha_k} \right] .$$

Justification of the above steps is possible, provided that

$$\int_A E_x\{ \left| \sum_{j=1}^{T_1} f(X_j) \right|^{2+\delta} \} \pi(dx) < +\infty,$$

for some $\delta > 0$ ([9]). For GSMP's, the RBM can be modified in the same way as the ERM so as to provide confidence intervals for continuous-time quantities of the form (3.5).

Observe that when $m = 0$, this technique reduces to the "approximate regenerative method" (ARM) of CRANE and IGLEHART (1975). Hence one can think of the RBM as a second-order refinement of the ARM. It should be noted, however, that the RBM involves the undersirable element of having to make an a priori judgment as to an appropriate value of the truncation index m. This is a problem to which we intend to devote more attention.

5. Multivariate Autoregressive Method

The last method we shall discuss is the multivariate autoregressive method (MARM). Let $\{X_n : n \geq 0\}$ be a vector-valued strictly stationary process with mean vector $\underset{\sim}{\mu} = E\{X_0\}$ and covariance function $\underset{\sim}{\Sigma}(h) = E\{[X_n - \underset{\sim}{\mu}] \, [X_{n+h} - \underset{\sim}{\mu}]'\}$. Then under some regularity conditions (see BILLINGSLEY (1968), Theorem 20.1) the following central limit theorem holds as $n \to \infty$:

$$\sqrt{n} \left[\frac{1}{n} \sum_{j=0}^{n-1} \underset{\sim}{X}_j - \underset{\sim}{\mu} \right] \Longrightarrow N(0, \underset{\sim}{\Sigma}) \, ,$$

where $\underset{\sim}{\Sigma} = \underset{\sim}{\Sigma}(0) + 2 \sum_{h=1}^{\infty} \underset{\sim}{\Sigma}(h)$.

To apply this result to simulation output analysis we need a method to estimate the covariance matrix $\underset{\sim}{\Sigma}$.

Let $\underset{\sim}{f}(\lambda)$ be the matrix-valued spectral density function for the process $\{X_n : n \geq 0\}$, namely,

$$\underset{\sim}{f}(\lambda) = \sum_{h=-\infty}^{\infty} \underset{\sim}{\Sigma}(h) \, e^{i\lambda h} \, , \quad \lambda \text{ real.}$$

Observe that $\underset{\sim}{\Sigma} = \underset{\sim}{f}(0)$. The function $\underset{\sim}{f}$ can be approximated arbitrarily closely by the corresponding spectral density of a multivariate autoregressive (MAR) process fitted to the process $\{\underset{\sim}{X}_n : n \geq 0\}$. To fit a MAR process of order p we must find matrices $\{\underset{\sim}{A}_k : 0 \leq k \leq p\}$ so that

$$\sum_{k=0}^{p} \underset{\sim}{A}_k \, \underset{\sim}{X}_{n-k} = \underset{\sim}{Z}_n \, ,$$

where the $\underset{\sim}{Z}_n$'s are i.i.d. normal with mean $\underset{\sim}{0}$ and known convariance $\underset{\sim}{B}$. We select the order p of the MAR process by applying the AIC-criterion of AKAIKE (1976), (1978). Once the model is fitted, it is a simple computation to find the spectral density function of the MAR process. This function evaluated at zero then provides our estimate for $\underset{\sim}{\Sigma}$.

This method and related ones will be developed in detail in JOW (1981). In Section 6 data will be presented from an application of the method to the lake model presented in Section 3.

6. Application to the Lake Model

In this section we illustrate the application of the three methods to the lake model presented in Example 2. Recall the model is generated by the recursion

$$X_n = \rho X_{n-1} + \varepsilon_n, \quad n \geq 0,$$

where the ε_n's are i.i.d. and $\rho < 1$. It can be shown that $X_n \Longrightarrow X$ as $n \to \infty$ with $E\{X\} = E\{Z_1\} = (1-\rho)^{-1} E\{\varepsilon_1\}$. For this simulation we have taken the ε_n's to have the distribution $P\{\varepsilon_n \leq x\} = 1 - (1-\rho)e^{-x}$, $x \geq 0$, which results in X being exponential with parameter 1. The simulations were carried out to estimate $E\{X\}$. For the ERM the return set $A = [0,b]$,

where $b = (-\log \rho)/(1-\rho)$. This value can be shown to maximize the expected number of regenerations. The value $\rho = 0.75$ was used which makes $b = 1.15$. For the RBM the return set $A = [.75, 1.25]$. The infinite sum of covariance terms in the RBM was truncated at m ($m = 0, 5, 10, 25$). For the MARM the vector of observations used was $(X_n, X_n^2, 1_{[0, \ln 2]}(X_n))$. A total of 10,000 observations were generated in each of 30 replications.

Table 6.1. Simulation Results for Lake Model

number of observations	value estimated	theoretical values	ERM	RBM m = 0	RBM m = 5	RBM m = 10	RBM m = 25	MARM
1000	E{X}	1.0	1.018+/- 0.023	1.022+/- 0.023	1.022+/- 0.023	1.022+/- 0.023	1.022+/- 0.023	1.018+/- 0.024
	σ^2	7.0	7.307+/- 0.656	7.057+/- 0.635	6.757+/- 0.797	6.674+/- 1.184	6.115+/- 1.897	7.230+/- 0.527
	coverage probability	.90	0.900+/- 0.090	0.900+/- 0.090	0.933+/- 0.075	0.900+/- 0.090	0.767+/- 0.127	0.900+/- 0.090
5000	E{X}	1.0	1.015+/- 0.009	1.016+/- 0.009	1.016+/- 0.009	1.016+/- 0.009	1.016+/- 0.009	1.015+/- 0.009
	σ^2	7.0	7.191+/- 0.297	7.037+/- 0.299	7.181+/- 0.487	7.335+/- 0.591	7.842+/- 1.002	7.107+/- 0.247
	coverage probability	0.9	0.933+/- 0.075	0.933+/- 0.075	0.933+/- 0.075	0.933+/- 0.075	0.900+/- 0.090	0.967+/- 0.055
10,000	E{X}	1.0	1.006+/- 0.007	1.007+/- 0.007	1.007+/- 0.007	1.007+/- 0.007	1.007+/- 0.007	1.007+/- 0.007
	σ^2	7.0	7.062+/- 0.210	6.919+/- 0.185	7.026+/- 0.313	6.989+/- 0.361	7.018+/- 0.643	7.001+/- 0.153
	coverage probability	0.9	0.933+/- 0.075	0.933+/- 0.075	0.933+/- 0.075	0.900+/- 0.090	0.867+/- 0.102	0.933+/- 0.076

7. References

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