A Dynamic Programming Approach for Pricing CDS and CDS Options.

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Abstract

We propose a general setting for pricing single-name knock-out credit derivatives. Examples include Credit Default Swaps (CDS), European and Bermudan CDS options. The default of the underlying reference entity is modeled within a doubly stochastic framework where the default intensity follows a CIR++ process. We estimate the model parameters through a combination of a cross sectional calibration-based method and a historical estimation approach. We propose a numerical procedure based on dynamic programming and a piecewise linear approximation to price American-style knock-out credit options. Our numerical investigation shows consistency, convergence and efficiency. We find that American-style CDS options can complete the credit derivatives market by allowing the investor to focus on spread movements rather than on the default event.

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1 Introduction

Credit derivatives are means of transferring credit risk (on a reference entity) between two parties by means of bilateral agreements. They can refer to a single credit instrument or a basket of instruments. In the last decade, credit derivatives have become increasingly popular. According to the International Swaps and Derivatives Association (ISDA), credit derivatives outstanding notionals grew 44 percent in the first half of 2005 to $12.4 trillion, up by more than 19 times from $631.5 billion at mid year 2001. Credit derivatives are traded over the counter, and many contracts are documented under ISDA swap documentation and the "1999 ISDA Credit Derivative Definitions," as amended by various supplements.

Credit Default Swaps (CDS) are the most important and widely used single-name credit derivatives. Under a CDS, the buyer of credit protection pays a periodic fee to an investor in return for protection against a potential credit event of a given firm known as the underlying *reference entity*. Credit events in practice are associated with credit-rating downgrading, firm restructuring, and default, among others. In this paper, the credit event refers only to the default of the reference entity. Recently, European options on CDS have been issued. They are also called *credit default swaptions*, a term borrowed from the interest rate derivatives market. CDS options give the investor the right, but not the obligation, to enter into a CDS contract at the option maturity. In general, a single-name default swaption is knocked out if the reference entity defaults during the life of the option. The knock-out feature marks the fundamental difference between a CDS option and a vanilla option. Following on the evolution of the interest rate derivatives market, we believe that there will be a need to trade CDS options with early exercise opportunities.

The aim of this paper is to price single name knock-out credit derivatives. In particular, we focus on European- and American-style CDS options. Credit derivatives, like many over-the-counter products, may suffer from a lack of liquidity, which often results in market mispricing. In this context, financial modeling and analysis is crucial in providing investors with rational asset prices, sensitivity measures, and optimal investment policies. Our work comes as a small contribution to this area.

Pricing derivatives can be considered as a Markov decision process and hence can be addressed as a stochastic *dynamic programming* (DP) problem. For a general overview on DP, we refer the reader to Bertsekas [1995]. Buttler and Waldvogel [1996] and Ben-Ameur et al. [2004] used DP to price default-free bonds with embedded options. We propose a numerical procedure based on a DP approach and piecewise linear approximations of value functions for pricing single-name knock-out derivatives. We adopt a reduced form approach where the default intensity, the *state variable*, is modeled through a CIR++ process (Brigo and Mercurio [2001] and Brigo and Alfonsi [2003]). In this context, the DP value function is the value of the credit derivative to be priced. Our numerical investigation shows consistency, robustness, and efficiency. For low-dimension cases, DP combined with piecewise linear approximations over-performs the least square Monte Carlo approach (Longstaff and Schwartz [2001], Tsitsiklis and Van-Roy [2001]); this is not surprising since the first only induces numerical errors while the second induces both numerical and statistical errors. For higher dimension problems, however, DP combined with Monte Carlo simulation is more convenient.

There are two main credit model families: structural models and reduced-form models. In the structural approach, the default time is the first instant where the firm value hits, from above, either a deterministic (Merton [1974] and Black and Cox [1976]) or a stochastic barrier (Giesecke [2001]). On the other hand, in reduced-form models, default time is modeled by means of an exogenous *doubly stochastic Poisson process* also known as a *Cox process* (See Section 3 for a formal definition). Here, unlike in the structural approach, default comes as a complete surprise. The reduced-form approach was adopted by a number of authors, including Jarrow et al. [1997], Lando [1998], Duffie and Singleton [1997], and Brigo and Alfonsi [2003].

The first attempts to price CDS options directly model the underlying credit spread. Schönbucher [2000] introduces a credit-risk model for credit derivatives, based on the "Libor market" framework for default-free interest rates. He provides formulas for CDS option prices under the so called survival measure (see also Schönbucher [2004]). In a similar setup, but with a different numéraire, Jamshidian [2004] provides CDS options prices, differently from Schoenbucher, under an equivalent measure. This approach is pursued further by Brigo [2005], who introduces a candidate market model for CDS options and callable defaultable floaters under an equivalent pricing measure. Hull and White [2003] use Black's formula (Black [1976]) to price CDS options and give numerical examples using data on quoted CDS spreads. A further results on CDS options is in Brigo [2005], where a variant of Jamshidian's decomposition for coupon bearing bond options or swaptions under affine short rate models is considered to derive a formula for CDS options under the CIR++ model.

The remainder of this paper is organized as follows. Section 2 charaterizes CDS and CDS options contracts. We set up the model in Section 3 and specify the dynamic programming procedure in Section 4. The model estimation step is addressed in Section 5. We provide a numerical investigation in Section 6. Finally, Section 7 concludes the paper.

2 CDS and CDS Options

2.1 CDS: Various Formulations

Under a typical default swap, the protection buyer pays the protection seller a regular and periodic premium, which is determined at the beginning of the transaction. If no default occurs during the life of the swap, these premium payments are the only cash flows. Following a default, the protection seller makes a payment to the protection buyer, which typically takes the form of a physical exchange between the two parties. The protection buyer provides the seller with a specific qualifying debt instrument, issued by the reference entity, in return for a cash payment corresponding to its full notional amount, i.e., par. The protection buyer stops paying the regular premium following the default. The *loss given default* to the protection seller is, therefore, par less the recovery value on the delivered bond. There are various CDS payoffs formulations resulting from different conventions and approximations.

Consider a CDS, incepted at time t_s, where the protection buyer pays the *premium rate* R

at times $T_{a+1}, \ldots, T_b$ in exchange for a single protection payment L_{GD} (loss given default) at the default time τ of the reference entity, provided that $T_a < \tau \le T_b$ (protection time window). In practice, the premium rate R is also known as the *CDS spread*. We assume that the recovery value and, hence, the protection payment are known at the inception of the contract. This is the prototype of the most diffused CDS contract, referred to as a *running CDS*. From the standpoint of the protection seller, the running CDS cash flows, discounted at time $t \in [t_s, T_{a+1})$, are

$$\begin{aligned} \Pi_{a,b}(R) &= D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau\ge T_i\}} \\ &\quad -\mathbf{1}_{\{T_a<\tau\le T_b\}} D(t,\tau)\text{L}_{\text{GD}}, \end{aligned} \tag{1}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function, $D(t,u)$ is the default-free discount factor over $[t,u]$, τ is the default time, $T_{\beta(\tau)}$ is the first date among the T_i's that follows τ, and α_i is the year fraction between T_{i-1} and T_i. The three elements of the right-hand side of (1) correspond respectively to the accrued premium term, the premium payments, and the protection payment given default. The total of discounted premium payments is known as the *premium leg*, whereas the discounted loss given default is known as the *protection leg*.

