A Correlation Bridge between Structural Models and Reduced Form Models for Multiname Credit Derivatives

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June 3, 2005

Abstract

We derive a consistent way to price multiname credit derivatives. The methodology proposed is based on copulas function to model the joint distribution of default times. The copula correlation coefficient are provided from the structural model.

Key words: Intensity models, structural models, Collateralized debt obligation, copulas

1 Introduction

"The Market for credit protection has an obvious appeal during times of economic downturn". When Mr. Montag, the Co-President of Goldman Sachs (Japan), made this statement in 2001, he based his reasoning on the fact that credit derivatives (mainly credit default swaps at that time) grew 45 percent to $918.9 billion whereas the equity and interest rate markets were having some difficulties. Credit derivatives were considered as products which are valuable to market participants in managing risk in times of volatility and uncertainty. The recent growth of the credit market had undermined this statement and placed credit derivatives as crucial components of the financial market regardless of the state of the economy. Indeed, according to the International Swaps and Derivatives Association, the notional outstanding in credit derivatives grew 44 percent in the first half of 2004 to $5.44 trillion outperforming the equity derivatives market which grew by 9.7 percent to $3.79 trillion. This surge of demand of credit derivatives, including credit default swaps and default swaps of baskets, urged the need for a more accurate modelling of firms default. Credit models can be divided into two main categories: (i) reduced form models and (ii) structural models.

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Structural models are based on the work by Merton (1974), in which a firm life is linked to its ability to pay back its debt. Let us suppose that a firm issues a bond to finance its activities and also that this bond has maturity T. At final time T, if the firm is not able to reimburse all the bondholders we can say that there has been a default event. In this context default may occur only at final time T and is triggered by the value of the firm being below the debt level. More realistic and sophisticated structural models (Black and Cox (BC) (1976)) have been proposed for which the default can happen also before maturity T and some times they are called *first passage time models*. In these models the default time is the first instant where the firm value hits from above either a deterministic (possibly time varying) or a stochastic barrier, ideally associated with the debt level of the firm in time. When the default barrier is exogenously fixed, as in Black and Cox (1976) and Longstaff and Schwartz (1995), it acts as a safety covenant to protect the bondholders. Alternatively it can be endogenously fixed as a result of the stockholders attempt to choose the default threshold which maximizes the value of the firm, e.g. Leland (1994) and Leland and Toft (1996).

For a summary of the literature on structural models, possibly with stochastic interest rates and default barriers, we refer for example to Chapter 3 of Bielecki and Rutkowski (2001). It is important to notice that structural models make some implicit but important assumptions: they assume that the firm value follows a random process similar to the one used to describe generic stocks in equity markets, and that it is possible to observe this value at any time. Therefore, unlike intensity models, here the default process can be completely monitored based on default free market information and comes less as a surprise. However, structural models in their basic formulations and with flat barriers (Merton, BC) have few parameters in their dynamics and cannot be calibrated exactly to structured data such as CDS quotes along different maturities. This problem was tackled recently by Brigo and Tarenghi (2004) who proposed a structural model with enough degrees of freedom to calibrate the CDS term structure. This model is trivial to be extended to multidimensional default: just insert equity correlation between the Brownian motions driving the firm values and through Monte Carlo or some other numerical tools you may price multiname credit derivatives.

Reduced form models (also called *intensity models* when a suitable context is possible) describe default by means of an exogenous jump process; more precisely, the default time is the first jump time of a Poisson process with deterministic or stochastic (Cox process) intensity. Here default is not triggered by basic market observable data but has an exogenous component that is independent of all the default free market information. Monitoring the default free market does not give complete information on the default process, and there is no economic rationale behind default. This family of models is particularly suited to model credit spreads and in its basic formulation is easy to calibrate to Credit Default Swap (CDS) data. This approach was adopted by a number of authors, including for example Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997), Madam and Unal (1998), Lando (1998), Duffie and Singleton (1999), Kijima and Muromachi (2000), Hughston and Turnbull (2001) and Brigo and Alfonsi (2003). See references at the beginning of Chapter 8 in Bielecki and Rutkowski (2001) for a summary of the literature on intensity models. There are different ways to extend this model to the multidimensional case. The first approach, the conditionally independent defaults (CID), introduces credit risk dependence between firms through the dependence of the firms intensity processes on a common set of state variables. We refer the reader to Duffie and Singleton (1999), Kijima (2000) and Kijima and Muromachi (2000) for a detailed treatment of this approach.

The second approach to model default correlation, contagion models, relies on the works by Davis and Lo (2001) and Jarrow and Yu (2001). It is based on the idea of default contagion in which, when a firm defaults, the default intensities of related firms jump upwards. In these models default dependencies arises from direct links between firms. A third approach, which we are more concern with in this paper, uses copula functions. To introduce multiname correlation, we introduce a copula on the jumps of the Poisson processes with those intensities for each name (Schonbucher and Schubert (2002) or Roncalli and al.(2003)) and obtain again a model for multiname credit derivatives. In this approach, however, estimation of the dependence parameter is less clear, and often dubious methods of forcing equity correlation into the copula are adopted.

