# THE REGULARIZING EFFECTS OF RESETTING IN A PARTICLE SYSTEM FOR THE BURGERS' EQUATION

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Abstract. We study the dissipation mechanism of a stochastic particle system for the Burgers' equation. The velocity field of the viscous Burgers' and Navier-Stokes equations can be expressed as an expected value of a stochastic process based on noisy particle trajectories (Constantin, Iyer, Comm. Pure Appl. Math, 2008). In this paper we study a particle system for the viscous Burgers' equations using a Monte-Carlo version of the above; we consider $N$ copies of the above stochastic flow, each driven by independent Wiener processes, and replace the expected value with $\frac{1}{N}$ times the sum over these copies. A similar construction for the Navier-Stokes equations was studied by J. Mattingly and the first author (`arXiv:0803.1222`, to appear in Nonlinearity).

Surprisingly, for any finite $N$, the particle system for the Burgers' equations shocks almost surely in finite time. In contrast to the full expected value, the empirical mean $\frac{1}{N}\sum_1^N$ does not regularize the system enough to ensure a time global solution. To avoid these shocks, we consider a resetting procedure, which at first sight should have no regularizing effect at all. We however prove that this procedure prevents the formation of shocks for any $N \geqslant 2$, and consequently as $N \to \infty$ we get convergence to the solution of the viscous Burgers' equations on long time intervals.

## 1. Introduction

The viscous Burgers' equation

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = 0 \tag{1.1}$$

has been studied extensively from several different points of view. Here $\nu > 0$ represents the viscosity, making the equation dissipative in nature. The inviscid Burgers' equation, (equation (1.1) with $\nu = 0$) is studied as the basic example of a scalar conservation law (see e.g. [5, 7]). The Burgers' equation is also linked to the KAM and Aubry-Mather theories [8, 12]. It is the simplest PDE that models the Euler and the Navier-Stokes nonlinearity. As such, it has been extensively studied as the first step in understanding the two key unresolved issues in fluid mechanics: turbulence and regularity of the Navier-Stokes equations in three dimensions. In the first category the objective is to characterize an statistical properties of turbulence [6]. In the second category the objective is to understand the regularizing mechanism of dissipation [1, 13]. This paper falls into the latter category: we study the regularising mechanism of a particle system for the Burgers' equations, analogous to the particle system for the Navier-Stokes equations developed in [4, 11].

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In [4], a class of second order non-linear transport equations (the Navier-Stokes and viscous Burgers' in particular) were formulated as the average of a stochastic process along noisy particle trajectories. Explicitly, for the viscous Burgers' equation consider the stochastic flow

$$dX_t = u_t(X_t) + \sqrt{2\nu}\, dW_t \tag{1.2}$$

with initial data $X_t(a) = a$. Here $W$ denotes a standard 1D Wiener process. If we require that the velocity $u$ satisfies

$$u_t = \boldsymbol{E}\left[u_0 \circ (X_t^{-1})\right] \tag{1.3}$$

where $\boldsymbol{E}$ denotes the expected value with respect to the Wiener measure, then $u$ satisfies[1] the viscous Burgers' equation (1.1) and initial data $u_0$. The formulation for Navier-Stokes developed in [4] involves recovering the velocity $u$ via the average of a non-local functional of the initial data.

Observe that when $\nu = 0$, the system (1.2)–(1.3) is exactly the method of characteristics for the inviscid inviscid Burgers' equation. Indeed trajectories of the flow $X$ are now characteristics, and equation (1.3) states that the velocity is transported along characteristics. Thus, the $\nu > 0$ case could be viewed as a stochastic generalization of the method of characteristics: we transport the initial data along noisy characteristics, and then average with respect to the Wiener measure.

The usual Monte-Carlo method of solving (1.2)–(1.3) numerically [18, 19] is to replace the flow $X$ with $N$ different copies $X^{i,N}$, each driven by an independent Wiener process $W^i$, and replace the expected value in (1.3) by the empirical mean: $\frac{1}{N}\sum_{i=1}^N$. Explicitly, the system in question becomes

$$dX_t^{i,N} = u_t^N(X_t)\, dt + \sqrt{2\nu}\, dW_t^i, \tag{1.4}$$

$$X_0^{i,N}(a) = a, \tag{1.5}$$

$$A_t^{i,N} = (X_t^{i,N})^{-1}, \tag{1.6}$$

$$u_t^N = \frac{1}{N}\sum_{i=1}^N u_0 \circ A_t^{i,N}, \tag{1.7}$$

where $u_0$ is the given initial data, $W^i$ a sequence of independent Wiener processes and $\nu > 0$ the viscosity. Throughout this paper, with the exception of section 3, we impose periodic boundary conditions on the above, and assume the initial data is periodic.

For the Navier-Stokes equations, the particle system in [11] involves using a higher dimensional Wiener process, and replacing (1.7) with the average of vorticity transport and Biot-Savart:

$$\omega_t^N = \boldsymbol{E}\left[\left((\nabla X_t^{i,N})\omega_0\right) \circ A_t^{i,N}\right], \tag{1.8}$$

$$u_t^N = (-\triangle)^{-1}\nabla \times \omega_t^N. \tag{1.9}$$

where $\omega_0 = \nabla \times u_0$ is the initial vorticity. In [11], the authors considered the system (1.4)–(1.6) & (1.8)–(1.9), with spatially periodic boundary conditions, and proved global existence in two dimensions, local existence in three dimensions, convergence to the correct limit as $N \to \infty$, and described the asymptotic behaviour for fixed $N$ as $t \to \infty$.

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[1] This is only valid for spatially periodic or decay at infinity boundary conditions.

The techniques of [11] however fail for the system (1.4)–(1.7). One heuristic explanation is as follows: The particle system (1.4)–(1.7) (and also the system (1.4)–(1.6) & (1.8)–(1.9)) is dissipative only for short time [11, Theorem 5.2]. Once the system (1.4)–(1.7) stops dissipating energy, the growth from the non-linear term forces the system to shock, in a manner similar to the inviscid Burgers' equation. In contrast, no dissipation is required to prove global existence (at least in 2D) for (1.4)–(1.6) & (1.8)–(1.9), as was shown in [11, Theorem 3.5]. This is to be expected, as (1.8)–(1.9) are structurally similar to Euler equations, for which 2D global existence is well known [21] (see also [2, 17]).

The main content of this paper is to show that the shocks in the system (1.4)–(1.7) can be *completely avoided* by resetting the Lagrangian maps. Namely, we solve (1.4)–(1.7) for short time $\delta_t$. Then we replace the initial data with $u^N_{\delta_t}$, and restart the system (1.4)–(1.7) with this new initial data. Explicitly, consider the system

$$dX^{i,N}_{k\delta_t,t} = u^N_t(X^{i,N}_{k\delta_t,t})\,dt + \sqrt{2\nu}dW^i_t, \tag{1.10}$$

$$X^{i,N}_{k\delta_t,k\delta_t}(a) = a, \tag{1.11}$$

$$A^{i,N}_{k\delta_t,t} = (X^{i,N}_{k\delta_t,t})^{-1}, \tag{1.12}$$

$$u^N_t = \frac{1}{N}\sum_{i=1}^{N} u^N_{k\delta_t} \circ A^{i,N}_{k\delta_t,t}, \tag{1.13}$$

where $k \in \mathbb{N}$, and $t$ is always assumed to be in the interval $[k\delta_t, (k+1)\delta_t)$.

If $\delta_t$ is small enough, we show that (time) global solutions to this system exist with arbitrarily large probability, and as $N \to \infty$ these solutions converge to the smooth solutions of the viscous Burgers' equation.

The fact that the shocks can be avoided by resetting is indeed unexpected! The system (1.2)–(1.3) is Markovian; if we reset it at regular intervals (as above), then the new solution obtained will be no different from the original solution without resetting. Fortunately, this is not true for the system (1.4)–(1.7). If we reset often enough, the generic short time dissipative effect is strong enough to overcome the nonlinear growth, and with large probability prevents the formation of shocks. We observed numerically that even for large $\delta_t$ (i.e. comparable to half the shock time of the inviscid Burgers' system) is enough to ensure that the system (1.10)–(1.13) is globally well posed. With the techniques in this paper however, we are only able to prove a global existence result for (1.10)–(1.13) when $\delta_t$ is small. The question for large $\delta_t$ remains open, and can not be addressed using techniques in this paper.

Finally, we mention that our technique can be used to show global existence of the analogue of (1.10)–(1.13) for the Navier-Stokes equations in *two dimensions*. As this is already known [11], *without resetting*, we do not carry out the details here.

One interesting application would be to the 3-dimensional Navier-Stokes equations. There are numerous results showing global existence of solutions to the Navier-Stokes equations with small initial data. One new, interesting question that can be asked in this framework is the existence of solutions for arbitrary initial data, which are time global for some small (non-zero) probability. The McKean type nonlinearity prevents us from asking this question for the stochastic Lagrangian formulation for the Navier-Stokes equations [4, equations (2.3)–(2.6)]. For the system (1.4)–(1.6) & (1.8)–(1.9), the empirical mean $\frac{1}{N}\sum_1^N$ provides no regularisation, so it is unlikely to expect small probability time global solutions. However, the repeatedly reset version of (1.4)–(1.6) & (1.8)–(1.9) is free of the McKean nonlinearity, *and* is dissipative, making it a better candidate for small probability time global solutions. Unfortunately, there are obstructions in proving this result directly with the techniques used here, and we are working on addressing this issue.

The plan of this paper is as follows: In Section 2 we establish our notational convention, and prove our main regularity result. This proof relies on a few auxiliary technical results, the proofs of which we postpone to end of the paper (Sections 5–7). In Section 3 we prove that the system (1.4)–(1.7) (without resetting) develops shocks almost surely, for any $N$. In section 4 we discuss why the classical conservation law / entropy solution techniques can not be used for (1.4)–(1.7).

## 2. The main theorem and proof.

Throughout this paper, we assume $N \geqslant 2$ to be fixed, and $W_t^1, \dots, W_t^N$ to be $N$ independent Wiener processes with filtration $\mathcal{F}_t$. We assume subsequently, without loss of generality, that $\nu = \frac{1}{2}$. We use $C^k$, to denote the space of all periodic functions on $\mathbb{R}$ (with period 1) which have $k$ continuous derivatives. We use $L^p$, $H^s$ to be the Lebesgue $p$-space, and the Sobolev space of order $s$ respectively, consisting of periodic functions. Statements about processes being $C^k$, $L^p$ or $H^s$ will refer only to the spatial regularity.

**Definition 2.1.** Given $t_0 \in \mathbb{R}$, we define $S^N_{t_0,\cdot}$ to be the solution operator of (1.10)–(1.13), starting at time $t_0$ and with periodic boundary conditions.