A slightly different CDS formulation is the *postponed running* CDS, under which, the protection payment L_{GD} is postponed to $T_{\beta(\tau)}$, instead to be paid at τ. The postponed running CDS cash flows, discounted at time $t \in [t_s, T_{a+1})$, are

$$\Pi_{a,b}(R) = \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau\ge T_i\}} - \sum_{i=a+1}^{b} \mathbf{1}_{\{T_{i-1}<\tau\le T_i\}} D(t,T_i)\text{L}_{\text{GD}}.$$

Recently, market participants have shown interest in *upfront* CDS characterized by a unique upfront premium payment Π_{UCDS}. The upfront CDS cash flows that we need to set to zero to get the upfront CDS premium, discounted at time t_s, are

$$\Pi_{a,b} = \Pi_{\text{UCDS}} - \sum_{i=a+1}^{b} \mathbf{1}_{\{T_{i-1}<\tau\le T_i\}} D(t_s,\tau)\text{L}_{\text{GD}}.$$

where Π_{UCDS} is the unique upfront premium.

Alternatively, the cash flows of a *postponed upfront* CDS, discounted at time t_s, are

$$\Pi_{a,b} = \Pi_{\text{PUCDS}} - \sum_{i=a+1}^{b} \mathbf{1}_{\{T_{i-1}<\tau\le T_i\}} D(t_s,T_i)\text{L}_{\text{GD}}, \tag{2}$$

where Π_{PUCDS} is the unique upfront premium.

2.2 CDS Options

A *European CDS option* is a claim that gives its holder the right, but not the obligation, to enter into a CDS with a specified spread at option maturity. The underlying CDS protection period, also known as the *CDS tenor*, starts at the option maturity. The option we consider here is knocked out if the reference entity defaults before the option maturity. The holder of a *payer* (or *receiver*) CDS option has the right to buy (or sell) a protection on a given CDS. This term convention is borrowed from the interest rate derivatives market. European CDS options are tradable over the counter, are quoted either with a knock-out or without knock-out feature, and generally have short maturities. Following on the evolution of the interest-rate derivatives market, we believe that a need will emerge for CDS options with early exercise features.

A *Bermudan CDS option* is an option on a CDS with a finite number of exercise opportunities. We indicate the latter dates by $t_0 = 0, \ldots, t_n = T$, where T is the option maturity. If the option is exercised at t_m, for $m = 0, \ldots, n$, the holder enters into a CDS with a protection period spanning $[t_m, T_b]$. That is, $t_s = T_a = t_m$. As an example, consider a Bermudan CDS option annually exercisable with a two-year maturity on a CDS with a protection maturity of $T_b = 10$ years. If the CDS option is exercised at $t_1 = 1$ year, the option holder enters into a CDS with a protection period of 9 years, whereas if the option is exercised at $t_2 = 2$ years, the underlying CDS has a protection period of 8 years. Note that a European CDS option is a particular Bermudan CDS option with $n = 1$.

3 Model Setup

3.1 The Default Intensity Process

Let $(\Omega, \mathcal{F}, \mathbb{F}, Q)$ be a filtered probability space endowed with the filtration $\mathbb{F} = \{\mathcal{F}_t,\, t \geq 0\}$, $\mathcal{F}_t \subset \mathcal{F}$, associated with a positive Markov process with left-limit right-continuous trajectories $\{\lambda_t,\, t \geq 0\}$. We consider the *Cox process* $\{N_t,\, t \geq 0\}$ associated with *intensity* $\{\lambda_t,\, t \geq 0\}$. Conditional on $\mathcal{F}_T$, the process $\{N_t,\, t \leq T\}$ verifies $N_0 = 0$, has independent increments, and the random variable $N_t - N_s$, for $0 \leq s \leq t \leq T$, has the Poisson distribution with parameter $\Lambda_t - \Lambda_s$, where $\Lambda_t - \Lambda_s = \int_s^t \lambda_u du$. We define the random variable $\tau = \inf\{t,\, N_t > 0\}$ (first jump-time of N) as the *default time* of a given reference entity. The intensity process $\{\lambda_t,\, t \geq 0\}$ is also known as the *hazard rate* of the default time τ.[1]

In practice, the hazard rate process $\{\lambda_t,\, t \geq 0\}$ is extracted from the quoted credit spreads. The filtration $\mathbb{F}$, hence, represents the information flow of quoted spreads in the market. Let $\mathbb{G} = \{\mathcal{G}_t,\, t \geq 0\}$ be the augmented filtration defined by $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau \leq u\}, u \leq t)$. Under the filtration $\mathbb{F}$, given $\mathcal{F}_t$ at the present time t, an investor cannot know whether default occurred before the present time, and if so when exactly. This information is instead contained in $\mathcal{G}_t$. The filtration $\mathbb{F}$ can be extended to include information from the market of default-free interest rates. In our framework, however, we assume a flat term structure of interest rates. The default-free discount factor over $[t, u]$ is therefore given by $D(t, u) = e^{-r(u-t)}$, where r is the risk-free rate.

[1] Although the terms "intensity" and "hazard rate" do not refer exactly to the same framework, since the former is linked to Poisson or Cox processes while the latter is more general, we refer to the two terms as equivalent in this paper. For a general discussion on hazard rates and intensities see for example Bielecki and Rutkowski [2002]

In this paper, we model the hazard rate with a CIR++ process (Brigo and Alfonsi [2003]). Under CIR++, the hazard rate λ_t is the sum of a positive deterministic function ψ_t and of a Markovian process y_t, that is,

$$\lambda_t = y_t + \psi_t,\ t \geq 0. \tag{3}$$

The process $\{y_t,\, t \geq 0\}$ follows the Cox et al. [1985] (CIR) dynamics:

$$dy_t = \kappa(\gamma - y_t)dt + \sigma\sqrt{y_t}dZ_t,$$

where $\{Z_t,\, t \geq 0\}$ is a standard Brownian motion under Q, and $\beta = (\kappa, \gamma, \sigma, y_0)$ is a vector of positive deterministic constants such that $2\kappa\gamma > \sigma^2$, to ensure that the origin is not accessible. Like CIR, CIR++ insures strictly positive and mean-reverting trajectories with the additional advantage of fully calibrating market data via the function ψ_t, for $t \geq 0$. Moreover, there are closed-form solutions for survival probabilities and zero-coupon bonds (Brigo and Alfonsi [2003]).

3.2 No-Arbitrage Pricing

We are interested in pricing European and Bermudan single-name credit derivatives with knock-out features. We focus now on pricing the European claims and deal with their Bermudan counterparts in the next subsection. We represent a European credit derivative by a future cash flow Y_u, $\mathcal{G}_u-$measurable, of the form $Y_u = \bar{Y}_u 1_{\{\tau>u\}}$, where $\bar{Y}_u$ is an $\mathcal{F}_u-$measurable random variable defining the "non-defaultable" (in that we omit the default indicator) part of Y_u.