In this paper we build the bridge between the two dependence paradigms and based only on market information: Choose as a first possible bridge a first to default contract with protection payment at maturity. Then the protection leg price is essentially the survival copula (the complementary copula) in the intensity model above, whereas we may resort to multivariate barrier option formulas for the first to default in the structural model. Set then the equity correlation rho in the structural model to a value, compute the first to default price, and then invert the survival copula parameter theta in the intensity model so that the price obtained is the same attained with the structural model. This leads to two models that are calibrated to the same single name CDS data and that agree on the first to default price. We do this for several values of rho and plot the rho-theta graph to see if the bridge is one to one or if it flattens in some situations, to see what is the impact of single name credit spreads on the shape of the bridge, to see which parametric copula gives a better one to one bridge, and so on. The remainder of this paper is organized as follows. Section 2 sets up the characteristics of the single name models. Section 3 describes the use of copula functions. Section 4 addresses the mutli-name credit derivatives issue. Section 5 explains how to build the bridge between the structural and intensity frameworks. Section 6 discusses some numerical results. Section 7 concludes.

2 Single Name Models and Credit Default Swaps

We recall briefly some basic definitions for CDS's. Consider a CDS where "A" (the protection buyer) buys protection from "B" (the protection seller) against possible default of a reference credit entity "C". "A" pays rates R at times $T_{a+1}, \ldots, T_b$ (the "premium leg") in exchange for a single protection payment L_{GD} (Loss Given Default of "C") at the default time $\tau = \tau_2$ of the reference entity "C" (the "protection leg"), provided that $T_a < \tau_2 \leq T_b$. This is called a "running CDS" (RCDS) discounted payoff.

$$\begin{array}{ccccc} \text{“B”} & \rightarrow & \text{protection } \text{L}_{\text{GD}} \text{ at default } \tau_2 \text{ if } T_a < \tau_2 \leq T_b & \rightarrow & \text{“A”} \\ \text{“B”} & \leftarrow & \text{rate } R \text{ at } T_{a+1}, \ldots, T_b \text{ or until default } \tau_2 & \leftarrow & \text{“A”} \end{array}$$

Formally, we may write the RCDS discounted value at time t seen from "A" as

$$\Pi_{\text{RCDS}_{a,b}}(t) := -D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}} - \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau\geq T_i\}} + \\ +\mathbf{1}_{\{T_a<\tau\leq T_b\}}D(t,\tau)\ \text{L}_{\text{GD}} \tag{1}$$

where $t \in [T_{\beta(t)-1}, T_{\beta(t)})$, i.e. $T_{\beta(t)}$ is the first date among the T_i's that follows t, and where α_i is the year fraction between T_{i-1} and T_i. The stochastic discount factor at time t for maturity T is denoted by $D(t,T) = B(t)/B(T)$, where $B(t) = \exp(\int_0^t r_u du)$ denotes the bank-account numeraire, r being the instantaneous short interest rate.

We explicitly point out that we are assuming the offered protection amount L_{GD} to be deterministic and, in particular, not to depend on the CDS rate but only on the reference entity. Typically $\text{L}_{\text{GD}} = 1 - \text{R}_{\text{EC}}$, where the recovery rate (of "C") is assumed to be deterministic and the notional is set to one.

We denote by $\text{CDS}(t, [T_{a+1}, \ldots, T_b], T_a, T_b, R, \text{L}_{\text{GD}})$ the price at time t of the above standard running CDS. At times some terms are omitted, such as for example the list of payment dates $[T_{a+1}, \ldots, T_b]$.

2.1 Single name CDS with intensity models

The pricing formulas for the above CDS payoffs depend on the assumptions on interest-rate dynamics and on the default time τ. Here we place ourselves in a stochastic intensity framework, where the intensity is an $\mathcal{F}_t$-adapted continuous positive process, $\mathcal{F}_t$ denoting the basic filtration without default, typically representing the information flow of interest rates, intensities and possibly other default-free market quantities. Default is modeled as the first jump time of a Cox process with the given intensity process. In the Cox process setting we have $\tau = \Lambda^{-1}(\xi)$, where Λ is the stochastic hazard function which we assume to be $\mathcal{F}_t$ adapted, absolutely continuous and strictly increasing, and ξ is exponentially distributed with parameter 1 and independent of $\mathcal{F}$. These assumptions imply the existence of a positive adapted process λ, which we assume also to be right continuous and limited on the left, such that $\Lambda(t) = \int_0^t \lambda_s ds$ for all t. In general, we can compute the CDS price according to risk-neutral valuation (see for example Bielecki and Rutkowski (2001)):

$$\text{CDS}(t, T_a, T_b, R, \text{L}_{\text{GD}}) = \mathbb{E}\left\{\Pi_{\text{RCDS}_{a,b}}(t)|\mathcal{G}_t\right\} \tag{2}$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \leq t)$. The second sigma field $\sigma(\{\tau < u\}, u \leq t)$ contributing to $\mathcal{G}_t$ represents the information on whether default occurred before t, and if so when exactly. Since this information is available to us when we price, we need to condition on $\mathcal{G}_t$ rather than on $\mathcal{F}_t$ alone. In the Cox process setting default is unpredictable and this is why observation of $\mathcal{F}_t$ alone does not imply observation of the default time, contrary to standard structural (Merton, Black and Cox, etc) models where instead $\mathcal{F}_t = \mathcal{G}_t$. At times we denote by $\mathbb{E}_t$ and $\mathbb{Q}_t$ the expectation and probability conditional on the default-free sigma field $\mathcal{F}_t$.

The above expected value can also be written as

$$\text{CDS}(t, T_a, T_b, R, \text{L}_{\text{GD}}) = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau > t|\mathcal{F}_t)}\mathbb{E}\left\{\Pi_{\text{RCDS}_{a,b}}(t)|\mathcal{F}_t\right\} \tag{3}$$

(see again Bielecki and Rutkowski (2001) formula (5.1) p. 143, or more in particular Jeanblanc and Rutkowski (2000)).