Explicitly, given $u_{t_0}$ (assumed to be $\mathcal{F}_{t_0}$-measurable), the process $u_t = S^N_{t_0,t} u_{t_0}$, $t \geqslant t_0$ is the unique fixed point of the system

$$dX^{i,N}_{t_0,t} = u_t(X^{i,N}_{t_0,t})\,dt + dW^i_t \tag{2.1}$$

$$X^{i,N}_{t_0,t_0}(a) = a \tag{2.2}$$

$$A^{i,N}_{t_0,t} = \left(X^{i,N}_{t_0,t}\right)^{-1} \tag{2.3}$$

$$u_t = \frac{1}{N}\sum_{i=1}^N u_{t_0} \circ A^{i,N}_{t_0,t}. \tag{2.4}$$

For notational convenience, we omit $N$ from the superscript superscripts for the remainder of this section. Given $u_0$ and $\delta_t > 0$, the system (1.10)–(1.13) can now be iteratively defined by

$$u^{\delta_t}_t = S_{k\delta_t,t} u^{\delta_t}_{k\delta_t} \quad \text{for } k \in \mathbb{N} \text{ such that } k\delta_t \leqslant t < (k+1)\delta_t. \tag{2.5}$$

Thus for times which are integer multiples of $\delta_t$ we have

$$u^{\delta_t}_{(k+1)\delta_t} = S_{k\delta_t,(k+1)\delta_t} u^{\delta_t}_{k\delta_t}. \tag{2.6}$$

We show global existence of (2.5), provided our resetting time is small enough.

**Theorem 2.2.** *Let $T > 0$, $\varepsilon > 0$, $s > 6 + \frac{1}{2}$, and suppose $u_0 \in H^s$. Then there exists $\delta_T = \delta_T(T, \varepsilon, s, \|u_0\|_{H^s})$ such that if $\delta_t < \delta_T$, then the solution of* (2.5) *has an $C^6$ solution on the interval $[0, T]$ with probability at least $1 - \varepsilon$.*

Note that the operator $S_{t_0,t}$ is *not* a smoothing operator. This is because the SPDE satisfied by $u_t = S_{0,t}u_0$ is (5.2) which is not a dissipative SPDE [14]. One can immediately verify this as the diffusive term in (5.2) does not necessarily dominate the noise.

Another (perhaps more intuitive) way of understanding the regularity properties of $S_{t_0,t}$ is via time splitting. The $S_{t_0}$ can be time split into two parts: $\bar{S}^1_{t_0}$, the non-linear solution operator associated with the *inviscid* Burgers' equations, and $\bar{S}^2_{t_0}$ the operator corresponding to resetting. By considering time split version of (1.10), one can see that $\bar{S}^2_{t_0}$ corresponds exactly to the operator

$$\bar{S}^2_{t_0,t_0+\delta_t}f(x) = \frac{1}{N}\sum_{j=1}^{N} f\left(x - \left(W^j_{t_0+\delta_t} - W^j_{t_0}\right)\right) \tag{2.7}$$

The operator $\bar{S}^1_{t_0}$ causes growth on the Fourier modes. It is well known that the damping provided by $\nu\partial_x^2$, for any $\nu > 0$, is enough to overcome this growth, and this gives us global existence for the viscous Burgers' equations for any strictly positive viscosity. Thus if the operator $\bar{S}^2_{t_0}$ provides damping comparable to $\nu\partial_x^2$, then the usual methods can be used to prove Theorem 2.2. However, the operator $\bar{S}^2_{t_0}$ provides no damping, as can immediately be checked from (2.7): the operator norm of $\bar{S}^2_{t_0}$ is *exactly* 1 (surely) in all Sobolev and Hölder spaces. This is the main difficulty in proving Theorem 2.2.

We overcome this difficulty by considering the limit as $\delta_t \to 0$. It turns out that the limit as $\delta_t \to 0$, of $u^{\delta_t}$ satisfies a dissipative SPDE. If the initial data is regular enough, we obtain convergence in a strong norm to this (dissipative) limit. This is the key to the proof of Theorem 2.2.

**Lemma 2.3** (Key Lemma). *Let $u^{\delta_t}$ be the solution of* (2.5) *with initial data $u_0$, and $v$ be the solution of the SPDE*

$$dv_t + v_t\partial_x v_t\,dt - \frac{1}{2}\partial_x^2 v_t\,dt + \frac{\partial_x v_t}{N}\sum_{j=1}^{N} dW^j_t = 0 \tag{2.8}$$

*with initial data $v|_{t=0} = u_0$, and spatially periodic boundary conditions. Let $\beta \in \mathbb{N}\cup\{0\}$, and assume*[2]

$$\sup_{t\leqslant T_0}\left\|u^{\delta_t}_t\right\|_{C^{4+\beta}} \leqslant U \quad a.s., \tag{2.9}$$

$$\sup_{t\leqslant T_0}\|v_t\|_{C^{4+\beta}} \leqslant U \quad a.s., \tag{2.10}$$

*and let $w^{\delta_t}_t = u^{\delta_t}_t - v_t$. Then there exists a constant $C = C(\beta, U, N, T_0)$ (independent of $\delta_t$) such that*

$$\max_{k\leqslant\frac{T_0}{\delta_t}} \boldsymbol{E}\left\|\partial_x^\beta w^{\delta_t}_{k\delta_t}\right\|^2_{L^2} \leqslant C\delta_t^{1/2}.$$

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[2] The assumptions (2.9)–(2.10) can be weakened slightly at the expense of a lengthier, more technical proof. The weakened assumptions however, still require more than $\beta$ derivatives. While replacing (2.9)–(2.10) with a condition involving only $\beta$ derivatives would be of sufficient interest to warrant a more technical proof, reducing $4+\beta$ to $4+\beta-\varepsilon$ only obscures the heart of the matter. Since sufficient regularity on our initial data will guarantee (2.9)–(2.10) anyway, we assume they hold and avoid unnecessary technicalities.

Our main interest in this lemma will be for $\beta = 2$, as it will enable us to obtain $C^1$ bounds on $u$ from a $C^1$ bound on $v$. A $C^1$ bound is all that is needed to continue a solution locally, thus a uniform in time $C^1$ bound will prove our theorem. Since (2.8) is dissipative, uniform in time bounds of strong norms are readily obtained.

**Lemma 2.4.** *Let $s \in \mathbb{N}$, $u_0 \in H^s$, and $v$ be a solution to the SPDE* (2.8) *with initial data $u_0$ and periodic boundary conditions. Then there exists a constant $V = V(\|u_0\|_{H^s}, s)$ such that*

$$\|v_t\|_{H^s} \leqslant V_s \quad a.s. \tag{2.11}$$

*for all $t \geqslant 0$.*

Lemma 2.4 and Lemma 2.3 will now allow uniform in time control a strong norm of $u^{\delta_t}$. The only remaining ingredient is obtain a $C^1$ local existence result, and guarantee that the inequality (2.9) is satisfied uniformly in $\delta_t$.

**Lemma 2.5.** *Suppose $u_0 \in C^n$, and let $u^{\delta_t}$ be the solution of* (2.5) *with initial data $u_0$. There exists $T_0 = T_0(\|u_0\|_{C^1})$ such that the solution to* (2.5) *exists on the interval $[0, T_0]$. Further, there exists constant $U_n = U_n(\|u_0\|_{C^n}, n)$ independent of $\delta_t$ such that*

$$\sup_{0 \leqslant t \leqslant T_0} \left\| u_t^{\delta_t} \right\|_{C^n} \leqslant U_n \quad a.s. \tag{2.12}$$

*for all $\delta_t < T_0$.*

As usual, the existence time $T_0$ only depends on a certain norm (in this case $C^1$) of the initial data. On this existence interval, any additional smoothness of the initial data is preserved.

We are now ready to prove the main theorem, and we postpone the proofs of the above propositions and lemmas to subsequent sections.

*Proof of Theorem 2.2.* Let $v_t$ be the solution of (2.8). By Lemma 2.4 and the Sobolev embedding theorem there is a constant $V_1$, such that $\|v_t\|_{C^1} \leqslant V_1$ for all $t \geqslant 0$. Let $T_0 = T_0(2V_1)$ be the local existence time in Lemma 2.5; namely for any initial data $u_0$ with $\|u_0\|_{C^1} \leqslant 2V_1$ the operator $S_{t_0,t}u_0$ is defined for all $t \in [t_0, t_0 + T_0]$.

Note that our assumption $u_0 \in H^{13/2+}$, Lemma 2.4 and the Sobolev embedding theorem implies the assumption (2.10) is valid for $\beta = 2$ (in this case, the supremum can in fact be taken over all $t \in \mathbb{R}$). Similarly Lemma 2.5 guarantees that the assumption (2.9) is valid for $\beta = 2$ and all $\delta_t < T_0$. Thus Lemma 2.3 can be applied.

Let $\Omega_1$ be the event $\{\|u_{T_0}^{\delta_t}\|_{C^1} \leqslant 2V_1\}$. Then

$$\begin{aligned}
P(\Omega_1) &\geqslant P\left(\left\|u_{T_0}^{\delta_t} - v_{T_0}\right\|_{C^1} \leqslant V\right) \\
&\geqslant P\left(\left\|u_{T_0}^{\delta_t} - v_{T_0}\right\|_{H^2} \leqslant \tfrac{V}{c_1}\right) && \text{[Sobolev embedding]} \\
&\geqslant 1 - \frac{c_1^2}{V^2}\boldsymbol{E}\left(\left\|u_{T_0}^{\delta_t} - v_{T_0}\right\|_{H^2}^2\right) && \text{[Chebyshev's inequality]} \\
&\geqslant 1 - \frac{C\delta_t^{1/2}}{V^2} && \text{[Lemma 2.3]},
\end{aligned}$$

where constant $c_1$ above is the constant arising in the Sobolev embedding theorem. Thus for small enough $\delta_t$, $P(\Omega_1)$ can be made arbitrarily close to 1. (Note that while the bound on $P(\Omega_1)$ is independent of $\delta_t$, the event $\Omega_1$ may itself depend on $\delta_t$.)