We consider a no-arbitrage intensity-based setting, as defined by Bielecki and Rutkowski [2002]. Under the usual regular conditions, there exists a risk-neutral probability measure Q under which the price of Y_u at time t, for $u \geq t$, is

$$\begin{aligned} v_t(\lambda) &= E\left[D(t,u)\, Y_u \mid \mathcal{G}_t\right] \\ &= 1_{\{\tau>t\}} \frac{E\left[D(t,u)\, Y_u \mid \mathcal{F}_t\right]}{Q(\tau > t \mid \mathcal{F}_t)} \\ &= 1_{\{\tau>t\}} E\left[D(t,u)\, \bar{Y}_u e^{-(\Lambda_u - \Lambda_t)} \mid \lambda_t = \lambda\right], \end{aligned} \tag{4}$$

where $E[\cdot]$ is the expectation operator under Q. Equation (4) comes from the following development (incorporating iterated expectation):

$$\begin{aligned} v_t(\lambda) &= 1_{\{\tau>t\}} \frac{E\left[D(t,u)\, \bar{Y}_u 1_{\{\tau>u\}} \mid \mathcal{F}_t\right]}{Q(\tau > t \mid \mathcal{F}_t)} = 1_{\{\tau>t\}} \frac{E\left[E\left\{D(t,u)\, \bar{Y}_u 1_{\{\tau>u\}} \mid \mathcal{F}_u\right\} \mid \mathcal{F}_t\right]}{Q(\tau > t \mid \mathcal{F}_t)} \\ &= 1_{\{\tau>t\}} \frac{E\left[D(t,u)\, \bar{Y}_u Q(\tau > u \mid \mathcal{F}_u) \mid \mathcal{F}_t\right]}{Q(\tau > t \mid \mathcal{F}_t)} = 1_{\{\tau>t\}} E\left[D(t,u)\, \bar{Y}_u e^{-(\Lambda_u - \Lambda_t)} \mid \lambda_t = \lambda\right], \end{aligned}$$

where $Q(\tau > t \mid \mathcal{F}_t) = Q(N_t - N_0 = 0 \mid \mathcal{F}_t) = e^{-\Lambda_t}$. We provide, below, some useful results that are relevant to price single-name credit derivatives with knock-out.

Example 1 *From the perspective of an investor at time t, the survival probability up to time T is*

$$\begin{aligned} S(t,T,\lambda) &= Q\left(\tau > T \mid \mathcal{G}_t\right) \\ &= 1_{\{\tau>t\}} \frac{Q\left(\tau > T \mid \mathcal{F}_t\right)}{Q\left(\tau > t \mid \mathcal{F}_t\right)} \\ &= 1_{\{\tau>t\}} E\left[e^{-(\Lambda_T-\Lambda_t)} \mid \lambda_t = \lambda\right] =: 1_{\{\tau>t\}} \bar{S}(t,T,\lambda) \end{aligned} \tag{5}$$

Under CIR++, the survival probabilities are known in closed form (Brigo and Alfonsi [2003]).

Example 2 *The price at time t of a defaultable, no-recovery, zero-coupon bond with maturity T and notional amount of 1 dollar is*

$$P\left(t,T,\lambda\right) = 1_{\{\tau>t\}} E\left[D\left(t,T\right) e^{-(\Lambda_T-\Lambda_t)} \mid \lambda_t = \lambda\right] =: 1_{\{\tau>t\}} \bar{P}(t,T,\lambda),$$

known in closed form for affine hazard rates $\{\lambda_t,\ t \geq 0\}$ and CIR++ in particular (Duffie et al. [2000], Duffie et al. [2003] and Brigo and Alfonsi [2003]).

Example 3 *For a given premium rate K, the value of a running CDS at time $t < T_a$ is*

$$\begin{aligned} CDS(t,K,\lambda) &= E\left[\Pi_{a,b}\left(K\right) \mid \mathcal{G}_t\right] =: 1_{\{\tau>t\}} \overline{CDS}(t,K,\lambda) = \\ &= 1_{\{\tau>t\}} \Bigg[R \int_{T_a}^{T_b} P(t,u)(T_{\beta(u)-1} - u) d_u \bar{S}(t,u,\lambda) + \\ &\quad + R \sum_{i=a+1}^{n} P(t,T_i)\alpha_i \bar{S}(t,T_i,\lambda) + L_{GD} \int_{T_a}^{T_b} P(t,u) d_u \bar{S}(t,u,\lambda) \Bigg], \end{aligned} \tag{6}$$

where the above integrals are computed with numerical integration. The premium rate R of a CDS is best computed by solving the following equation:

$$\overline{CDS}(t_s,R,\lambda) = 0 \quad (\Rightarrow CDS(t_s,R,\lambda) = 0). \tag{7}$$

As an example the premium rate of a postponed running CDS is

$$R = R_{a,b}(t) = \frac{\sum_{i=a+1}^{b} \alpha_i E\left[D(t_s,T_i) e^{-\left(\Lambda_{T_i}-\Lambda_{t_s}\right)} \mid \lambda_{t_s} = \lambda\right]}{\sum_{i=a+1}^{b} E\left[D(t_s,T_i)\left(e^{-\left(\Lambda_{T_{i-1}}-\Lambda_{t_s}\right)} - e^{-\left(\Lambda_{T_i}-\Lambda_{t_s}\right)}\right) \mid \lambda_{t_s} = \lambda\right]}.$$

In the same line, the premium payment Π_{UCDS} of a postponed upfront CDS is obtained in a closed form:

$$\Pi_{PUCDS} = 1_{\{\tau>t_s\}} L_{GD} \sum_{i=a+1}^{b} D\left(t_s,T_i\right) E\left[e^{-\left(\Lambda_{T_{i-1}}-\Lambda_{t_s}\right)} - e^{-\left(\Lambda_{T_i}-\Lambda_{t_s}\right)} \mid \lambda_{t_s} = \lambda\right] \tag{8}$$

For more details on a suitable definition of CDS forward rates in general see Brigo [2005].

3.3 Pricing Bermudan CDS Options

The main concern in pricing Bermudan CDS options is to identify the optimal strategy to enter into the underlying CDS. We assume, without loss of generality, that the decision dates $t_0, \ldots, t_n$ are a subset of the CDS payment schedule $T_a, \ldots, T_b$, where the option maturity t_n is strictly less than the CDS maturity T_b. We define the following entities:

- The option strike as the strike CDS premium rate (or strike CDS spread) K;
- The value, exercise value, and holding value of the CDS option at time t_m, for $m = 0, \ldots, n$, respectively as $v_m(\lambda)$, $v_m^e(\lambda)$, and $v_m^h(\lambda)$, where $\lambda = \lambda_{t_m}$ is the current hazard rate;
- The "non-defaultable" counterpart of the value, exercise value, and holding value of the CDS option at time t_m, for $m = 0, \ldots, n$, respectively as $\bar{v}_m(\lambda)$, $\bar{v}_m^e(\lambda)$, and $\bar{v}_m^h(\lambda)$, where $\lambda = \lambda_{t_m}$ is the current hazard rate.