For an explicit and tractable stochastic intensity/ interest-rate model with automatic analytical and separable calibration to interest rate derivatives and CDS's see Brigo and Alfonsi (2003), where an analytical formula for CDS options based on Jamshidian's decomposition is also presented. In this paper we focus more on multiname credit derivatives and we will take intensity to be deterministic. Before doing so, however, let us see some possible definitions of forward CDS rates.

Let us deal with the definition of (running) CDS forward rate $R_{a,b}(t)$. This can be defined as that R that makes the CDS value equal to zero at time t, so that

$$\mathrm{CDS}(t, T_a, T_b, R_{a,b}(t), \mathrm{L{\scriptstyle GD}}) = 0.$$

The idea is then solving this equation in $R_{a,b}(t)$. In doing this one has to be careful. It is best to use expression (3) rather than (2). Equate this expression to zero and derive R correspondingly. Strictly speaking, the resulting R would be defined on $\{\tau > t\}$ only, since elsewhere the equation is satisfied automatically thanks to the indicator in front of the expression, regardless of R. Since the value of R does not matter when $\{\tau < t\}$, the equation being satisfied automatically, we need not worry about $\{\tau < t\}$ and may define, in general,

$$R_{a,b}(t) = \frac{\mathrm{L{\scriptstyle GD}}\ \mathbb{E}[D(t,\tau)\mathbf{1}_{\{T_a<\tau\leq T_b\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i) + \mathbb{E}\left\{D(t,\tau)(\tau - T_{\beta(\tau)-1})\mathbf{1}_{\{T_a<\tau<T_b\}}|\mathcal{F}_t\right\}}, \tag{4}$$

where $\bar{P}(t,T) := \mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t]/\mathbb{Q}(\tau > t|\mathcal{F}_t)$ is the "no survival-indicator" part of the defaultable T-maturity (no recovery) zero coupon bond, i.e.

$$\mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t] = \mathbf{1}_{\{\tau>t\}}\mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t]/\mathbb{Q}(\tau > t|\mathcal{F}_t) = \mathbf{1}_{\{\tau>t\}}\bar{P}(t,T)$$

is the price at time t of a defaultable zero-coupon bond maturing at time T. Notice that replacing $\{\tau > T\}$ by $\{\tau \geq T\}$, as we implicitly do in (4), does not change anything since we are assuming continuous processes for the short rate and the stochastic intensity. We will denote by $P(t,T)$ the default-free zero coupon bond at time t for maturity T.

This approach to define $R_{a,b}$ amounts to equating to zero only the expected value in (3), and in a sense is a way of privileging $\mathcal{F}_t$ expected values to $\mathcal{G}_t$ ones. The technical tool allowing us to do this is the above-mentioned Jeanblanc Rutkowski (2000) result, and this is the spirit of part of the work in Jamshidian (2002).

2.1.1 Market quoting mechanism and implied hazard functions

Now we explain shortly how the market quotes running CDS prices. First we notice that typically the T's are three- months spaced. Let us begin with running CDS's. Usually at time $t = 0$, provided default has not yet occurred, the market sets R to a value $R^{\mathrm{MID}}_{a,b}(0)$ that makes the CDS fair at time 0, i.e. such that $\mathrm{CDS}(0, T_a, T_b, R^{\mathrm{MID}}_{a,b}(0), \mathrm{L{\scriptstyle GD}}) = 0$. In fact, in the market running CDS's used to be quoted at a time 0 through a bid and an ask value for this "fair" $R_{a,b}^{\text{MID}}(0)$, for CDS's with $T_a = 0$ and with T_b spanning a set of canonical final maturities, $T_b = 1y$ up to $T_b = 10y$. As time moves on to say $t = 1day$, the market shifts the T's of t, setting $T_a = 0 + t, \ldots, T_b = 10y + t$, and then quotes $R_{a,b}^{\text{MID}}(t)$ satisfying $\text{CDS}(t, T_a, T_b, R_{a,b}^{\text{MID}}(t), \text{L}_{\text{GD}}) = 0$. This means that as time moves on, the maturities increase and the times to maturity remain constant.

Recently, the quoting mechanism has changed and has become more similar to the mechanism of the futures markets, see for example Brigo (2004) for the details. However, provided a small approximation is accepted, once we fix the quoting time (say to 0) the method to strip implied hazard functions is the same under the two quoting paradigms. For example, Brigo and Alfonsi (2003) present a more detailed section on the "constant time-to-maturity" paradigm, and illustrate the notion of implied deterministic intensity (hazard function), satisfying

$$\mathbb{Q}\{s < \tau \leq t\} = \exp(-\Gamma(s)) - \exp(-\Gamma(t)).$$

The market Γ's are obtained by inverting a pricing formula based on the assumption that τ is the first jump time of a Poisson process with deterministic intensity $\gamma(t) = d\Gamma(t)/dt$. In this case one can derive a formula for CDS prices based on integrals of γ, and on the initial interest-rate curve, resulting from the above expectation:

$$\text{CDS}(t, T_a, T_b, R, \text{L}_{\text{GD}}; \Gamma(\cdot)) = \mathbf{1}_{\{t<\tau\}} \left[-R \int_{T_a}^{T_b} P(t,u)(T_{\beta(u)-1} - u) d(e^{-(\Gamma(u)-\Gamma(t))}) - \right. \tag{5}$$
$$\left. \sum_{i=a+1}^{n} P(t,T_i) R \alpha_i e^{\Gamma(t)-\Gamma(T_i)} - \text{L}_{\text{GD}} \int_{T_a}^{T_b} P(t,u) d(e^{-(\Gamma(u)-\Gamma(t))}) \right].$$

By equating to zero the above expression in γ for $t = 0, T_a = 0$, after plugging in the relevant market quotes for R, one can extract the γ's corresponding to CDS market quotes for increasing maturities T_b and obtain market implied γ^{mkt} and Γ^{mkt}'s. It is important to point out that usually the actual model one assumes for τ is more complex and may involve stochastic intensity either directly or through stochastic modeling of the future R dynamics itself. Even so, the γ^{mkt}'s are retained as a mere quoting mechanism for CDS rate market quotes, and may be taken as inputs in the calibration of more complex models.