Now this procedure can be iterated: Let $\tilde{u}^{\delta_t}_{T_0} = \chi_{\Omega_1} u^{\delta_t}_{T_0}$, and for $t > T_0$ define $\tilde{u}^{\delta_t}_t$ by

$$\tilde{u}^{\delta_t}_t = S_{k\delta_t, t} \tilde{u}^{\delta_t}_{k\delta_t} \quad \text{for } k \in \mathbb{N} \text{ such that } k\delta_t \leqslant t < (k+1)\delta_t.$$

Note that $\|\tilde{u}^{\delta_t}_{T_0}\|_{C^1} \leqslant 2V_1$ almost surely. Thus by Lemma 2.5, and our choice of $T_0$, $\tilde{u}^{\delta_t}_{T_0}$ is defined (and spatially $C^6$) for all $t \in [T_0, 2T_0]$. Further, since the equations (2.1)–(2.4) are all almost sure relations (specifically, since they don't involve the law of the processes), uniqueness of solutions will guarantee that the processes $\tilde{u}^{\delta_t} = u^{\delta_t}$ are indistinguishable on $\Omega_1$, on the time interval $[T_0, 2T_0]$.

Using Sobolev embedding, Chebyshev's inequality and Lemma 2.3 as above, we can find an event $\Omega_2 \subset \Omega_1$ such that $P(\Omega_2)$ is arbitrarily close to $P(\Omega_1)$, and the solution $u^{\delta}_t$ can be continued to the interval $[2T_0, 3T_0]$. Iterating, finishes the proof. □

Finally we address the question of $N \to \infty$. For this purpose, we introduce the superscript of $N$ to indicate the dependence on $N$ of the process considered. Using techniques similar to [11], we show that the solution $v^N$ of (2.8) converges to the solution of the viscous Burgers' equations as $N \to \infty$.

**Proposition 2.6.** *Let $v^N$ be the solution of* (2.8) *with initial data $u_0$, and $u^b_t$ be the solution of the viscous Burgers' equation* (1.1) *with the same initial data. If $u_0 \in H^s$, $s > \frac{3}{2}$, then for any $T > 0$*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \boldsymbol{E} \left\| u^b_t - v^N_t \right\|_{L^2} = 0.$$

Now Proposition 2.6, Lemma 2.3 and an argument similar to the proof of Theorem 2.2 will show that for small enough $\delta_t$, as $N \to \infty$, $u^{N,\delta_t}$ converges to the same limit on an event of almost full probability.

## 3. Almost sure existence of shocks without resetting

In this section we show that the system (1.4)–(1.7) develops shocks almost surely, for any $N$. For simplicity, we work with functions on the line, instead of on the Torus. For this section we will assume all functions spaces refer to functions defined on the real line.

**Definition 3.1.** We say a stopping time $\tau$ is an $C^1$ existence time of the system (1.4)–(1.7) if there exists a $C^1$ solution of (1.4)–(1.7) defined on the (random) interval $[0, \tau)$.

**Definition 3.2.** We say $\tau$ is a maximal $C^1$ existence time of the system (1.4)–(1.7) if for any $C^1$ existence time $\tau'$, we have $\tau' \leqslant \tau$.

**Proposition 3.3.** *Suppose $u_0 \in C^1(\mathbb{R})$ is a decreasing function, and $\tau'$ be the maximal $C^1$ existence time of* (1.4)–(1.7)*. Then*

$$\tau' < \frac{N}{\|\nabla u_0\|_{L^\infty}} \tag{3.1}$$

*almost surely.*

*Remark* 3.4. The numerically observed shock time is of the order $\frac{1}{\|\nabla u_0\|_{L^\infty}}$ with large probability, indicating that our bound (3.1) is far from optimal. Further, this is also (numerically) observed without the assumption that the initial data is monotone.

*Remark.* One can show[3] that as $N \to \infty$ the solution to (1.4)–(1.7) approaches the solution to (1.1) at a rate of $\frac{1}{\sqrt{N}}$. However, it is well known that the solution to (1.1) is smooth for all time and no shock develops, provided the initial data is for instance $C^1$ and bounded [7].

Thus Proposition 7.1 (or more precisely Remark 3.4) shows that no matter how large $N$ is, the system (1.4)–(1.7) will only be a good approximation to the true solution of (1.1) for short time in the order of $\frac{1}{\|\nabla u_0\|_{L^\infty}}$.

*Proof of Proposition 3.3.* Assume for simplicity that $\|\nabla u_0\|_{L^\infty} = -\nabla u_0(0) = 1$. We first show that $\nabla X_t^{i,N}$ becomes 0 almost surely in finite time. Differentiating (1.4) in space gives

$$d(\nabla X_t^{i,N}) = \nabla u_t^N\Big|_{X_t^{i,N}} \nabla X_t^{i,N}\, dt \tag{3.2}$$

For $i = 1$, equation (1.7) gives

$$\begin{aligned}\nabla u\Big|_{X_t^{1,N}} \nabla X_t^{1,N} &= \frac{1}{N}\Big[\nabla(u_0 \circ A_t^{1,N})\Big|_{X_t^{1,N}} \nabla X_t^{1,N} + \\ &\qquad + \sum_{i=2}^{N} \nabla(u_0 \circ A_t^{i,N})\Big|_{X_t^{1,N}} \nabla X^{1,N}\Big] \\ &= \frac{1}{N}\Big[\nabla u_0 + \sum_{i=2}^{N} \nabla u_0\Big|_{A_t^{i,N}\circ X_t^{1,N}} \cdot \nabla\left(A_t^{i,N} \circ X_t^{1,N}\right)\Big] \end{aligned} \tag{3.3}$$

Now note that as long as we consider a $C^1$ solution of the system (1.4)–(1.7), the flow $X_t^{i,N} : \mathbb{R} \to \mathbb{R}$ is homotopic to the identity map via $C^1$ diffeomorphisms of $\mathbb{R}$. The same can be said about $A_t^{i,N}$, and thus $\nabla(A_t^{i,N} \circ X_t^{i,N}) > 0$. Finally since $u_0$ is assumed to be decreasing, we know that $\nabla u_0 < 0$. Using this along with equations (3.2), (3.3) yield

$$\partial_t \nabla X_t^{1,N}(0) \leqslant \frac{-1}{N}.$$

Since at time 0, $\nabla X_0^{1,N} \equiv 1$, $\nabla X_t^{1,N}(0)$ becomes 0 in time at most $N$. Consequently $\|\nabla A_t^{1,N}\|_{L^\infty}$ approaches infinity in time at most $N$.

Now differentiating (1.7) gives

$$\nabla u_t^N = \frac{1}{N}\sum_{i=1}^{N} \nabla u_0\Big| A_t^{i,N}.$$

We know the first term (when $i = 1$) approaches $-\infty$ in time at most $N$. The remaining terms are all negative, so $\nabla u_t^N$ approaches $-\infty$ in time at most $N$. □

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[3] See for instance [11, Theorem 4.1], where the analogous result is proved for the Navier-Stokes equations.

## 4. The formal failure of classical techniques to continue past shocks

In this section show that the classical techniques used to continue solutions of the inviscid Burgers' equations past shocks will not apply here. As we shall see, the Rankin-Hugonit [7] condition now involves $\partial_x u$ on the shock, making it impossible to use here.

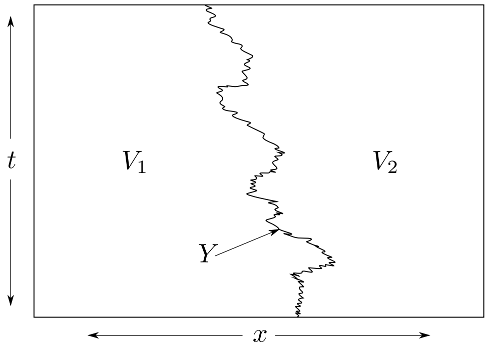

Figure 1. A possible shock separating two regions where the solution is classical.

Using the computations from [4, 11] (described briefly in Lemma 5.1), we see that for a classical solution of the system (1.4)–(1.7), the velocity $u^N$ satisfies the SPDE

$$du_t^N + u_t^N \partial_x u_t^N \, dt - \tfrac{1}{2}\partial_x^2 u_t^N \, dt + \frac{1}{N}\sum_{j=1}^N \partial_x u_t^{N,j} \, dW_t^j = 0 \tag{4.1}$$

where $u_t^{j,N} = u_0 \circ A_t^{j,N}$. For notational convenience, we subsequently assume $N$ is fixed, and omit it as a superscript.

If we impose a decay at infinity boundary condition on (4.1), the natural notion of a weak solution to (4.1) would be to require that for all $v \in C_c^\infty(\mathbb{R}\times[0,T])$,

$$\int_0^T \int_{\mathbb{R}} \left( u_t \partial_t v_t + \left[ \frac{u_t^2}{2} - \nu \partial_x u_t \right] v_t \right) dx \, dt + \frac{\sqrt{2\nu}}{N} \sum_{n=1}^N \int_0^T \int_{\mathbb{R}} u_t^i \partial_x v_t \, dx \, dW_t^i = 0$$

almost surely. Proceeding in the classical manner, we divide $\mathbb{R}\times[0,T]$ into two regions $V_1$ and $V_2$ (as in Figure 4), along a (random) shock parametrized by by $x = Y_t$. We assume that $u$ is a classical solution of (4.1) on the interior of the regions $V_1$ and $V_2$, and $u$ is a weak solution of (4.1) on $V_1 \cup V_2 = \mathbb{R}\times\mathbb{R}^+$. Now formally treating $dW_t = \dot{W}_t \, dt$ and applying Stokes theorem[4] gives

$$\iint_{V_1} u_t \partial_t v_t + \left[ \frac{u_t^2}{2} - \nu \partial_x u_t + \frac{\sqrt{2\nu}}{N} \sum_{n=1}^N u_t^i \dot{W}_t^i \right] \partial_x v_t =$$

---

[4] This can of course be made rigorous by splitting the boundary of $V_1$ into piecewise linear segments, and using the Itô isometry.

$$= \int_0^T \left[ \frac{u_t^2}{2} - \nu \partial_x u_t + \frac{\sqrt{2\nu}}{N} \sum_{n=1}^N u_t^i \dot{W}_t^i \right] v_t \, dt - u_t v_t \, dY_t$$

$$= \int_0^T v_t \left( \left[ \frac{u_t^2}{2} - \nu \partial_x u_t \right] dt + \frac{\sqrt{2\nu}}{N} \sum_{n=1}^N u_t^i \, dW_t^i - u_t \, dY_t \right)$$

almost surely. Repeating the same computation for $V_2$, and using the fact that $\iint_{V_1 \cup V_2} (\cdots) = 0$, we have

$$\int_0^T \left[ \tfrac{1}{2} J(u_t^2) - \nu J(\partial_x u_t) \right] dt + \frac{\sqrt{2\nu}}{N} \sum_{n=1}^N \int_0^T J(u_t^i) \, dW_t^i - \int_0^T J(u_t) \, dY_t = 0 \tag{4.2}$$

almost surely, where $J(v)$ denotes the jump in the function $v$ across the random shock $x = Y_t$.