The value functions $\bar{v}_m(\cdot)$, $\bar{v}_m^e(\cdot)$, and $\bar{v}_m^h(\cdot)$ verify the properties $v_m(\cdot) = \bar{v}_m(\cdot) 1_{\{\tau > t_m\}}$, $v_m^e(\cdot) = \bar{v}_m^e(\cdot) 1_{\{\tau > t_m\}}$, and $v_m^h(\cdot) = \bar{v}_m^h(\cdot) 1_{\{\tau > t_m\}}$, where $\bar{v}_m(\cdot)$, $\bar{v}_m^e(\cdot)$, and $\bar{v}_m^h(\cdot)$ are $\mathcal{F}_{t_m}-$measurable random variables.

Proposition 4 *Consider a payer CDS Bermudan option with strike K, exercise dates $t_0 = 0, \ldots, t_n < T_b$, and CDS option payment schedules nested in $T_a, \ldots, T_b$. The cash flows of the underlying CDS discounted at time t_m, for $m = 0, \ldots, n$, are indicated by $\Pi_{m,b}(K)$. The value function of the option at maturity is*

$$\begin{aligned} v_n(\lambda) &= 1_{\{\tau > t_n\}} \bar{v}_n(\lambda) \\ &= 1_{\{\tau > t_n\}} \bar{v}_n^e(\lambda), \end{aligned} \tag{9}$$

where

$$\bar{v}_n^e(\lambda) = \left(-E\left[\Pi_{n,b}(K) \mid \mathcal{F}_{t_n}\right]\right)^+ . \tag{10}$$

For $m = 0, ..., n-1$, the option value is

$$\begin{aligned} v_m(\lambda) &= 1_{\{\tau > t_m\}} \bar{v}_m(\lambda) \\ &= 1_{\{\tau > t_m\}} \max\left\{\bar{v}_m^e(\lambda), \bar{v}_m^h(\lambda)\right\}, \end{aligned} \tag{11}$$

where

$$\bar{v}_m^e(\lambda) = \left(-E\left[\Pi_{m,b}(K) \mid \mathcal{F}_{t_n}\right]\right)^+ , \tag{12}$$

and

$$\bar{v}_m^h(\lambda) = E\left[D_m e^{-\left(\Lambda_{t_{m+1}} - \Lambda_{t_m}\right)} \bar{v}_{m+1}\left(\lambda_{t_{m+1}}\right) \mid \lambda_{t_m} = \lambda\right],$$

with $D_m = D(t_m, t_{m+1})$.

Proof. We provide a proof by induction. At the option maturity t_n, the value function is

$$v_n(\lambda) = 1_{\{\tau > t_n\}} \max\left\{\bar{v}_n^e(\lambda), \bar{v}_n^h(\lambda)\right\}$$

with the convention that $\bar{v}_n^h(\cdot) = 0$. This results in

$$\begin{aligned} v_n(\lambda) &= 1_{\{\tau > t_n\}} \bar{v}_n^e(\lambda) \\ &= 1_{\{\tau > t_n\}} \left(E\left[\Pi_{n,b}(R) \mid \mathcal{F}_{t_n}\right] - E\left[\Pi_{n,b}(K) \mid \mathcal{F}_{t_n}\right]\right)^+, \end{aligned}$$

where R is the fair CDS premium rate prevailing at time t_n. By equation (7), or by definition of CDS premium rate, having the CDS net present value vanishing for that premium rate, we have

$$v_n(\lambda) = 1_{\{\tau > t_n\}} \left(-E\left[\Pi_{n,b}(K) \mid \mathcal{F}_{t_n}\right]\right)^+.$$

Suppose that the value function $v_{m+1}(\cdot)$ is known. From the perspective of an investor at time t_m, the value function $v_{m+1}(\cdot) = 1_{\{\tau > t_{m+1}\}} \bar{v}_{m+1}(\cdot)$ can be interpreted as a European knock-out option. By section (3.2) the holding value is

$$\begin{aligned} v_m^h(\lambda) &= E\left[D_m v_{m+1}\left(\lambda_{t_{m+1}}\right) \mid \mathcal{G}_{t_m}\right] \\ &= E\left[D_m 1_{\{\tau > t_{m+1}\}} \bar{v}_{m+1}\left(\lambda_{t_{m+1}}\right) \mid \mathcal{G}_{t_m}\right] \\ &= 1_{\{\tau > t_m\}} E\left[D_m e^{-\left(\Lambda_{t_{m+1}} - \Lambda_{t_m}\right)} \bar{v}_{m+1}\left(\lambda_{t_{m+1}}\right) \mid \lambda_{t_m} = \lambda\right]. \end{aligned}$$

On the other hand, the exercise value is

$$v_m^e(\lambda) = 1_{\{\tau > t_m\}} \left(-E\left[\Pi_{m,b}(K) \mid \mathcal{F}_{t_m}\right]\right)^+. \tag{13}$$

The optimal exercise strategy is the following: exercise at time t_m and state λ if, and only if, $v_m^e(\lambda) > v_m^h(\lambda)$, otherwise hold the option up to t_{m+1}. The value function at time t_m is therefore

$$v_m(\lambda) = 1_{\{\tau > t_m\}} \max\left(\bar{v}_m^e(\lambda), \bar{v}_m^h(\lambda)\right).$$

■

4 Dynamic Programming Approach

American-style derivatives cannot, in general, be priced in closed form. We propose a numerical procedure based on dynamic programming (DP) and piecewise linear approximations of value functions to price Bermudan CDS options. In our context, the DP value function is the value of the credit derivative to be priced and, at each decision date, the state variable is the hazard rate.

Let $a_0 = 0 < a_1 < ... < a_{p+1} = +\infty$ be a set of points to which we associate a sequence of intervals $R_i = [a_i, a_{i+1})$, for $i = 0, ..., p$. Given an approximation $\tilde{v}_m(\cdot)$ of the value function $\bar{v}_m(\cdot)$ at time t_m, fully determined at each point of the grid, we introduce a piecewise linear interpolation $\hat{v}_m(\cdot)$ of $\tilde{v}_m(\cdot)$ that extends $\tilde{v}_m(\cdot)$ everywhere:

$$\hat{v}_m(\lambda) = \sum_{i=0}^{p} \left(\alpha_i^m + \beta_i^m \lambda\right) 1_{\{\lambda \in R_i\}}. \tag{14}$$

The coefficients α_i^m and β_i^m are obtained by solving

$$\hat{v}_m(a_i) = \tilde{v}_m(a_i), \text{ for } i = 1, ..., p-1,$$

which implies the continuity of $\hat{v}_m(\cdot)$. Therefore, we obtain

$$\beta_i^m = \frac{\tilde{v}_m(a_{i+1}) - \tilde{v}_m(a_i)}{a_{i+1} - a_i} \text{ and } \alpha_i^m = \frac{a_{i+1}\tilde{v}_m(a_i) - a_i\tilde{v}_m(a_{i+1})}{a_{i+1} - a_i}.$$