2.2 Single name CDS with the Structural model

Brigo and Tarenghi (2004) introduced a structural model with enough degrees of freedom to calibrate the term structure of CDS data. The model is summarized in the following

Proposition 1. (Analytically-Tractable First Passage (AT1P) Model) *Assume the risk neutral dynamics for the value of the firm equity S is characterized by a risk free rate r_t, a dividend yield q_t and an instantaneous volatility σ_t, according to*

$$dS_t = S_t\,(r_t - q_t)\,dt + S_t\,\sigma_t\,dW(t)$$

and assume a default barrier $\widehat{H}(t)$ of the form

$$\widehat{H}(t) = H \exp\left(- \int_0^t \left(q_s - r_t + (1 + 2\beta)\frac{\sigma_s^2}{2} \right) ds \right)$$

for a given constant β, and let the default time τ be defined as the first time where S hits $\widehat{H}$ from above, starting from $S_0 > H$,

$$\tau = \inf\{t \geq 0 : S_t \leq \widehat{H}(t)\}.$$

Then the survival probability is given analytically by

$$\mathbb{Q}\{\tau > T\} = \left[\Phi\left(\frac{\log\frac{S_0}{H} + \beta\int_0^T \sigma_s^2 ds}{\sqrt{\int_0^T \sigma_s^2 ds}}\right) - \left(\frac{H}{S_0}\right)^{2\beta}\Phi\left(\frac{\log\frac{H}{S_0} + \beta\int_0^T \sigma_s^2 ds}{\sqrt{\int_0^T \sigma_s^2 ds}}\right)\right]. \tag{6}$$

Formula (6) is easily obtained by barrier option techniques. This formula can be used to fit the model parameters to CDS market data; but now the only parameter that can account for time dependence is the volatility. If we use exogenous equity volatility (historical or implicit) we are left with no freedom. However, we may infer the first year volatility $\sigma(0 \div 1y) := \{\sigma(t) : t \in [0,1]\}$ from equity data and use H as a fitting parameter, and then use the remaining later volatilities $\sigma(1y \div 2y), \sigma(2y \div 3y)$ etc as fitting parameters. Therefore $\sigma(2y \div \cdot)$ will be determined by credit quality as implicit in CDS data rather than by equity data. To sum up, we can choose piecewise constant volatility, and look for those volatility values after the first year that make the quoted CDS's fair when inserting in their premium legs the market quoted R's. In this way we find as many volatilities as many CDS's we consider. This procedure is justified by the fact that in the end we are not interested in the real process of the equity firm value underlying the contract, but only in risk neutral default probabilities. A useful remark is that in formula (6) the firm value appears always in a ratio with the barrier parameter, so we can imagine to re-scale S (considering S/S_0 so that $S_0/S_0 = 1$) and also the barrier parameter H (H/S_0). In principle, in the calibration, we can arbitrarily choose the value H (Brigo and Tarenghi (2004) showed two procedures to assign a significant value to this parameter) and β, leaving all the unknown information in the calibration of the volatility.

Finally, notice that in the context of this model, as for traditional structural models, $\mathcal{F}_t = \mathcal{G}_t$, since here default is triggered by information contained in $\mathcal{F}$ and thus no explicit monitoring of default has to be added to $\mathcal{F}_t$.

3 Copula Functions: Introduction and Motivation

Consider a random vector $X = (X_1, ..., X_n)$, and suppose that we wish to analyze the dependence between its components. The whole information on the distribution of the vector is given by the joint cumulative distribution function of X. If $\mathbb{Q}$ denotes the probability measure in our setting, such function in the point $(x_1, \ldots, x_n)$ is given by $\mathbb{Q}(X_1 \leq x_1, ..., X_n \leq x_n)$. However, this function mixes information on the dependence between the different components of the vector with information on the distribution of the single components themselves. Copula functions have been introduced in order to allow a separation between the marginal cumulative distribution functions (cdf for short) and the dependence structure. The former concerns single components, taken one at the time, and is given by the cdf's $F_i(x) := \mathbb{Q}(X_i \leq x)$, $i = 1, \ldots, n$, which we assume to be continuous. The latter is entirely represented by the copula function we introduce now. It is well known that $U_1 = F_1(X_1), ..., U_n = F_n(X_n)$ are uniformly distributed random variables on $[0,1]$. The joint cumulative distribution function of $(U_1, ..., U_n)$, that we denote by

$$C(u_1, ..., u_n) = \mathbb{Q}(U_1 \leq u_1, ..., U_n \leq u_n),$$

is called the copula function of $(X_1, ..., X_n)$ and has the following link with the multivariate cdf:

$$\mathbb{Q}(X_1 \leq x_1, ..., X_n \leq x_n) = C(\mathbb{Q}(X_1 \leq x_1), ..., \mathbb{Q}(X_n \leq x_n)). \tag{7}$$

One can easily check that a copula has the following properties:

1. $C(u_1, .., u_{i-1}, 0, u_{i+1}, .., u_n) = 0$
2. $C(1, .., 1, u_k, 1, .., 1) = u_k$
3. $\partial_{u_1 \ldots u_n} C$ is a positive measure in the sense of Schwartz distributions. This means concretely that for any hypercube $H = [a_1, b_1] \times ... \times [a_n, b_n] \subset [0,1]^n$,

   $$\mathbb{Q}[(U_1, .., U_n) \in H] \geq 0.$$

   When $n = 2$, this can be written as
   $C(b_1, b_2) - C(a_1, b_2)x - C(b_1, a_2) + C(a_1, a_2) \geq 0.$

Conversely, one can show that any function that satisfies these three conditions can be viewed as the joint cdf of a vector of uniform variables on $[0,1]$ and is thus a copula. This is known as Sklar's theorem, see for example Joe (1997) or Nelsen (1999).