Now if there is a jump in $u$ across the shock, then $\partial_x u$ will involve a delta function and it will be meaningless to talk about a jump in $\partial_x u$. Thus to balance (4.2), we must have $J(u) = 0$. In this case the remaining nonzero terms in (4.2) are $J(u_t^i) \, dW_t^i$, and $J(\partial_x u_t) \, dt$. Since the former are all independent martingales, and the latter is the only nonzero term of bounded variation, they must all be 0 individually. This forces our solution $u$ to in fact be classical! Thus a theory of entropy solutions can not be developed using these techniques.

## 5. Proof of the key lemma (Lemma 2.3).

In this section we prove convergence of $u^{\delta_t}$ to $v$ as $\delta_t \to 0$. The basic idea is to show that the velocity in our reset system (1.10)–(1.13) satisfies the limiting SPDE (2.8) with a small error which is controlled as $\delta_t \to 0$.

For this section, we always assume $u_0$ is fixed, $u_t$ is defined by (2.5), and the processes $X^i_{k\delta_t,t}$, $A^i_{k\delta_t,t}$ are all as in (1.10)–(1.13), and $v$ is the solution of the SPDE (2.8) with initial data $u_0$ (and periodic boundary conditions). For notational clarity, we will omit the $N$ and $\delta_t$ as superscripts throughout this section.

We need a few lemmas before we can prove Lemma 2.3. Our first lemma computes an SPDE satisfied by $u_t$ on the interval $[k\delta_t, (k+1)\delta_t)$.

**Lemma 5.1.** *For $t \in [k\delta_t, (k+1)\delta_t)$, we let $u_t^i = u_{k\delta_t} \circ A^i_{k\delta_t,t}$ be the $i^{th}$ summand in equation* (1.13). *Then $u_t$, $u_t^i$ satisfy the SPDE's*

$$du_t^i + u_t \partial_x u_t^i \, dt - \tfrac{1}{2} \partial_x^2 u_t^i \, dt + \partial_x u_t^i \, dW_t^i = 0 \tag{5.1}$$

$$du_t + u_t \partial_x u_t \, dt - \tfrac{1}{2} \partial_x^2 u_t \, dt + \frac{1}{N} \sum_{j=1}^N \partial_x u_t^j \, dW_t^j = 0 \tag{5.2}$$

*on the interval $[k\delta_t, (k+1)\delta_t)$.*

*Proof.* From [4,11] (see also [15,20]) we know that $A^i$ satisfies the SPDE

$$dA^i_{k\delta_t,t} + u_t \partial_x A^i_{k\delta_t,t} \, dt - \tfrac{1}{2} \partial_x^2 A^i_{k\delta_t,t} \, dt + \partial_x A^i_{k\delta_t,t} \, dW_t^i.$$

Equations (5.1) and (5.2) are now a direct application of Itô's formula. □

Now we show that $u$ satisfies the SPDE (2.8), with a small error, and obtain bounds on this error. Let $\boldsymbol{E}_{\mathcal{F}_{k\delta_t}} Y$ denote the conditional expectation of $Y$ given $\mathcal{F}_{k\delta_t}$. Given any function of time $f$, we define the increment $\Delta_k f$ by

$$\Delta_k f = f_{(k+1)\delta_t} - f_{k\delta_t}.$$

Finally, let $L$ be the (non-linear) operator defined by

$$Lu = u\partial_x u - \tfrac{1}{2}\partial_x^2 u.$$

**Lemma 5.2.** *Suppose* (2.9) *holds for some* $\beta \in \mathbb{N} \cup \{0\}$*, and let* $\varepsilon'_k$ *be defined by*

$$\varepsilon'_k(x) = u_{(k+1)\delta_t}(x) - u_{k\delta_t}(x) + Lu_{k\delta_t}(x)\,\delta_t + \partial_x u_{k\delta_t}(x)\biggl(\frac{1}{N}\sum_{j=1}^{N}\Delta_k W^j\biggr).$$

*Then there exists a constant* $C = C(\beta, U, T_0)$ *(independent of* $k, \delta_t$*) such that*

$$\sup_{x\in[0,1]} \boldsymbol{E}\left|\partial_x^\beta \varepsilon'_k(x)\right|^2 \leqslant C\delta_t^2, \tag{5.3}$$

$$\sup_{x\in[0,1]} \boldsymbol{E}\left|\boldsymbol{E}_{\mathcal{F}_{k\delta_t}}\partial_x^\beta \varepsilon'_k(x)\right|^2 \leqslant C\delta_t^3, \tag{5.4}$$

*for all* $k \leqslant \frac{T_0}{\delta_t}$.

*Remark.* Since $u$ and all derivatives of $u$ are a priori uniformly bounded almost surely, the proof of this lemma is straightforward. Without this a priori bound, we would only obtain similar bounds on $\boldsymbol{E}\|\partial_x^\beta \varepsilon'_{k\delta_t}\|_{L^2}^2$ and $\boldsymbol{E}\|\boldsymbol{E}_{\mathcal{F}_{k\delta_t}}\partial_x^\beta \varepsilon'_{k\delta_t}\|_{L^2}^2$, which is still sufficient for Lemma 2.3.

*Proof of Lemma 5.2.* We assume throughout this section that $C$ is a constant only depending on $U$ and $T_0$ which could change from line to line. Our assumption (2.9) and local existence (Proposition 7.1) will give short time almost sure $C^{4+\beta}$ bounds on the displacement of the flows $X^i_{k\delta_t,t}$, $A^i_{k\delta_t,t}$ and hence a short time almost sure bound on $\|u^i\|_{C^{4+\beta}}$. Thus we can assume (2.9) holds for $u^i$ as well.

For any $t \in [k\delta_t, (k+1)\delta_t)$,

$$u_t - u_{k\delta_t} = -\int_{k\delta_t}^{t} Lu_s\,ds - \frac{1}{N}\sum_{j=1}^{N}\int_{k\delta_t}^{t}\partial_x u^j_s\,dW^j_s.$$

Hence for any $x \in [0,1]$, we have

$$\boldsymbol{E}\left|\partial_x^n u_t(x) - \partial_x^n u_{k\delta_t}(x)\right|^2 \leqslant C\left(\|u\|_{C^{n+2}}, \left\|u^j\right\|_{C^{n+1}}\right)\delta_t. \tag{5.5}$$

for any $n \in \mathbb{N}\cup\{0\}$. Similarly for any $n \in \mathbb{N}\cup\{0\}$, we obtain

$$\boldsymbol{E}\left|\partial_x^n u^i_t(x) - \partial_x^n u_{k\delta_t}(x)\right|^2 \leqslant C\left(\|u\|_{C^{n}}, \left\|u^j\right\|_{C^{n+2}}\right)\delta_t \tag{5.6}$$

as $u_{k\delta_t} = u^i_{k\delta_t}$. We will only use (5.5) and (5.6) for $n \leqslant 2+\beta$.

Now,

$$\begin{aligned}\varepsilon'_k &= u_{(k+1)\delta_t} - u_{k\delta_t} + Lu_{k\delta_t}\,\delta_t + \partial_x u_{k\delta_t}\biggl(\frac{1}{N}\sum_{j=1}^{N}\Delta_k W^j_t\biggr)\\ &= \int_{k\delta_t}^{(k+1)\delta_t}\left(Lu_{k\delta_t} - Lu_s\right)ds + \frac{1}{N}\sum_{j=1}^{N}\int_{k\delta_t}^{(k+1)\delta_t}\left(\partial_x u_{k\delta_t} - \partial_x u^j_s\right)dW^j_s. \qquad (5.7)\end{aligned}$$

Hence

$$\boldsymbol{E}\left|\boldsymbol{E}_{\mathcal{F}_{k\delta_t}}\partial_x^\beta \varepsilon'_k\right|^2 = \boldsymbol{E}\left(\partial_x^\beta\int_{k\delta_t}^{(k+1)\delta_t}\left(Lu_{k\delta_t} - Lu_s\right)ds\right)^2$$

$$\begin{aligned}&\leqslant \delta_t \int_{k\delta_t}^{(k+1)\delta_t} \boldsymbol{E}\left[\partial_x^\beta\left(Lu_{k\delta_t} - Lu_s\right)\right]^2 ds\\ &\leqslant C\delta_t^3\end{aligned}$$

where the last inequality follows from (5.5) with $n = 2 + \beta$. This proves (5.4).

For (5.3), note that the expected value of the square of the first term in (5.7) has already been bounded by $C\delta_t^3 < C\delta_t^2$. For the second term, the Itô isometry gives

$$\boldsymbol{E}\left(\frac{1}{N}\sum_{j=1}^N \int_{k\delta_t}^{(k+1)\delta_t} \partial_x^\beta\left(\partial_x u_{k\delta_t} - \partial_x u_s^j\right) dW_s^j\right)^2 = \\ = \frac{1}{N}\sum_{j=1}^N \int_{k\delta_t}^{(k+1)\delta_t} \boldsymbol{E}\left[\partial_x^\beta\left(\partial_x u_{k\delta_t} - \partial_x u_s^j\right)\right]^2 ds$$

and using (5.6) with $n = 1 + \beta$ the proof is complete. ☐

We now prove that a time split version of the SPDE (2.8) satisfies the same error estimates as in Lemma 5.2.

**Lemma 5.3.** *Suppose* (2.10) *holds for some* $\beta \in \mathbb{N} \cup \{0\}$, *and let* $\varepsilon_k''$ *be defined by*

$$\varepsilon_k'' = v_{(k+1)\delta_t} - v_{k\delta_t} + Lv_{k\delta_t}\,\delta_t + \partial_x v_{k\delta_t}\left(\frac{1}{N}\sum_{j=1}^N \Delta_k W^j\right). \tag{5.8}$$

*Then the bounds* (5.3) *and* (5.4) *hold for* $\varepsilon_k''$.

*Proof.* First note that

$$v_t - v_{k\delta_t} = -\int_{k\delta_t}^t Lv_s\,ds - \frac{1}{N}\sum_{j=1}^N \int_{k\delta_t}^{(k+1)\delta_t} \partial_x v_s\,dW_s^j.$$

Thus for any $x \in [0,1]$, $t \in [k\delta_t, (k+1)\delta_t]$ we have

$$\boldsymbol{E}\left|\partial_x^n v_t(x) - \partial_x^n v_{k\delta_t}(x)\right|^2 \leqslant C(\|v\|_{C^{n+2}})\delta_t. \tag{5.9}$$

for any $n \in \mathbb{N}$. We will only use (5.9) for $n \leqslant 2 + \beta$.