We add the following restrictions $\alpha_0^m = \alpha_1^m$, $\beta_0^m = \beta_1^m$, $\alpha_p^m = \alpha_{p-1}^m$, and $\beta_p^m = \beta_{p-1}^m$. Assume now that $\hat{v}_{m+1}(\cdot)$ is known. An approximation of the holding value at time t_m is

$$\begin{aligned} \tilde{v}_m^h(a_k) &= E\left[D_m e^{-\left(\Lambda_{t_{m+1}} - \Lambda_{t_m}\right)} \hat{v}_{m+1}\left(\lambda_{t_{m+1}}\right) \mid \lambda_{t_m} = a_k\right] \\ &= D_m \sum_{i=0}^{p} \left(\alpha_i^{m+1} A_{ki}^m + \beta_i^{m+1} B_{ki}^m\right), \end{aligned} \tag{15}$$

where

$$\begin{aligned} A_{ki}^m &= E\left[e^{-\left(\Lambda_{t_{m+1}} - \Lambda_{t_m}\right)} 1_{\left\{\lambda_{t_{m+1}} \in R_i\right\}} \mid \lambda_{t_m} = a_k\right] \\ B_{ki}^m &= E\left[e^{-\left(\Lambda_{t_{m+1}} - \Lambda_{t_m}\right)} \lambda_{t_{m+1}} 1_{\left\{\lambda_{t_{m+1}} \in R_i\right\}} \mid \lambda_{t_m} = a_k\right], \end{aligned}$$

for $m = 0, ..., n-1$. Here, A and B could be interpreted as transition matrices. For example, A_{ki}^m represents the probability that the hazard rate migrates from the state a_k at time t_m to the interval R_i at time t_{m+1}. Closed-form solutions for A and B under CIR are given in Ben-Ameur et al. [2004]. The following proposition gives an extension under CIR++.

Proposition 5 *The transition coefficients A_{ki}^m are*

$$A_{ki}^m = S_m(a_k)\left(\sum_{n=0}^{\infty} e^{-\frac{(\delta_k/2)^n}{n!}} \frac{(\delta_k/2)^n}{n!}\left(F_{d+2n}\left(\frac{a_{i+1}^m}{\eta}\right) - F_{d+2n}\left(\frac{a_i^m}{\eta}\right)\right)\right)$$

where $a_i^m = a_i - \psi_{t_{m+1}}$, $S_m(a_k)$ is the survival probability over $[t_m, t_{m+1}]$ when the initial state λ_{t_m} is a_k, given by equation (5), and

$$\eta = \frac{\sigma^2\left(e^{h\Delta_m} - 1\right)}{\left(2(h+\kappa)\left(e^{h\Delta_m} - 1\right) + 2h\right)}, \quad \delta_k = \frac{8h^2 e^{h\Delta_m} a_k}{\sigma^2\left((h+\kappa)\left(e^{h\Delta_m} - 1\right) + 2h\right)},$$

with $\Delta_m = t_{m+1} - t_m$. The transition coefficients B_{ki}^m are

$$\begin{aligned} B_{ki}^m &= S_m(a_k)\,\eta\left(\sum_{n=0}^{\infty} e^{-\frac{(\delta_k/2)^n}{n!}} \frac{(\delta_k/2)^n}{n!} Q_n\left(a_i^m, a_{i+1}^m\right)\right) + \\ &\quad S_m(a_k)\,\psi_{t_{m+1}}\left(\sum_{n=0}^{\infty} e^{-\frac{(\delta_k/2)^n}{n!}} \frac{(\delta_k/2)^n}{n!}\left(F_{d+2n}\left(\frac{a_{i+1}^m}{\eta}\right) - F_{d+2n}\left(\frac{a_i^m}{\eta}\right)\right)\right), \end{aligned}$$

where

$$Q_n\left(a_i^m, a_{i+1}^m\right) = -2\left(a_{i+1}^m f_{d+2n}\left(\frac{a_{i+1}^m}{\eta}\right) - a_i f_{d+2n}\left(\frac{a_i^m}{\eta}\right)\right) + (d+2n)\left(F_{d+2n}\left(\frac{a_{i+1}^m}{\eta}\right) - F_{d+2n}\left(\frac{a_i^m}{\eta}\right)\right),$$

with $h = \sqrt{\kappa^2 + 2\sigma^2}$ *and* $d = \frac{4\kappa}{\gamma}$. *The functions* $F_{d+2n}(\cdot)$ *and* $f_{d+2n}(\cdot)$*are respectively the cumulative distribution and density of a chi-2 with* $d+2n$ *degrees of freedom.*

At this point, we specify the step-by-step procedure to be implemented:

1. Set $m = n$;
2. Compute $\tilde{v}_n(a_k) = v_n^e(a_k)$, for $k = 1, \ldots, p$, by (6) substituted in (13);
3. Compute $\hat{v}_n(\lambda)$, for $\lambda > 0$, by (14);
4. Set $m = m - 1$;
5. Compute $v_m^e(a_k)$, for $k = 1, \ldots, p$, by (6) substituted in (13);
6. Compute $\tilde{v}_m^h(a_k)$, for $k = 1, \ldots, p$, by (15);
7. Compute $\tilde{v}_m(a_k) = \max\left(v_m^e(a_k), \tilde{v}_m^h(a_k)\right)$, for $k = 1, \ldots, p$;
8. Identify the optimal decision at time t_m;
9. Compute $\hat{v}_m(\lambda)$, for $\lambda > 0$, by (14);
10. If $m = 0$ stop, else go to step 4.

At steps 2 and 5, numerical integration is required to compute the expected value of the discounted accrued term for a running CDS.

5 Model Estimation

The model estimation step alternates between the implicit method based on quoted spreads and the historical approach based on maximum likelihood. We use the implicit calibration to extract the time series of the "unobservable" hazard rates from observable CDS premium rates. Then we optimize the likelihood function over the set of admissible model parameters in correspondence of the "unobservable" hazard rate series we estimated from CDS quotes. The two steps are repeated until convergence.

The implicit method is based on matching the theoretical CDS premium rate with the market spread R^M, which is done implicitly by solving in λ the equation:

$$\overline{\text{CDS}}\left(t_s, R^M, \lambda\right) = 0. \tag{16}$$

From equation (6), finding $\lambda = \lambda_{t_s}$ amounts to computing survival probabilities. Under CIR++, the survival probability is obtained by means of (Brigo and Alfonsi [2003])

$$\begin{aligned} \bar{S}(t,T,\lambda) &= E\left[e^{-(\Lambda_T-\Lambda_u)}\right] \\ &= e^{-(\Psi_T-\Psi_t)}E\left[e^{-(Y_T-Y_t)}\right] \\ &= e^{-(\Psi_T-\Psi_t)}P^{\text{CIR}}(t,T,y), \end{aligned} \tag{17}$$

where $\Psi_t = \int_0^t \psi(s)ds$, $Y_t = \int_0^t y_s ds$, and $P^{\text{CIR}}(t,T,y)$ is the price at time t of a zero-coupon bond with maturity T under CIR. In this paper, we choose $\psi(\cdot)$ to be piecewise constant over a number J of intervals, i.e., $\psi_t = \sum_{j=0}^{J-1} \psi_{t_j} 1_{\{t_j \le t < t_{j+1}\}}$ with the convention that $t_J = \infty$.