Among the different ways to define specific copula functions, there are following two. The first one consists in seeking functions C satisfying the three above properties. Archimedean copulas are an example of this approach. Indeed, Archimedean copulas come from the remark that if φ is a convex decreasing function such that $\varphi(1) = 0$, then

$$C(u_1, .., u_n) = \mathbf{1}_{\{\varphi(u_1)+..+\varphi(u_n) \leq \varphi(0)\}} \varphi^{-1}(\varphi(u_1) + .. + \varphi(u_n))$$

has the above three properties and is thus a copula. Therefore, by specifying families of decreasing convex functions that vanish in 1 we specify families of copulas (e.g. Gumbel, Joe, Frank...), see Bouyé et al. (2000), Nelsen (1999) and Joe (1997).

The second method consists in working directly with joint cdf's $F(x_1, ..., x_n)$ and the related marginal cdf's F_i. The associated copula is then defined as $F(F_1^{-1}(u_1), ..., F_n^{-1}(u_n))$. Even if this method does not always lead to analytically tractable copulas, it can provide us copulas that are easy to simulate. Indeed, the main example of this kind of construction is the well known fundamental family of Gaussian copulas. A Gaussian copula is defined as the copula of a joint Gaussian random vector X with standard Gaussian marginals and correlation matrix ρ, and is thus given by $N_\rho(N^{-1}(u_1), ..., N^{-1}(u_n))$ where N is the cdf of a standard normal variable and N_ρ is the joint cdf of X. This copula cannot be computed explicitly. The simulation is however straightforward: it is sufficient to consider $(N(X_1), ..., N(X_n))$ where $X = (X_1, ..., X_n)$ is a Gaussian vector with correlation ρ that can be easily simulated by resorting to a standard Gaussian simulator and to a Cholesky decomposition of ρ. A similar approach leads to Student's copula (see Bouyé et al. (2000) and Genz and Bretz (2002)).

In recent years, copula functions have received a great deal of attention, see for example the papers of Genz and Bretz (2002), Hürlimann (2002, 2003), Juri and Wüthrich (2002), Nelsen et al. (2001), Wei and Hu (2002), and the books of Joe (1997) and Nelsen (1999). For financial and insurance applications, recent applications on copulas include for example Bouyé et al. (2000), Cherubini et al. (2002), Embrechts et al. (2001), Jouanin et al. (2001), Klugman and Parsa (1999), Prampolini (2003), and Schönbucher and Schubert (2001).

4 Multi name default models and n-th to default

4.1 Multi name intensity models: Treshold copula approach

We describe the two dimensional case to contain notation, and we will hint at generalizations later on. Throughout the paper we assume a deterministic default intensities (hazard rates) γ_1 and γ_2 for names "1" and "2" respectively. Let Γ_1 and Γ_2 be the corresponding hazard functions, related to the single-name default probabilities by

$$\mathbb{Q}\{\tau_1 > T\} = e^{-\Gamma_1(T)}, \quad \mathbb{Q}\{\tau_2 > T\} = e^{-\Gamma_2(T)}.$$

We are thus assuming for the default times a Cox process structure with deterministic intensities. Assume we have calibrated such intensities from CDS quotes on names "B" and "C" with the methodology seen earlier. For the default dependence we take a Copula function on the basic exponential random variables driving the default jumps in the two names. This "threshold copula" approach, described for example in Schönbucher and Schubert (2001) or in Jouanin et al. (2001), leads to

$$\mathbb{E}_t\{\mathbf{1}_{\{\tau_1>S\}}\mathbf{1}_{\{\tau_2>U\}}\} = \mathbb{Q}\{\tau_1 > S, \tau_2 > U\} = \mathbb{Q}\{\Gamma_1(\tau_1) > \Gamma_1(S), \Gamma_2(\tau_2) > \Gamma_2(U)\} =$$

$$= \mathbb{Q}\{\xi_1 > \Gamma_1(S), \xi_2 > \Gamma_2(U)\} = \mathbb{Q}\{e^{-\xi_1} < e^{-\Gamma_1(S)}, e^{-\xi_2} < e^{-\Gamma_2(U)}\} =$$

where ξ's are two unit mean exponential random variables and E_t denotes expectation with respect to $\mathcal{F}_t$. Remembering that $\exp(-\xi)$'s are uniform variables, imposing a joint distribution function to them amounts to imposing a copula $\bar{C}$. This will be the survival copula for the jump times τ. We obtain

$$\mathbb{E}_t\{\mathbf{1}_{\{\tau_1>S\}}\mathbf{1}_{\{\tau_2>T\}}\} = \bar{C}_\zeta(e^{-\Gamma_1(S)}, e^{-\Gamma_2(T)}),$$

where ζ is a parameter in a parametric family of copulas where we assume our survival copulas to belong in. If we wish to express the expected values directly in terms of the copula C (instead of survival copula) between the τ's, this amounts to compute

$$\mathbb{E}_t\{\mathbf{1}_{\{\tau_2>T\}}\mathbf{1}_{\{\tau_1>S\}}\} = \mathbb{Q}\{1 - e^{-\xi_1} > 1 - e^{-\Gamma_1(S)}, 1 - e^{-\xi_2} > 1 - e^{-\Gamma_2(T)}\}$$