By definition (5.8),

$$\varepsilon_k'' = \int_{k\delta_t}^{(k+1)\delta_t} \left(Lv_{k\delta_t} - Lv_s\right) ds + \frac{1}{N}\sum_{j=1}^N \int_{k\delta_t}^{(k+1)\delta_t} \left(\partial_x v_{k\delta_t} - \partial_x v_s\right) dW_s^j.$$

The remainder of the proof is now identical to the proof of Lemma 5.2. ☐

We are now ready to prove Lemma 2.3. Before beginning the proof, we remark that the constant $C$ can be chosen independent of $N$, and Lemma 2.3 is true even when $N = 1$. However the assumptions (5.10) and (5.11) will not be satisfied when $N = 1$.

We also remark that, the assumptions (2.9) and (2.10) are stronger than we require for the Lemma. We only need

$$\max_{0\leqslant k\leqslant \frac{T_0}{\delta_t}} \left(\|u_{k\delta_t}\|_{C^{1+\beta}} + \|v_{k\delta_t}\|_{C^{1+\beta}}\right) \leqslant U, \quad \text{a.s.} \tag{5.10}$$

$$\sup_x \max_{0\leqslant k\leqslant \frac{T_0}{\delta_t}} \boldsymbol{E}\left(\left|\partial_x^{2+\beta}u_{k\delta_t}(x)\right|^2+\left|\partial_x^{2+\beta}v_{k\delta_t}(x)\right|^2\right)\leqslant U^2, \tag{5.11}$$

and the bounds on $\varepsilon'$, $\varepsilon''$ provided by Lemmas 5.2 and 5.3 above (we wrote $U^2$ in (5.11) above only for dimensional correctness). The proof we provide below depends only on these weaker assumptions.

*Proof of Lemma 2.3.* Let $\varepsilon_k=\varepsilon'_k-\varepsilon''_k$, where $\varepsilon'_k$, $\varepsilon''_k$ are defined by Lemmas 5.2 and 5.3 respectively. Using equations (5.3), (5.4) and the corresponding estimates for $\varepsilon''_k$, we have

$$\sup_{x\in[0,1]} \boldsymbol{E}\partial_x^\beta \varepsilon_k(x)^2 \leqslant C\delta_t^2 \tag{5.12}$$

$$\sup_{x\in[0,1]} \boldsymbol{E}\left|\boldsymbol{E}_{\mathcal{F}_{k\delta_t}}\partial_x^\beta \varepsilon_k(x)\right|^2 \leqslant C\delta_t^3 \tag{5.13}$$

for all $k\leqslant \frac{T_0}{\delta_t}$. We remark that we only require the corresponding bounds on $\boldsymbol{E}\|\varepsilon_k\|^2_{H^\beta}$ and $\boldsymbol{E}\|\boldsymbol{E}_{\mathcal{F}_{k\delta_t}}\varepsilon_k\|^2_{H^\beta}$ for the proof of this Lemma.

We know, $w_0=0$ and

$$\begin{aligned}\partial_x^\beta\Delta_k w &= \partial_x^\beta\Delta_k u-\partial_x^\beta\Delta_k v\\ &= -\partial_x^\beta\left(Lu_{k\delta_t}-Lv_{k\delta_t}\right)\delta_t-\partial_x^{\beta+1}w_{k\delta_t}\left(\frac{1}{N}\sum_{j=1}^N\Delta_k W^j\right)+\partial_x^\beta\varepsilon_k.\end{aligned} \tag{5.14}$$

We first estimate $\boldsymbol{E}(\partial_x^\beta\Delta_k w)^2$ where $k$ is any integer such that $k\delta_t\leqslant T_0$.

By independence of increments,

$$\begin{aligned}\boldsymbol{E}\left[\partial_x^{\beta+1}w_{k\delta_t}\left(\frac{1}{N}\sum_{j=1}^N\Delta_k W^j\right)\right]^2 &= \boldsymbol{E}(\partial_x^{\beta+1}w_{k\delta_t})^2\boldsymbol{E}\left(\frac{1}{N}\sum_{j=1}^N\Delta_k W^j\right)^2\\ &= \frac{\delta_t}{N}\boldsymbol{E}(\partial_x^{\beta+1}w_{k\delta_t})^2\end{aligned}$$

On squaring and taking the expectation of (5.14), Lemma 5.2 and Cauchy-Schwartz show that the product of the last and second last term is bounded by $C\delta_t^{3/2}$. Thus

$$\boldsymbol{E}(\partial_x^\beta\Delta_k w)^2\leqslant \frac{\delta_t}{N}\boldsymbol{E}(\partial_x^{\beta+1}w_{k\delta_t})^2+C\delta_t^{3/2}. \tag{5.15}$$

as the remaining terms are of order $\delta_t^2$ or higher. Note that the quadratic term in $Lu$, $Lv$ appearing in (5.14) will contain at most $\beta+1$ derivatives of $u$, $v$, and thus can be bounded using (5.10). The second order term in $Lu$, $Lv$ appearing in (5.14) only needs to be controlled in the mean square sense to obtain (5.15), and thus can be bounded by (5.11).

Now for any $K\leqslant\frac{T_0}{\delta_t}$,

$$\begin{aligned}\left(\partial_x^\beta w_{K\delta_t}\right)^2 &= 2\sum_{k=0}^{K-1}\partial_x^\beta w_{k\delta_t}\partial_x^\beta\Delta_k w+\sum_{k=0}^{K-1}\left(\partial_x^\beta\Delta_k w\right)^2\\ &= 2\sum_k \partial_x^\beta w_{k\delta_t}\Bigg(-\partial_x^\beta\left(Lu_{k\delta_t}-Lv_{k\delta_t}\right)\delta_t-\\ &\qquad -\partial_x^{\beta+1}\Delta_k w\left(\frac{1}{N}\sum_{j=1}^N\Delta_k W^j\right)+\partial_x^\beta\varepsilon_k\Bigg)+\end{aligned}$$

$$+\sum_k \left(\partial_x^\beta \Delta_k w\right)^2 .$$

Taking expected values, integrating in space and using (5.15) gives
(5.16)

$$\begin{aligned}\boldsymbol{E}\left\|\partial_x^\beta w_{K\delta_t}\right\|_{L^2}^2 \leqslant & -2\delta_t \sum_{k=0}^{K-1} \boldsymbol{E}\int_x \partial_x^\beta w_{k\delta_t}\partial_x^\beta\left(u_{k\delta_t}\partial_x u_{k\delta_t} - v_{k\delta_t}\partial_x v_{k\delta_t}\right)\,dx + \\ & + 2\sum_{k=0}^{K-1}\boldsymbol{E}\int_x \partial_x^\beta w_{k\delta_t}\partial_x^\beta \varepsilon_k\,dx - \\ & -\left(1-\frac{1}{N}\right)\delta_t\sum_{k=0}^{K-1}\boldsymbol{E}\int_x (\partial_x^{\beta+1}w_{k\delta_t})^2\,dx + CK\delta_t^{3/2}\end{aligned}$$

For the first term,

$$\partial_x^\beta w_{k\delta_t}\partial_x^\beta\left(u_{k\delta_t}\partial_x u_{k\delta_t} - v_{k\delta_t}\partial_x v_{k\delta_t}\right) = \partial_x^\beta w_{k\delta_t}\partial_x^\beta\left(w_{k\delta_t}\partial_x u_{k\delta_t} - v_{k\delta_t}\partial_x w_{k\delta_t}\right).$$

For the term involving $u$,

$$\left|\int_x \partial_x^\beta w_{k\delta_t}\partial_x^\beta\left(w_{k\delta_t}\partial_x u_{k\delta_t}\right)\right| \leqslant C\left\|\partial_x^\beta w_{k\delta_t}\right\|_{L^2}^2 \left\|u_{k\delta_t}\right\|_{C^{\beta+1}}.$$

For the term involving $v$, note

$$\partial_x^\beta w_{k\delta_t} v_{k\delta_t}\partial_x^{\beta+1} w_{k\delta_t} = \tfrac{1}{2} v_{k\delta_t}\partial_x\left(\partial_x^\beta w_{k\delta_t}\right)^2$$

and so we can integrate by parts to avoid the extra derivative on $w$. Thus

$$\left|\int_x \partial_x^\beta w_{k\delta_t}\partial_x^\beta\left(v_{k\delta_t}\partial_x w_{k\delta_t}\right)\right| \leqslant C\left\|\partial_x^\beta w_{k\delta_t}\right\|_{L^2}^2 \left\|v_{k\delta_t}\right\|_{C^{\beta+1}}$$

and hence using (5.10) we have

$$-2\delta_t\boldsymbol{E}\int_x \partial_x^\beta w_{k\delta_t}\partial_x^\beta\left(u_{k\delta_t}\partial_x u_{k\delta_t} - v_{k\delta_t}\partial_x v_{k\delta_t}\right)\,dx \leqslant C\delta_t\boldsymbol{E}\left\|\partial_x^\beta w_{k\delta_t}\right\|_{L^2}^2. \tag{5.17}$$

For the second term in (5.16), we know $w_{k\delta_t}$ is $\mathcal{F}_{k\delta_t}$-measurable. Thus using 5.13 and Cauchy-Schwartz gives

$$\left|\boldsymbol{E}\partial_x^\beta w_{k\delta_t}\partial_x^\beta\varepsilon_k\right| = \left|\boldsymbol{E}\left(\partial_x^\beta w_{k\delta_t}\boldsymbol{E}_{\mathcal{F}_{k\delta_t}}\partial_x^\beta\varepsilon_k\right)\right| \leqslant C\delta_t^{3/2}.$$

Replacing $K$ with $\frac{T_0}{\delta_t}$, and using equations (5.17) and (5.18) in equation (5.16) give

$$\boldsymbol{E}\left\|\partial_x^\beta w_{K\delta_t}\right\|_{L^2}^2 \leqslant C\delta_t^{1/2} + C\sum_{k=0}^{K-1}\boldsymbol{E}\left\|\partial_x^\beta w_{k\delta_t}\right\|_{L^2}^2\delta_t.$$

The remainder of the proof is an elementary discrete Gronwall argument: Let

$$y_K = C\delta_t^{1/2} + C\sum_{k=0}^{K-1}\boldsymbol{E}\left\|\partial_x^\beta w_{k\delta_t}\right\|_{L^2}^2\delta_t.$$

Then

$$y_{k+1} - y_k = C\delta_t\boldsymbol{E}\left\|\partial_x^\beta w_{k\delta_t}\right\|_{L^2}^2 \leqslant C\delta_t y_k$$

and hence

$$y_{k+1} \leqslant (1 + C\delta_t)\,y_k.$$

Iterating, and using $y_0 = C\delta_t^{1/2}$ gives

$$y_k \leqslant (1 + C\delta_t)^k \, C\delta_t^{1/2}.$$

Since $k \leqslant \frac{T_0}{\delta_t}$ this gives

$$\max_{k \leqslant \frac{T_0}{\delta_t}} y_k \leqslant C\delta_t^{1/2} \sup_{\delta_t' > 0} (1 + C\delta_t')^{T_0/\delta_t'}$$

concluding the proof. □

## 6. Proof of Lemma 2.4 and Proposition 2.6

In this section we establish uniform in time bounds for the solution of (2.8), and prove Proposition 2.6.