To combine the implicit and the historical approaches, we need to specify the relationship between the dynamics of the intensity under the physical probability measure P and the risk-neutral probability measure Q. Define the *risk premium*[2] $\eta\sqrt{y_t}$, where η is a positive real parameter. Under the probability measure P, the process $\{y_t,\, t \ge 0\}$ is given by

$$dy_t = \bar{\kappa}(\bar{\gamma} - y_t)dt + \sigma\sqrt{y_t}dZ_t^P,$$

where $\bar{\kappa} = \kappa - \sigma\eta$ and $\bar{\gamma} = \kappa\gamma/(\kappa - \sigma\eta)$. Let $\theta = (\beta, \eta)$ be the vector of parameters to estimate, where $\beta = (\kappa, \gamma, \sigma, y_0)$ is the risk-neutral set of parameters.

Assume at this step that the time series $\overrightarrow{y}_T = (y_0, ..., y_T)$ is known, where T is the length of the time series. The likelihood function is

$$L_T(\theta) = \Pi_{t=0}^{T-1} f(y_{t+1} \mid y_t; \theta) f(y_0),$$

where

$$f(y_{t+1} \mid y_t; \theta) = ce^{(-u-v)}\left(\frac{v}{u}\right)^{\frac{q}{2}} I_q\left(2\sqrt{uv}\right)$$

$c = 2\bar{\kappa}/\left(\sigma^2\left(1 - \exp\left(-\bar{\kappa}\right)\right)\right)$, $u = cy_t e^{-\bar{\kappa}}$, $v = cy_{t+1}$, $q = 2\bar{\kappa}\bar{\gamma}/\sigma^2 - 1$ and $I_q(.)$ is the modified Bessel function of the first kind of order q. The density function at time t_0 is

$$f(y_0) = \frac{1}{b^a\Gamma(a)} y_0^{a-1} e^{\frac{y_0}{b}},$$

where $a = 2\bar{\kappa}\bar{\gamma}/\sigma^2$, $b = \sigma^2/(2\bar{\kappa})$, and $\Gamma(\cdot)$ is the gamma function. The maximum likelihood estimator of θ is

$$\hat{\theta} = \arg\max_{\theta} L_T(\theta). \tag{18}$$

The estimation procedure runs as follows:

1. Set $\psi(t) = 0$ for all t, that is, $\{\lambda_t = y_t,\, t \ge 0\}$ is a CIR process;
2. Set the parameter vector θ at an initial value θ^0 and define $\hat{\theta}^0 = \theta^0$;
3. Set $m = 0$;

[2] This is the standard approach in specifying the risk premium, we refer the reader to Pan and Singleton [2005] for a more general and detailed treatment of the subject

4. Extract the intensity time series λ_t^m, for $t = 0, \ldots, T$, by solving (16), where the survival probabilities in the CDS price to be set to zero come from the CIR hazard-rate model with parameter $\hat{\theta}^m$;

5. Compute the maximum likelihood estimator $\hat{\theta}^{m+1}$ by (18);

6. If $\left|\hat{\theta}^{m+1} - \hat{\theta}^m\right| < \varepsilon$, for a given strictly positive ε, stop and proceed to the next point; else set $m = m + 1$, $\hat{\theta}^m = \hat{\theta}^{m+1}$, and go to step 4;

7. At a given time $u \geq T$, determine $\psi(\cdot)$ by equating (6) to zero with $S(u, T_b, \lambda)$, for selected CDS maturities T_b, given by (17) with CIR parameters $\hat{\theta}^m$.

In practice $u = T$ is the current date and the CDS protection periods $T_b - u$ refer to the most liquid CDS contracts.

6 Numerical Investigation

Our numerical investigation focuses on the reference entity Ford Motor Credit Corporation (from now on "Ford Credit"). The estimation step is based on five-year CDS contracts on Ford Credit provided by Deutsche Bank Securities Inc. The spread is paid quarterly and quoted in basis points. We choose a five-year maturity CDS since it is the most liquid contract. The data sample covers the period from November 29, 2002, to February 18, 2005, with 562 daily observations. To reduce noise from the daily observations, we construct the associated weekly data series by picking spreads on each Friday (or on a Thursday if there is no data for a particular Friday). Figure (1) shows the evolution of the quoted spread R^M over the sample period. We choose a piecewise constant function $\psi(\cdot)$ with the key maturities being $1, 3, 5, 7$, and 10 years. Tables (1) and (3) show the data used for March 10, 2004, and September 8, 2004, respectively, as given by the credit desk of Banca IMI. Tables (2) and (4) provide the piecewise constant hazard rates and their associated survival probabilities. Tables (5) and (6) show the estimates of the CIR model parameters and the pieces of $\psi(\cdot)$ on September 8, 2004, for the key maturities.

Table 1: CDS Ford Credit quotes R^M in basis points from 03/10/04

date	**Mid**	**Bid**	**Ask**
21-mar-05	110.5	110	111
20-mar-07	116	115.9	116.3
20-mar-09	204.5	204	205
21-mar-11	212	207	217
20-mar-14	226	221	231

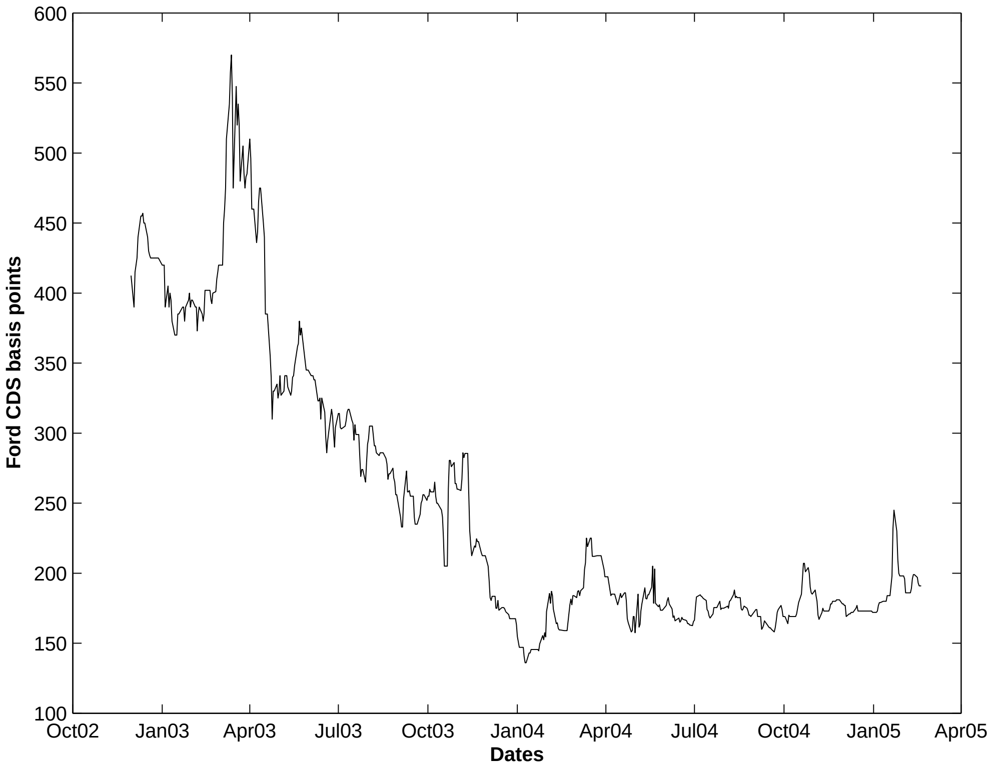

Figure 1: 5-year maturity FORD Credit CDS quotes

Table 2: Calibration of the Intensity Model to FORD Credit CDS quotes from 03/10/04