Now we take a joint distribution C_θ on the uniforms $1 - e^{-\xi}$'s (this is the copula of default times $[\tau_1, \tau_2]$, which we assume to lay in a parametric family with parameter θ) and obtain

$$\begin{aligned} \mathbb{E}_t\{\mathbf{1}_{\{\tau_1>S\}}\mathbf{1}_{\{\tau_2>T\}}\} &= \mathbb{Q}(\tau_1 > S, \tau_2 > T) \\ = e^{-\Gamma_1(S)} + e^{-\Gamma_2(T)} - 1 + C_\theta(1 - e^{-\Gamma_1(S)}, 1 - e^{-\Gamma_2(T)}) &=: J(S, T, \Gamma_1, \Gamma_2; \theta) \end{aligned} \tag{8}$$

("J" stands for "Joint survival probability"). We will omit the arguments Γ_1, Γ_2 at times when obvious from the context.

4.1.1 First to default protection leg

An First to Default (FtD) is a claim that provides a protection against the first reference entity that experiences a credit event from a basket of more than one reference entity. The protection seller, therefore, assumes the "first-to-default" risk on a basket of credits. The protection seller is motivated primarily by receiving a higher premium than any individual default swap to reflect the higher level of risk. The protection buyer views a basket swap as a lower cost method of hedging multiple credits. However, since the seller is exposed to the notional amount of only one (the first-to-default) credit, the buyer retains the residual risk of multiple defaults. This residual risk represents the imperfect hedge for the buyer. The potential cost of managing the hedge could determine the price the buyer is willing to pay for the basket.

Now, we show the mathematics behind pricing FtD claims. Denote by $\tau^{(1)}$ (notice the upper index) the first among the default times τ_1, τ_2: $\tau^{(1)} = \min(\tau_1, \tau_2)$. If the protection payment occurs at the final maturity T, the price of the protection leg of the first to default is simply the survival copula, computed as follows, under deterministic interest rates:

$$\begin{aligned} \mathrm{FtD}(t, T; T) = \mathbb{E}\{D(t, T)\mathbf{1}_{\{\tau^{(1)}\leq T\}}|\mathcal{F}_t, \tau^{(1)} > t\} &= P(t, T)\frac{\mathbb{Q}(t < \tau^{(1)} \leq T)}{\mathbb{Q}(\tau^{(1)} > t)} \\ &= P(t, T)\left(1 - \frac{\mathbb{Q}(\tau^{(1)} \geq T)}{\mathbb{Q}(\tau^{(1)} > t)}\right) \end{aligned}$$

so that, since we have

$$\mathbb{Q}(\tau^{(1)} > t) = \mathbb{Q}(\tau_1 > t, \tau_2 > t) = \bar{C}_\zeta(e^{-\Gamma_1(t)}, e^{-\Gamma_2(t)}),$$

as we have seen earlier, we find that, in general

$$\mathrm{FtD}(t, T; T) = \mathbf{E}(D(t, T)\mathbf{1}_{\{\tau^{(1)}\leq T\}}|\mathcal{F}_t, \tau^{(1)} > t) = \frac{P(0, T)}{P(0, t)}\left(1 - \frac{\bar{C}(e^{-\Gamma^1(T)}, .., e^{-\Gamma^n(T)})}{\bar{C}(e^{-\Gamma^1(t)}, .., e^{-\Gamma^n(t)})}\right)$$

4.2 Multi name structural models: Equity correlation

4.2.1 First to default protection leg

As it has been shown in section 4.1.1 if the protection payment occurs at the final maturity T, the price of the protection leg of the first to default, under deterministic interest rates, is computed as follows :

$$\begin{aligned} \mathrm{FtD}(t,T;T) &= \mathbb{E}\{D(t,T)\mathbf{1}_{\{\tau^{(1)}\leq T\}}|\mathcal{F}_t,\tau^{(1)}>t\} = P(t,T)\frac{\mathbb{Q}(t<\tau^{(1)}\leq T)}{\mathbb{Q}(\tau^{(1)}>t)} \\ &= P(t,T)\frac{\mathbb{Q}(\tau^{(1)}\leq T)-\mathbb{Q}(\tau^{(1)}\leq t)}{\mathbb{Q}(\tau^{(1)}>t)} \end{aligned}$$

So finding the protection leg of the first to default amounts to computing the cumulative distribution of $\tau^{(1)}$ which could be computed as follows :

$$\begin{aligned} \mathbb{Q}(\tau^{(1)} &\leq t) = \mathbb{Q}(\tau_1\leq t \text{ or } \tau_2\leq t) \qquad (9)\\ &= \mathbb{Q}(\tau_1\leq t)+\mathbb{Q}(\tau_2\leq t)-\mathbb{Q}(\tau_1\leq t \text{ and } \tau_2\leq t) \end{aligned}$$

Now, recall that in the single name structural model as presented in section 2.2 the barrier has the following form :

$$\hat{H}_t = H\exp\left(-\int_0^t\left(q_s-r_s+(1+2\beta)\frac{\sigma_s^2}{2}\right)ds\right) \qquad (10)$$

for a given constant β.