We prove Lemma 2.4, via the following two Lemmas:

**Lemma 6.1.** *Let $u_0 \in H^{\bar{s}}$, $\bar{s} \in \mathbb{Z}^+$ and $v$ be the solution to* (2.8) *with initial data $u_0$ and periodic boundary conditions. Then for any $T > 0$, there exists a constant $V = V(\bar{s}, T, \|u_0\|_{H^{\bar{s}}})$ such that*

$$\sup_{0 \leqslant t \leqslant T} \|v_t\|_{H^{\bar{s}}} \leqslant V$$

*almost surely.*

**Lemma 6.2.** *Let $u_0 \in L^2$, $\bar{s} \in \mathbb{Z}^+$ and $v$ be the solution to* (2.8) *with initial data $u_0$. Then for any $T > 0$, there exists a constant $V_s = V_s(T)$ such that*

$$\sup_{t \geqslant T} \|v_t\|_{H^{\bar{s}}} \leqslant V_{\bar{s}}$$

*almost surely.*

*Proof of Lemma 2.4.* From Lemmas 6.1 and 6.2. □

*Proof of Lemma 6.1.* We prove Lemma 6.1 via energy estimates. Since $\bar{s} \geqslant 1$ by assumption, $u_0 \in L^\infty$ by the Sobolev embedding theorem. By the maximum principle [16] we obtain $\|v_t\|_{L^\infty} \leqslant \|u_0\|_{L^\infty}$ almost surely. Differentiating (2.8) with respect to $x$ and applying Itô's formula for $(\partial_x v_t)^2$ we obtain

$$\begin{aligned} d(\partial_x v_t)^2 &= 2\partial_x v_t \, d(\partial_x v_t) + \frac{1}{N} \left|\partial_x^2 v_t\right|^2 \, dt \\ &= -2\partial_x v_t \left( \partial_x (v_t \partial_x v_t) \, dt - \frac{1}{2}\partial_x^3 v_t \, dt + \frac{\partial_x^2 v_t}{N} \sum_{j=1}^{N} dW_t^j \right) + \frac{1}{N} \left|\partial_x^2 v_t\right|^2 \, dt \end{aligned}$$

Integrating with respect to $x$ on $[0,1]$, and noting that $\int_0^1 \partial_x v_t \partial_x^2 v_t \, dx = 0$, gives

$$\begin{aligned} d \|\partial_x v_t\|_{L^2}^2 &= -\left(1 - \frac{1}{N}\right) \left\|\partial_x^2 v_t\right\|_{L^2}^2 \, dt + \left( 2 \int_0^1 \partial_x^2 v_t (v_t \partial_x v_t) \, dx \right) dt \\ \implies \partial_t \|\partial_x v_t\|_{L^2}^2 &\leqslant -\frac{1}{4} \left\|\partial_x^2 v_t\right\|_{L^2}^2 + 8 \|v_t \partial_x v_t\|_{L^2}^2 \\ &\leqslant -\frac{1}{4} \left\|\partial_x^2 v_t\right\|_{L^2}^2 + 8 \|v_t\|_{L^\infty}^2 \|\partial_x v_t\|_{L^2}^2 , \\ &\leqslant -\frac{1}{4} \left\|\partial_x^2 v_t\right\|_{L^2}^2 + 8 \|u_0\|_{L^\infty}^2 \|\partial_x v_t\|_{L^2}^2 , \end{aligned} \tag{6.1}$$

which gives us

$$\|v_t\|_{H^1} \leqslant C_1 e^{c_0 t} \quad \text{a.s.}$$

for some constants $C_1 = C_1(\|u_0\|_{H^1})$ and $c_0 = c_0(\|u_0\|_{L^\infty})$. We also obtain

$$\int_0^t \|v_{t'}\|_{H^2}^2 \, dt' \leqslant C_1 e^{c_0 t}$$

almost surely.

For the remainder of this proof we adopt the convention that $c$, $C$ denote absolute constants, $C_s = C_s(s, \|u_0\|_{H^s})$ denotes a constant depending only on $s$, $\|u_0\|_{H^s}$ and $c_0$ denotes a constant depending only on $\|u_0\|_{L^\infty}$. The exact value of these constants are immaterial, and we will allow them to change from line to line.

Differentiating (2.8) with respect to $x$ twice, applying Itô's formula for $(\partial_x^2 v_t)^2$, and integrating gives

$$\begin{aligned}
\partial_t \left\|\partial_x^2 v_t\right\|_{L^2}^2 &\leqslant -\left(1 - \frac{1}{N}\right) \left\|\partial_x^3 v_t\right\|_{L^2}^2 + 2 \left\|\partial_x^3 v_t\right\|_{L^2} \left\|\partial_x \left(v_t \partial_x v_t\right)\right\|_{L^2} \\
&\leqslant -c \left\|\partial_x^3 v_t\right\|_{L^2}^2 + C \left(\|\partial_x v_t\|_{L^4}^4 + \|v_t\|_{L^\infty}^2 \left\|\partial_x^2 v_t\right\|_{L^2}^2\right) \\
&\leqslant -c \left\|\partial_x^3 v_t\right\|_{L^2}^2 + C \left(\|\partial_x v_t\|_{L^\infty}^2 + \|v_t\|_{L^\infty}^2\right) \left\|\partial_x^2 v_t\right\|_{L^2}^2 \\
&\leqslant -c \left\|\partial_x^3 v_t\right\|_{L^2}^2 + C \|v_t\|_{H^2}^2 \left\|\partial_x^2 v_t\right\|_{L^2}^2 ,
\end{aligned}$$

where the last inequality is obtained by Sobolev embedding. This gives

$$\|v_t\|_{H^2} \leqslant C_2 e^{\int_0^t \|v_t\|_{H^2}^2 dt} \leqslant C_2 e^{C_1 e^{c_0 t}}$$

and

$$\int_0^t \|v_{t'}\|_{H^3}^2 \, dt' \leqslant C_2 e^{C_1 e^{c_0 t}},$$

where $C_2 = C_2(\|u_0\|_{H^2})$ and $C_1 = C_1(\|u_0\|_{H^1})$ and $c_0 = c_0(\|u_0\|_{L^\infty})$. Proceeding inductively, suppose we know

$$\|v_t\|_{H^s} \leqslant C_s \exp\left(C_{s-1} \exp\left(C_{s-2} \cdots \exp\left(c_0 t\right) \cdots\right)\right) \tag{6.2}$$

and

$$\int_0^t \|v_{t'}\|_{H^{s+1}}^2 \, dt' \leqslant C_s \exp\left(C_{s-1} \exp\left(C_{s-2} \cdots \exp\left(c_0 t\right) \cdots\right)\right)$$

hold almost surely for some $s < \bar{s}$. Differentiating (2.8) $s+1$ times with respect to $x$, applying Itô's formula for $(\partial_x^{s+1} v_t)^2$, and integrating we obtain

$$d \left\|\partial_x^{s+1} v_t\right\|_{L^2}^2 = -\left(1 - \frac{1}{N}\right) \left\|\partial_x^{s+2} v_t\right\|_{L^2}^2 \, dt + 2 \left\|\partial_x^{s+2} v_t\right\|_{L^2} \left\|\partial_x^s \left(v_t \partial_x v_t\right)\right\|_{L^2} \, dt$$

and hence

$$\begin{aligned}
\partial_t \left\|\partial_x^{s+1} v_t\right\|_{L^2}^2 \leqslant -c \left\|\partial_x^{s+2} v_t\right\|_{L^2}^2 + C\Big(&\|\partial_x^s v_t\|_{L^\infty}^2 \|\partial_x v_t\|_{L^2}^2 + \cdots + \\
&+ \|\partial_x v_t\|_{L^\infty}^2 \|\partial_x^s v_t\|_{L^2}^2 + \|v_t\|_{L^\infty}^2 \left\|\partial_x^{s+1} v_t\right\|_{L^2}^2\Big) \\
\leqslant -c \left\|\partial_x^{s+2} v_t\right\|_{L^2}^2 + C\Big(&\|\partial_x^s v_t\|_{L^\infty}^2 + \cdots + \|\partial_x v_t\|_{L^\infty}^2 + \\
&+ \|v_t\|_{L^\infty}^2\Big) \left\|\partial_x^{s+1} v_t\right\|_{L^2}^2 \\
\leqslant -c \left\|\partial_x^{s+2} v_t\right\|_{L^2}^2 + C &\|v_t\|_{H^{s+1}}^2 \left\|\partial_x^{s+1} v_t\right\|_{L^2}^2
\end{aligned}$$

almost surely. Thus by Gronwall's Lemma

$$\begin{aligned}\|v_t\|_{H^{s+1}} &\leqslant C_{s+1}\exp\left(\int_0^t \|v_{t'}\|^2_{H^{s+1}}\ dt'\right)\\ &\leqslant C_{s+1}\exp\left(C_s\exp\left(C_{s-1}\cdots\exp\left(c_0 t\right)\cdots\right)\right)\end{aligned}$$

almost surely. Further

$$\int_0^t \|v_{t'}\|^2_{H^{s+2}}\ dt' \leqslant C_{s+1}\exp\left(C_s\exp\left(C_{s-1}\cdots\exp\left(c_0 t\right)\cdots\right)\right) \quad \text{a.s.}$$

completing the inductive step. By induction, (6.2) holds for $s = \bar{s}$, which immediately implies the Lemma. □

*Proof of Lemma 6.2.* We prove Lemma 6.2 via a bootstrapping argument. Taking the Fourier transform of (2.8) gives the system of SDE's

$$d\hat{v}_t(n) + \frac{2\pi i n}{N}\hat{v}_t(n)\sum_{j=1}^N dW_t^j + 2\pi^2 n^2\hat{v}_t(n)\,dt + \pi i n\sum_{m\in\mathbb{Z}}\hat{v}_t(n-m)\hat{v}_t(m)\,dt = 0 \tag{6.3}$$

on the Fourier coefficients.