Default Intensity Curve		
date	**intensity (%)**	**survival pr (%)**
21-mar-05	1.837	98.100
20-mar-07	3.141	92.055
20-mar-09	4.694	83.685
21-mar-11	3.929	77.268
20-mar-14	4.593	67.194

Table 3: CDS Ford Credit quotes R^M in basis points from 09/08/04

date	**Mid**	**Bid**	**Ask**
20-Sep-05	37.74	28.8	46.67
20-Sep-07	132.5	95.2	169.7
21-Sep-09	175.1	147.6	202.6
20-Sep-11	175.1	147.6	202.6
22-Sep-14	175.1	147.6	202.6

Table 4: Calibration of the Intensity Model to FORD Credit CDS quotes from 09/08/04

Default Intensity Curve		
date	**intensity (%)**	**survival pr (%)**
20-Sep-05	0.627	99.346
20-Sep-07	2.340	94.742
21-Sep-09	4.460	86.527
20-Sep-11	2.697	81.929
22-Sep-14	2.695	75.463

Table 5: CIR parameters estimation using MLE

κ	θ	σ	y_0	η
.44178	0.0348468	0.23264	0.015	-0.0002
(0.070078)	(0.0092541)	(0.04282)	(0.04245)	(0.000001)

Table 6: Estimation of ψ on September 08,2004

Maturity	**1**	**3**	**5**	**7**	**10**
ψ	0.00007	0.00429	0.00334	0.00208	0.00351

A postponed upfront CDS can be priced in a closed form, as shown in equation (8). Table (7) shows the convergence of the DP price of the upfront postponed CDS to its closed-form limit. We can see that even with a low number of grid points ($p = 500$), our algorithm is accurate to the fourth digit. Accuracy can be enhanced with an increase in the number of grid points.

Table 7: Price of 5 years maturity upfront postponed CDS with yearly payments, $\mathrm{R_{EC}} = 40\%$.

λ_0	**p =100**	**p=500**	**p=1000**	**p=2000**	**Closed Formula**
0.01	0.062161	0.063246	0.063290	0.063301	0.063305
0.02	0.071711	0.072854	0.072898	0.072909	0.072909
0.03	0.081261	0.082293	0.082331	0.082341	0.082345
0.04	0.090352	0.091546	0.091589	0.091600	0.091604
0.05	0.099429	0.100637	0.100679	0.100690	0.100694
0.06	0.108507	0.109568	0.109605	0.109614	0.109618
CPU (sec)	0.05	2.11	8.03	47.86	

Table (8) gives the price of a one-year-maturity European and Bermudan options on a CDS with a protection maturity $T_b = 6$ years as a function of the option strike. The Bermudan option can be exercised quarterly. For example, if the option is exercised in 3 months, the holder enters a CDS of 5 years and 9 months. From Tables (8) and (9) we can see that the difference in price between European and Bermudan options increases with the option maturity. Table (10) shows how the frequency of the exercise opportunity impacts the price of the Bermudan option. As expected, the price of the option increases with the exercise frequency.

Table 8: Price of CDS Bermudan and European options for different strikes. Option maturity = 1 year, protection maturity = 6 years, exercise quarterly, default recovery $\mathrm{R_{EC}} = 0.4$, λ_0 = 600 bps, risk free rate $r = 2\%$

Strike (bps)	**Bermudan Option (bps)**	**European Option (bps)**
700	1.02	0.94
600	3.21	2.91
500	10.11	8.93
400	31.63	26.98
300	97.27	78.88

Table (11) sheds more light on how maturity affects the pricing of European and Bermudan CDS options. For Bermudan options, the price rises with an increase in the maturity. This obvious characteristic is due to the fact that shorter maturity Bermudan options are nested in longer maturity options; hence, there is always a gain in having a longer option. This feature is not guaranteed for European options. There are actually two effects: the *default risk effect* and the *economic cycle effect*. Since European options are knocked out at default, the longer the option maturity, the riskier the option, and the less attractive it is. The economic cycle effect refers to the different cycles affecting the solvency of the reference entity. For example, for a financially stressed company, short-maturity European options are more attractive,

Table 9: Price of CDS Bermudan European options for different strikes. Option maturity = 5 year, protection maturity = 6 years, exercise quarterly, default recovery $\mathrm{R}_{\mathrm{EC}} = 0.4$, $\lambda_0 =$ 600 bps, risk free rate $r = 2\%$

Strike (bps)	**Bermudan Option (bps)**	**European Option (bps)**
700	23.71	6.57
600	38.33	9.77
500	63.34	14.61
400	107.72	22.01
300	190.74	33.5

Table 10: Price of CDS Bermudan options for different exercise opportunities. Option maturity = 5 year, protection maturity = 6 years, exercise quarterly, default recovery $\mathrm{R}_{\mathrm{EC}} =$ 0.4, $\lambda_0 = 600$ bps, risk free rate $r = 2\%$

Strike (bps)	**Exercise quarterly**	**Exercise annually**
700	23.71	17.75
600	38.33	29.33
500	63.34	49.73
400	107.72	86.95
300	190.74	157.32

whereas, for a healthier company, long-maturity options are more attractive. The choice of the CIR++ process in this context is relevant. The economic cycle effect may either enhance or offset the default risk effect. In Table (11), with $\lambda_0 = 800$ bps while $\gamma = 348$ bps, the default intensity is expected to decrease. Thus, the cycle effect strengthens the default risk effect. In Table (12), with $\lambda_0 = 200$ bps while $\gamma = 348$ bps, the default intensity is expected to increase. The cycle effect offsets the default risk effect.

The most traded European CDS options have either three- or six- month maturities. We think that the lack of liquidity in longer maturity European options is due to the default risk effect. CDS Bermudan options may offer an interesting alternative for longer maturities, the default risk effect being less relevant. CDS Bermudan options can then complete the credit derivatives market by allowing investors to focus on spread movements without worrying about default risk.