Let us define β to be 0. Then the default probability is simply given by :

$$\mathbb{Q}^i\{\tau<T\} = 2\Phi\left(-\frac{\ln\frac{S_0^i}{H^i}}{\sqrt{\int_0^T(\sigma_s^i)^2\,ds}}\right) \text{ for } i=1,...,n. \qquad (11)$$

where n defines the number of securities.

Proposition 2. *Under the barrier shape assumption defined in* (10)*, the joint probability of default is given by :*

$$\begin{aligned} \mathbb{Q}(\tau_1 &\leq t \text{ and } \tau_2\leq t) = 4\int_{b_1}^{\infty}\int_{b_2}^{\infty}\frac{1}{2\pi\sqrt{\left(\bar{\sigma}_t^1\bar{\sigma}_t^2\right)^2-\rho^2\int_0^t\sigma_s^1\sigma_s^2ds}} \qquad (12)\\ &\exp\left(-\frac{\left(\bar{\sigma}_t^1\right)^2x^2-2xy\rho\int_0^t\sigma_s^1\sigma_s^2ds+\left(\bar{\sigma}_t^2\right)^2y^2}{2\left(\left(\bar{\sigma}_t^1\bar{\sigma}_t^2\right)^2-\rho^2\int_0^t\sigma_s^1\sigma_s^2ds\right)}\right)dxdy \qquad (13) \end{aligned}$$

where $b_i=\ln\frac{S_0^i}{H^i}$ *and* $\bar{\sigma}_t^i=\sqrt{\int_0^t(\sigma_s^i)^2\,ds}$ *for* $i\in\{1,2\}$

Proof. To carry out this proof we follow closely Bielecki and Rutkowski (2002). For ease of notation, define $\mu_s^i = r_s - q_s^i - \frac{\sigma_{i,s}^2}{2}$. Then, the barrier has the following form :

$$\hat{H}_t^i = H^i\exp\left(\int_0^t\mu_s^i ds\right)$$

Now note that $\tau_i = \inf\left\{t \geq 0 : S_t^i \leq H^i \exp\left(\int_0^t \mu_s^i ds\right)\right\} = \inf\left\{t \geq 0 : X_t^i \geq b_i\right\}$, where $X_t^i = -\ln\left(\frac{S_t^i \exp\left(-\int_0^t \mu_s^i ds\right)}{S_0^i}\right) = \int_0^t \sigma_s^i dW_s^i$ for $i \in \{1,2\}$ by symmetry of the brownian motion. The two-dimensional vector $\left(X_t^1, X_t^2\right)^T$ follows a gaussian distribution with mean $(0,0)^T$ and variance covariance matrix $\Sigma = \begin{bmatrix} \int_0^t \left(\sigma_s^1\right)^2 ds & \rho \int_0^t \sigma_s^1 \sigma_s^2 ds \\ \rho \int_0^t \sigma_s^1 \sigma_s^2 ds & \int_0^t \left(\sigma_s^2\right)^2 ds \end{bmatrix}$.

Let $M_t^i = \max\limits_{0 \leq u \leq t} \left\{X_u^i\right\}$, then

$$\mathbb{Q}(\tau_1 \leq t\ ,\ \tau_2 \leq t) = \mathbb{Q}(M_t^1 \geq b_1\ ,\ M_t^2 \geq b_2)$$

We need now to compute $\mathbb{Q}(M_t^1 \geq b_1\ ,\ M_t^2 \geq b_2)$. For $a_1 \leq b_1$, $a_2 \leq b_2$, we have

$$\begin{aligned}
\mathbb{Q}(X_t^1 & \leq a_1, M_t^1 \geq b_1, X_t^2 \leq a_2, M_t^2 \geq b_2) = \mathbb{Q}(X_t^1 \leq a_1, \tau_1 \leq t, X_t^2 \leq a_2, \tau_2 \leq t) \\
&= \int_0^t \int_0^t \mathbb{Q}(X_t^1 \leq a_1, X_t^2 \leq a_2 | \tau_1 = s_1, \tau_2 = s_2)\mathbb{Q}(\tau_1 \in ds_1, \tau_2 \in ds_2) \\
&= \int_0^t \int_0^t \mathbb{Q}(X_{t-s}^1(b_1) \leq a_1, X_{t-s}^2(b_2) \leq a_2)\mathbb{Q}(\tau_1 \in ds_1, \tau_2 \in ds_2) \\
&= \int_0^t \int_0^t \mathbb{Q}(X_{t-s}^1(b_1) \geq b_1 + (b_1 - a_1), X_{t-s}^2(b_2) \geq b_2 + (b_2 - a_2))\mathbb{Q}(\tau_1 \in ds_1, \tau_2 \in ds_2) \\
&= \int_0^t \int_0^t \mathbb{Q}\left(X_t^1 \geq 2b_1 - a_1, X_t^2 \geq 2b_2 - a_2 | \tau_1 = s_1, \tau_2 = s_2\right)\mathbb{Q}(\tau_1 \in ds_1, \tau_2 \in ds_2) \\
&= \mathbb{Q}(X_t^1 \geq 2b_1 - a_1, \tau_1 \leq t, X_t^2 \geq 2b_2 - a_2, \tau_2 \leq t) \\
&= \mathbb{Q}(X_t^1 \geq 2b_1 - a_1, X_t^2 \geq 2b_2 - a_2)
\end{aligned}$$