By Itô's formula applied to (6.3)

$$\begin{aligned} d\left|\hat{v}_t(n)\right|^2 &= \overline{\hat{v}_t(n)}\,d\hat{v}_t(n)\,dt + \hat{v}_t(n)\,d\overline{\hat{v}_t(n)} + \frac{4\pi^2 n^2}{N}\left|\hat{v}_t(n)\right|^2\ dt\\ &= -4\pi^2 n^2\left(1-\frac{1}{N}\right)\left|\hat{v}_t(n)\right|^2\ dt + \pi i n\left(\overline{\hat{v}_t(n)}B_t(n) - \hat{v}_t(n)\overline{B_t(n)}\right)dt, \end{aligned} \tag{6.4}$$

where

$$B_t(n) = \sum_{m\in\mathbb{Z}}\hat{v}_t(n-m)\hat{v}_t(m),$$

is the non-linear Fourier coupling in (6.3). Using Young's inequality in (6.4) gives

$$\begin{aligned}\partial_t\left|\hat{v}_t(n)\right|^2 &\leqslant -4\pi^2 n^2\left(1-\frac{1}{N}\right)\left|\hat{v}_t(n)\right|^2 + 2\pi n\left|\hat{v}_t(n)\right|\left|B_t\right|\\ &\leqslant -2\pi^2 n^2\left(1-\frac{1}{N}\right)\left|\hat{v}_t(n)\right|^2 + \left|B_t(n)\right|^2\end{aligned} \tag{6.5}$$

almost surely. By Parseval's identity we know $|B_t(n)| \leqslant \|v_t\|^2_{L^2} \leqslant \|u_0\|^2_{L^2}$. Thus,

$$\begin{aligned}\left|\hat{v}_t(n)\right|^2 \leqslant \left|\hat{u}_0(n)\right|^2 \exp\left[-2\pi^2 n^2\left(1-\frac{1}{N}\right)t\right] +\\ + \int_0^t \exp\left[-2\pi^2 n^2\left(1-\frac{1}{N}\right)(t-t')\right]\left|B_{t'}(n)\right|^2\ dt'\end{aligned}$$

almost surely. Integrating out, we can arrange

$$\left|\hat{v}_t(n)\right|^2 \leqslant \frac{C_0}{n^2} \quad \text{a.s.} \tag{6.6}$$

provided $t > \frac{T}{2}$ say, for some constant $C_0 = C_0(\|u_0\|_{L^2}, T)$. This is because we only need to bound the term that comes from initial conditions. This can of course be done provided $t$ is bounded away from 0, since

$$\left|\hat{u}_0(n)\right|^2 \exp\left[-2\pi^2 n^2\left(1-\frac{1}{N}\right)t\right] \leqslant \|u_0\|^2_{L^2} e^{-cn^2} \leqslant \frac{C_0}{n^2} \quad \text{a.s.}$$

and our choice $t > \frac{T}{2}$ above was arbitrary. For convinience, we let $t_1 = \frac{T}{2}$. Here $c = c(T)$ denoted a constant depending only on $T$.

Now we bootstrap, and use (6.6) to obtain a better estimate on $B_t$. Note

$$\begin{aligned}
|B_t(n)| &\leqslant \sum_{m \in \mathbb{Z}} |\hat{v}_t(n-m)| \, |\hat{v}_t(m)| \\
&\leqslant 2 \sum_{|m| \geqslant |n|/2} |\hat{v}_t(n-m)| \, |\hat{v}_t(m)| \\
&\leqslant 2 \|u_0\|_{L^2} \left( \sum_{|m| \geqslant |n|/2} |\hat{v}_t(m)|^2 \right)^{1/2} \\
&\leqslant 2 \|u_0\|_{L^2} \left( \sum_{|m| \geqslant |n|/2} \frac{C_0}{m^2} \right)^{1/2} \\
&\leqslant \frac{C_0}{\sqrt{|n|}}, \qquad (6.7)
\end{aligned}$$

for all $t > t_1$.

Using the estimate (6.7) in the differential inequality (6.5), we obtain

$$|\hat{v}_t(n)|^2 \leqslant \frac{C_0}{|n|^3}, \quad \text{a.s.}$$

for all $t > t_2$, where $t_2$ can be chosen to be any number strictly larger than $t_1$ (say $t_2 = \frac{2T}{3}$). Note that this is better than our previously obtained decay (6.6) of Fourier coefficients. In general, if we know that

$$|\hat{v}_t(n)|^2 \leqslant \frac{C_0}{|n|^{\alpha+1}}, \quad \text{a.s.}$$

then

$$|B_t(n)| \leqslant 2 \|u_0\|_{L^2} \left( \sum_{|m| \geqslant |n|/2} \frac{C_0}{|m|^{\alpha+1}} \right)^{1/2} \leqslant \frac{C_0}{|n|^{\alpha/2}}, \quad \text{a.s.}$$

and hence

$$\begin{aligned}
|\hat{v}_t(n)|^2 &\leqslant O(e^{-cn^2 t}) + \int_0^t \exp\left[ -2\pi^2 n^2 \left(1 - \frac{1}{N}\right)(t - t') \right] |B_{t'}(n)|^2 \, dt' \\
&\leqslant \frac{C_0}{|n|^{\alpha+2}}
\end{aligned}$$

almost surely. Thus, for any $\alpha \in \mathbb{Z}^+$, we choose $t_\alpha = \frac{\alpha T}{\alpha+1}$, and we can find a constant $C_0 = C_0(\|u_0\|_{L^2}, T, \alpha)$, so that

$$|\hat{v}_t(n)|^2 \leqslant \frac{C_0}{|n|^{\alpha+1}},$$

for all $t > t_\alpha$. Note that the last inequality implies that for $\alpha = 2s + 1$ we have almost surely

$$\|v_t\|_{H^s}^2 \leqslant C_0 \sum_m |m|^{2s} |\hat{v}_t(m)|^2 \leqslant C_0 \sum_m \frac{|m|^{2s}}{|m|^{2(s+1)}} = C_0 \sum_m \frac{1}{|m|^2} \leqslant C_0,$$

for all $t > t_\alpha$. $\square$

We now reintroduce an $N$ as a superscript to explicitly keep track of the dependence of our processes on $N$, and prove convergence as $N \to \infty$.

*Proof of Proposition 2.6.* Consider the difference $w_t^N = v_t^N - u_t^b$. It satisfies the SPDE

$$dw_t^N + w_t^N \partial_x v_t^N \, dt + u_t^b \partial_x w_t^N \, dt - \frac{1}{2}\partial_x^2 w_t^N \, dt + \frac{\partial_x v_t^N}{N} \sum_{j=1}^N dW_t^j = 0. \tag{6.8}$$

Thus, by Itô's formula

$$\frac{1}{2} d \left\| w_t^N \right\|_{L^2}^2 + \left( \int_x (w_t^N)^2 \partial_x v_t^N dx \right) dt + \frac{1}{2} \left\| \partial_x w_t^N \right\|_{L^2}^2 dt + \\ + \left( \int_x w_t^N \partial_x v_t^N dx \right) \left( \frac{1}{N} \sum_{j=1}^N dW_t^j \right) dt = \frac{1}{2N} \left\| \partial_x v_t^N \right\|_{L^2}^2 dt.$$

Taking expectations we obtain

$$\partial_t \boldsymbol{E} \left\| w_t^N \right\|_{L^2}^2 = -\boldsymbol{E} \left\| \partial_x w_t^N \right\|_{L^2}^2 + \frac{\boldsymbol{E} \left\| \partial_x v_t^N \right\|_{L^2}^2}{N} - 2\boldsymbol{E} \left( \int_0^1 (w_t^N)^2 \partial_x v_t^N dx \right).$$

By Lemma 2.4 and Sobolev embedding, $\|\partial_x v_t^N\|_{L^\infty}$ is (almost surely) bounded uniformly in time. Using this, and Poincare's inequality we obtain

$$\partial_t \boldsymbol{E} \left\| w_t^N \right\|_{L^2}^2 \leqslant C \left( \boldsymbol{E} \left\| w_t^N \right\|_{L^2}^2 + \frac{1}{N} \right).$$

for some constant $C = C(\|u_0\|_{H^s})$. Gronwall's lemma, applied to the previous equation gives

$$\boldsymbol{E} \left\| w_t^N \right\|_{L^2}^2 \leqslant \frac{C}{N} e^{Ct},$$

which concludes the proof. $\square$

## 7. Proof of Lemma 2.5

In this section, we prove the almost sure $C^n$ bounds on $u$ stated in Lemma 2.5. We need a few preliminary results first.

**Proposition 7.1** (Local existence without resetting)**.** *Suppose $u_{t_0} \in C^1$ be $\mathcal{F}_{t_0}$-measurable, and $\|u_{t_0}\|_{C^1}$ is bounded almost surely. There exists $T_0 = T_0(\|u_{t_0}\|_{C^1})$, independent of $N$, such that the solution to* (2.1)–(2.4) *exists on the interval* $[t_0, t_0 + T_0]$. *Further if for some $n \geqslant 1$, $\|u_{t_0}\|_{C^n}$ is almost surely bounded, then there exists $U_n = U_n(\|u_{t_0}\|_{C^n}, n)$ such that*

$$\sup_{t_0 \leqslant t \leqslant t_0 + T_0} \|u_t\|_{C^n} \leqslant U_n \tag{7.1}$$

*almost surely.*

Proposition 7.1 can be proved using standard iterative techniques. A similar result for the Navier-Stokes equations appeared in the appendix of [11] (see also [9, 10]), and we do not provide the proof here.

**Lemma 7.2.** *Let $I$ denote the identity function, and $\lambda \in C^n$ be a periodic function such that $\|\partial_x \lambda\|_{L^\infty} \leqslant d < 1$. Then there exist a constant $c_{n-1}$ depending only on $\|\partial_x^{n-1}\lambda\|_{L^\infty}$, $d$ and $n$ such that*

$$\|\partial_x^n \left[f \circ (I + \lambda)\right]\|_{L^\infty} \leqslant \|\partial_x^n f\|_{L^\infty} \left(1 + \|\partial_x \lambda\|_{L^\infty}\right)^n + c_{n-1} \|\partial_x^n \lambda\|_{L^\infty} , \tag{7.2}$$

$$\left\|\partial_x^n (I + \lambda)^{-1}\right\|_{L^\infty} \leqslant c_{n-1} \|\partial_x^n \lambda\|_{L^\infty} , \tag{7.3}$$

*for $n > 1$.*

*Proof.* First note that $\lambda$ is periodic, and for any $k \geqslant 1$, $\partial_x^k \lambda$ has mean 0. Thus for any $k \in \{1, \dots, n\}$, we have $|\partial_x^k \lambda| \leqslant c(n) \|\partial_x^n \lambda\|_{L^\infty}$, for some constant $c(n)$ depending only on $n$ and the period of $\lambda$, which is always assumed to be 1. (For $k = 0$, we need to subtract the mean of $\lambda$ for this bound to be valid.)