Table 11: Price of CDS Bermudan options for different maturities opportunities. Option maturity = protection maturity - 1 year, exercise quarterly , default recovery $\mathrm{R}_{\mathrm{EC}} = 0.4$, strike = 500 bps, $\lambda_0 = 800$ bps, risk free rate $r = 2\%$

Protection Maturity (years)	**Bermudan Option (bps)**	**European Option (bps)**
2	58.31	39.10
3	78.02	33.14
4	81.40	25.23
5	84.58	19.54
6	84.65	15.74

Table 12: Price of CDS Bermudan options for different maturities opportunities. Option maturity = protection maturity - 1 year, exercise quarterly , default recovery $R_{EC} = 0.4$, strike = 500 bps, $\lambda_0 = 200$ bps, risk free rate $r = 2\%$

Protection Maturity (years)	**Bermudan Option (bps)**	**European Option (bps)**
2	4.74	4.16
3	14.58	9.23
4	23.63	11.4
5	31.02	12.11
6	36.67	12.15

7 Conclusion

In this paper, we use a DP approach for pricing CDS and CDS options. Our main contribution is twofold. First, we address the pricing of single-name knock-out credit derivatives as a DP problem. Second, we propose a numerical procedure to efficiently solve the Bellman equation. The numerical investigation yields the following results. For Bermudan credit derivatives, the value of the early exercise opportunity increases with the option maturity and the exercise frequency. The impact of maturity on European CDS options is less obvious. Depending on the strength of the default risk effect relative to the economic cycle effect, the option price might increase or decrease with the option maturity.
In the examples presented in Brigo and Cousot [2006] it is shown that with the CIR++ model the large ranges of possible CDS volatilities never exceed levels about 30%. Since Brigo [2005] shows that market implied volatilities may easily exceed 50%, we may need to include jumps in the core CIR process for λ in order to attain high enough levels of implied CDS volatilities, according to the suggestion in El-Bachir [2005]. The dynamic programming approach presented here can be extended to a more general jump-diffusion affine setting for stochastic intensities where we can incorporate stochastic interest rates as well. Finally, we can extend our work to price other class for credit derivatives such as defaultable bonds with embedded options. We leave these extensions for future research.

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8 Appendix

Proof of proposition 5. Under the forward measure $\tilde{\mathbb{Q}}$, $(y_{t_{m+1}})/\eta$ has a non-central chi-square distribution with non-centrality parameter δ and with d degrees of freedom where

$$\delta = \frac{8h^2 e^{h(\Delta_m)}\lambda}{\sigma^2\left((h+\kappa)\left(e^{h(\Delta_m)}-1\right)+2h\right)}$$

and where h, d, η and Δ are defined as in the theorem's statement.

Therefore

$$\begin{aligned}
A_{ki}^m &= E\left[e^{-\left(\Lambda_{t_{m+1}}-\Lambda_{t_m}\right)}1_{\left\{\lambda_{t_{m+1}}\in R_i\right\}}\,|\lambda_{t_m}=a_k\right] \\
&= e^{-\left(\Psi_{t_{m+1}}-\Psi_{t_m}\right)}E_{m,a_k}\left[e^{-\left(Y_{t_{m+1}}-Y_{t_m}\right)}1_{\left\{a_i-\psi_{t_{m+1}}\le y_{t_{m+1}}\le a_{i+1}-\psi_{t_{m+1}}\right\}}\right] \\
&= S_m\left(a_k\right)P^{\tilde{Q}}\left[a_i-\psi_{t_{m+1}}\le y_{t_{m+1}}\le a_{i+1}-\psi_{t_{m+1}}\right] \\
&= S_m\left(a_k\right)\sum_{n=0}^{\infty}e^{-\frac{(\delta_k/2)^n}{n!}}\frac{(\delta_k/2)^n}{n!}\left(F_{d+2n}\left(\frac{a_{i+1}^m}{\eta}\right)-F_{d+2n}\left(\frac{a_i^m}{\eta}\right)\right),
\end{aligned}$$

and

$$
\begin{aligned}
B_{ki}^{m} &= E_{m,a_k}\left[e^{-\left(\Lambda_{t_{m+1}}-\Lambda_{t_m}\right)}\lambda_{t_{m+1}}1_{\left\{\lambda_{t_{m+1}}\in R_i\right\}}\right] \\
&= S_m\left(a_k\right)\eta E_{m,a_k}^{\tilde{Q}}\left[\frac{\lambda_{t_{m+1}}}{\eta}1_{\left\{\lambda_{t_{m+1}}\in R_i\right\}}\right] \\
&= S_m\left(a_k\right)\eta E_{m,a_k}^{\tilde{Q}}\left[\frac{y_{t_{m+1}}}{\eta}1_{\left\{\frac{a_i^m}{\eta}\leq\frac{y_{t_{m+1}}}{\eta}\leq\frac{a_{i+1}^m}{\eta}\right\}}\right]+S_m\left(a_k\right)\psi_{t_{m+1}}E_{m,a_k}^{\tilde{Q}}\left[1_{\left\{\frac{a_i^m}{\eta}\leq\frac{y_{t_{m+1}}}{\eta}\leq\frac{a_{i+1}^m}{\eta}\right\}}\right]
\end{aligned}
$$

Notice that the properties of the non-central $\mathcal{X}^2$ distribution imply

$$
\begin{aligned}
E_{m,a_k}^{\tilde{Q}}\left[\frac{y_{t_{m+1}}}{\eta}1_{\left\{\frac{a_i^m}{\eta}\leq\frac{y_{t_{m+1}}}{\eta}\leq\frac{a_{i+1}^m}{\eta}\right\}}\right] &= \sum_{n=0}^{\infty}e^{-\frac{(\lambda_k/2)^n}{n!}}\frac{(\lambda_k/2)^n}{n!}\int_{\frac{a_i^m}{\eta}}^{\frac{a_{i+1}^m}{\eta}}yf_{d+2t}dy \\
&= Q_n\left(a_i^m,a_{i+1}^m\right),
\end{aligned}
$$

where

$$
\begin{aligned}
Q_n\left(a_{i+1},a_i\right) &= -2\left(a_{i+1}f_{d+2n}\left(\frac{a_{i+1}}{\eta}\right)-a_if_{d+2n}\left(\frac{a_i}{\eta}\right)\right) \\
&\quad +(d+2n)\times\left(F_{d+2n}\left(\frac{a_{i+1}}{\eta}\right)-F_{d+2n}\left(\frac{a_i}{\eta}\right)\right),
\end{aligned}
$$

with $h=\sqrt{\kappa^2+2\sigma^2}$ and where, as before, $d=\frac{4\kappa}{\theta}$ and F_{d+2n} and f_{d+2n} are respectively the cumulative distribution and density functions of a chi-2 random variable with $d+2n$ degrees of freedom. This implies

$$
\begin{aligned}
B_{k,i}^{m} &= S_m\left(a_k\right)\eta\left(\sum_{n=0}^{\infty}e^{-\frac{(\delta_k/2)^n}{n!}}\frac{(\delta_k/2)^n}{n!}\times Q_n\left(a_i^m,a_{i+1}^m\right)\right)+ \\
&\quad S_m\left(a_k\right)\psi_{t_{m+1}}\sum_{n=0}^{\infty}e^{-\frac{(\delta_k/2)^n}{n!}}\frac{(\delta_k/2)^n}{n!}\left(F_{d+2n}\left(\frac{a_{i+1}^m}{\eta}\right)-F_{d+2n}\left(\frac{a_i^m}{\eta}\right)\right).
\end{aligned}
$$