Now for any two $C^n$ functions $f, g$, we have

$$\partial_x^n (f \circ g) = \sum_{m=1}^{n} (\partial_x^m f) \circ g \sum_{\substack{k_1 + \cdots + k_m = n \\ k_i \geqslant 1}} \prod_{i=1}^{m} \partial_x^{k_i} g. \tag{7.4}$$

To prove (7.2), we set $g = I + \lambda$. The term in 7.4 corresponding to $m = n$ gives the first term of 7.2. When $m < n$, we notice that $k_i > 1$ for at least one $i$, and $k_j \leqslant n - 1$ for all other $j$. Thus $\partial_x^{k_i}(I + \lambda) = \partial_x^{k_i} \lambda \leqslant c(n) \|\partial_x^n \lambda\|_{L^\infty}$, and the remaining terms in the product can be bounded by $c_{n-1}$. This proves (7.2).

For (7.2), set $X = I + \lambda$ and $A = X^{-1}$. Since $n > 1$, $\partial_x^n (A \circ X) = 0$, and using (7.4) gives

$$\partial_x^n A\Big|_X = \frac{-1}{(\partial_x X)^n} \sum_{m=1}^{n-1} \partial^m A\Big|_X \sum_{\substack{k_1 + \cdots + k_m = n \\ k_i \geqslant 1}} \prod_{i=1}^{m} \partial_x^{k_i} A.$$

By induction, one can assume that $\partial_x^k A \leqslant c_{n-1}$ for all $k \leqslant n - 1$. Since $d < 1$, $\frac{1}{|\partial_x X|} \leqslant \frac{1}{1-d}$, and remaining terms can be bounded by the same argument as before. This proves (7.2). ☐

**Lemma 7.3.** *Let $u_{t_0} \in C^n$, and $u$ the solution of* (2.1)–(2.4)*. If $n > 1$, there exists $T_0 = T_0(\|u_{t_0}\|_{C^1}) > 0$ and a constant $c_{n-1} = c_{n-1}(\|u_{t_0}\|_{C^{n-1}}, n)$ such that*

$$\|\partial_x^n u_t\|_{L^\infty} \leqslant \|\partial_x^n u_{t_0}\|_{L^\infty} \left(1 + c_{n-1}(t - t_0)\right) \tag{7.5}$$

*almost surely for $t \in [t_0, t_0 + T_0]$. For $n = 1$,* (7.5) *holds with $c_0$ to be an absolute constant.*

*Proof.* For simplicity, we assume $t_0 = 0$. As can be seen below, it does not affect the proof. We first obtain $C^1$ estimates on the Eulerian and Lagrangian displacements. Throughout this section, we use the convention that $c_{n-1}$ is a constant depending only on $n$ and $\|u_0\|_{C^{n-1}}$ (or an absolute constant for $n = 1$), which could change from line to line.

Let $I$ denote be the identity map, $X^i$, $A^i$ be as in (2.1), (2.3) respectively, and define $\lambda_t^i = X_t^i - I$, $\ell_t^i = A_t^i - I$. Let $T_0 = T_0(\|u_0\|_{C^1})$ be the local existence time, and $c_1 = c_1(\|u_0\|_{C^1})$ the almost sure bound on $\|u_t\|_{C^1}$ provided by Proposition 7.1.

Now

$$\left\|\partial_x \lambda_t^i\right\|_{L^\infty} \leqslant \int_0^t \|\partial_x u_s\|_{L^\infty} \left(1 + \left\|\partial_x \lambda_t^i\right\|_{L^\infty}\right)$$

and by Gronwall's lemma,

$$\left\|\partial_x \lambda^i_t\right\|_{L^\infty} \leqslant c_0 \int_0^t \|\partial_x u_s\|_{L^\infty}\, ds \quad \text{a.s.} \tag{7.6}$$

for $t \in [0, T_0]$, where $c_0 = \exp(\int_0^t \|\partial_x u_s\|_{L^\infty}\, ds)$. Note that $T_0$ is allowed to depend on $\|u_0\|_{C^1}$, which for short time controls $\|u_t\|_{C^1}$. So making $T_0$ smaller if necessary, one can make $c_0$ an absolute constant, and $\int_0^t \|\partial_x u_s\|_{L^\infty}\, ds < \frac{1}{2c_0}$ almost surely. This will ensure $\|\partial_x \lambda^i_t\|_{L^\infty} \leqslant \frac{1}{2}$ almost surely for $t \leqslant T_0$. Now

$$\partial_x \ell^i_t = \frac{1}{\left(\partial_x X^i_t\right) \circ A^i_t} - 1 = -\frac{\left(\partial_x \lambda^i_t\right) \circ A^i_t}{\left(\partial_x X^i_t\right) \circ A^i_t}$$

so we must have

$$\left\|\partial_x \ell^i_t\right\|_{L^\infty} \leqslant 2 \left\|\partial_x \lambda^i_t\right\|_{L^\infty} \tag{7.7}$$

almost surely for $t \in [0, T_0]$. Using (2.4) we have

$$\begin{aligned}\|\partial_x u_t\|_{L^\infty} &\leqslant \frac{1}{N} \sum_{i=1}^N \|\partial_x u_0\|_{L^\infty} \left(1 + \left\|\partial_x \ell^i_t\right\|_{L^\infty}\right) \\ &\leqslant \|\partial_x u_0\|_{L^\infty} + 2c_0 \int_0^t \|\partial_x u_s\|_{L^\infty}\end{aligned}$$

almost surely for $t \in [0, T_0]$. This proves (7.5) for $n = 1$.

For $n > 1$, local existence (Proposition 7.1) guarantees that $\|u_t\|_{C^{n-1}} \leqslant c_{n-1}$ almost surely on $[0, T_0]$, where $c_{n-1} = c_{n-1}(\|u_0\|_{C^{n-1}}, n)$. We subsequently allow $c_{n-1}$ to vary from line to line, provided it only depends on $n$ and $\|u_0\|_{C^{n-1}}$. Our bound on $\|u_t\|_{C^{n-1}}$ and equation (2.1) immediately implies that $\|\partial_x \lambda^i_t\|_{C^{n-2}} \leqslant c_{n-1}$ almost surely for $t \in [0, T_0]$. Equations (7.6) and (7.3) will imply $\|\partial_x \ell^i_t\|_{C^{n-2}} \leqslant c_{n-1}$ almost surely for $t \in [0, T_0]$.

Thus using equations (2.1) and (7.2) we obtain

$$\begin{aligned}\left\|\partial^n_x \lambda^i_t\right\|_{L^\infty} &\leqslant \int_0^t \left\|\partial^n_x \left[u_s \circ (I + \lambda^i_s)\right]\right\|_{L^\infty}\, ds \\ &\leqslant c_{n-1} \int_0^t \left[\|\partial^n_x u_s\|_{L^\infty} + \left\|\partial^n_x \lambda^i_s\right\|_{L^\infty}\right]\, ds\end{aligned}$$

almost surely. Using Gronwall's lemma this implies

$$\left\|\partial^n_x \lambda^i_t\right\|_{L^\infty} \leqslant c_{n-1} \int_0^t \|\partial^n_x u_s\|_{L^\infty}\, ds \tag{7.8}$$

almost surely. Here we absorbed the constant $e^{c_{n-1} t}$ into $c_{n-1}$, which is valid as $t \leqslant T_0 = T_0(\|u_0\|_{C^1})$. Now

$$\begin{aligned}\|\partial^n_x u_t\|_{L^\infty} &\leqslant \frac{1}{N} \sum_{i=1}^N \left(\|\partial^n_x u_0\|_{L^\infty} \left(1 + \left\|\partial_x \ell^i_t\right\|_{L^\infty}\right)^n + c_{n-1} \left\|\partial^n_x \ell^i_t\right\|_{L^\infty}\right) \\ &\leqslant \frac{1}{N} \sum_{i=1}^N \left(\|\partial^n_x u_0\|_{L^\infty} (1 + 2\left\|\partial_x \lambda^i_t\right\|_{L^\infty})^n + c_{n-1} \left\|\partial^n_x \lambda^i_t\right\|_{L^\infty}\right) \\ &\leqslant \|\partial^n_x u_0\|_{L^\infty} (1 + c_{n-1} t) + c_{n-1} \int_0^t \|\partial^n_x u_s\|_{L^\infty}\, ds\end{aligned}$$

where we used (7.3) and (7.7) to obtain the first inequality, and equations (7.6) and (7.8) to obtain the second. Now Gronwall's Lemma gives (7.5), where we again absorb the exponential factor $e^{c_{n-1}t}$ into $(1+c_{n-1}t)$, by replacing $c_{n-1}$ with a larger constant, which by our convention we still denote by $c_{n-1}$. □

*Proof of Lemma 2.5.* By Proposition 7.1, existence will follow if we establish (2.12) for $n = 1$. We prove (2.12) by induction. Since the constant $c_0$ in Lemma 7.3 is absolute, the proof for $n = 1$ is identical to the proof of the inductive step. Thus we only prove the inductive step.

Assume that (2.12) holds for $n-1$, choose $c_{n-1} = c_{n-1}(U_{n-1})$ to be the constant from Lemma 7.3. Thus whenever $\delta_t < T_0$,

$$\left\|\partial_x^n u^{\delta_t}_{(k+1)\delta_t}\right\|_{L^\infty} \leqslant (1 + c_{n-1}\delta_t) \left\|\partial_x^n u^{\delta_t}_{k\delta_t}\right\|_{L^\infty} \quad \text{a.s.} \tag{7.9}$$

holds for all $k \leqslant \frac{T_0}{\delta_t}$. Iterating this we have

$$\left\|\partial_x^n u^{\delta_t}_{t}\right\|_{L^\infty} \leqslant (1 + c_{n-1}\delta_t)^{T_0/\delta_t} \left\|\partial_x^n u_0\right\|_{L^\infty} \quad \text{a.s.}$$

for all $t \leqslant T_0$. Thus we choose $U_n$ to be given by

$$U_n = \left\|\partial_x^n u_0\right\|_{L^\infty} \sup_{\delta>0} (1 + c_{n-1}\delta)^{T_0/\delta}.$$

From (5.2) we see that $\int_x u^{\delta_t}_t$ is conserved almost surely. Since $u^{\delta_t}_t$ is periodic, a bound on $\|\partial_x^n u^{\delta_t}_t\|_{L^\infty}$ will give us a bound on $\|u^{\delta_t}_t\|_{C^n}$. □

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