Noticing that : $\mathbb{Q}(X_t^1 \leq a_1, M_t^1 = b_1, X_t^2 \leq a_2, M_t^2 = b_2)db_1db_2 = \frac{\partial^2}{\partial b_1 b_2}\mathbb{Q}(X_t^1 \geq 2b_1 - a_1, X_t^2 \geq 2b_2 - a_2)$ and that $\mathbb{Q}(X_t^1 \leq b_1, M_t^1 = b_1, X_t^2 \leq b_2, M_t^2 = b_2)db_1db_2 = \mathbb{Q}(M_t^1 = b_1, M_t^2 = b_2)db_1db_2$, then

$$\mathbb{Q}(M_t^1 \geq b_1, M_t^2 \geq b_2) = 4\int_{b_1}^{\infty}\int_{b_2}^{\infty} \frac{1}{2\pi\sqrt{\left(\bar{\sigma}_t^1\bar{\sigma}_t^2\right)^2 - \rho^2\int_0^t \sigma_s^1\sigma_s^2 ds}} \exp\left(-\frac{\left(\bar{\sigma}_t^1\right)^2 x^2 - 2xy\rho\int_0^t \sigma_s^1\sigma_s^2 ds + \left(\bar{\sigma}_t^2\right)^2 y^2}{2\left(\left(\bar{\sigma}_t^1\bar{\sigma}_t^2\right)^2 - \rho^2\int_0^t \sigma_s^1\sigma_s^2 ds\right)}\right) dxdy$$

□

Proposition 3. *Under the structural model framework, the distribution of $\tau^{(1)}$ is given by*

$$\begin{aligned}
\mathbb{Q}(\tau^{(1)} & \leq t) = 2\Phi\left(-\frac{\ln\frac{S_0^1}{H^1}}{\bar{\sigma}_t^1}\right) + 2\Phi\left(-\frac{\ln\frac{S_0^2}{H^2}}{\bar{\sigma}_t^2}\right) \\
& -4\int_{b_1}^{\infty}\int_{b_2}^{\infty} \frac{1}{2\pi\sqrt{\left(\bar{\sigma}_t^1\bar{\sigma}_t^2\right)^2 - \rho^2\int_0^t \sigma_s^1\sigma_s^2 ds}} \exp\left(-\frac{\left(\bar{\sigma}_t^1\right)^2 x^2 - 2xy\rho\int_0^t \sigma_s^1\sigma_s^2 ds + \left(\bar{\sigma}_t^2\right)^2 y^2}{2\left(\left(\bar{\sigma}_t^1\bar{\sigma}_t^2\right)^2 - \rho^2\int_0^t \sigma_s^1\sigma_s^2 ds\right)}\right) dxdy
\end{aligned}$$

Proof. The proof is straightforward by using (9) and (11) □

Notice that in a high dimension cases ($n \geq 4$), the protection leg of the first to default could not be computed analytically and we resort in that case to Monte Carlo Simulation.

5 Building the bridge

After having addressed in the previous section pricing of multiname credit derivatives with structural and intensity models, we tackle in this section the issue of building the bridge between these two models. To do so, we price First to Default (or any security based on the joint default distribution) with the structural model with a specific correlation as an input and then find out what parameter should we put in the copula based intensity model so we come up with the same price. More precisely recall, that the current price of First to Default maturing at time T in the structural model framework is given by :

$$\begin{aligned}
\mathrm{FtD}^{\mathrm{Struct}}(0,T;T) &= P(0,T)\ \mathbb{Q}(\tau^{(1)} \le T) \\
&= P(0,T)\left[\ \mathbb{Q}(\tau_1 \le T) + \ \mathbb{Q}(\tau_2 \le T) - \ \mathbb{Q}(\tau_1 \le T, \tau_2 \le T)\right] \\
&= P(0,T)\Bigg[2\Phi\left(-\frac{\ln \frac{S_0^1}{H^1}}{\bar{\sigma}_T^1}\right) + 2\Phi\left(-\frac{\ln \frac{S_0^2}{H^2}}{\bar{\sigma}_T^2}\right) \\
&\quad -4\int_{b_1}^{\infty}\int_{b_2}^{\infty} \frac{1}{2\pi\sqrt{\left(\bar{\sigma}_T^1\bar{\sigma}_T^2\right)^2 - \rho^2\int_0^T \sigma_s^1\sigma_s^2 ds}} \exp\left(-\frac{\left(\bar{\sigma}_T^1\right)^2 x^2 - 2xy\rho\int_0^T \sigma_s^1\sigma_s^2 ds + \left(\bar{\sigma}_T^2\right)^2 y^2}{2\left(\left(\bar{\sigma}_T^1\bar{\sigma}_T^2\right)^2 - \rho^2\int_0^T \sigma_s^1\sigma_s^2 ds\right)}\right) dxdy
\end{aligned}$$

where $b_i = \ln \frac{S_0^i}{H^i}$ for $i \in \{1,2\}$.

Whereas the price of the same First to Default in the intensity framework is given by :

$$\begin{aligned}
\mathrm{FtD}^{\mathrm{Intensity}}(0,T;T) &= P(0,T)\,\mathbb{Q}(\tau^{(1)} \le T) \\
&= P(0,T)\left[1 - \bar{C}_\theta\left(e^{-\Gamma_1(T)}, e^{-\Gamma_2(T)}\right)\right] = P(0,T)\left[1 - \bar{C}_\theta(u,v)\right] \\
&= P(0,T)\left[1 - \int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\theta^2}} \exp\left(-\frac{x^2 - 2\theta xy + y^2}{2(1-\theta^2)}\right) dxdy\right]
\end{aligned}$$

where $\bar{C}_\theta$ is the survival copula, $\Gamma_i(T) = \int_0^T \gamma_i(s)\,ds$ for $i \in \{1,2\}$ and Φ^{-1} is the inverse cumulative normal distribution.

To make comparison easier we can write the price of First to Default as follows :

$$\begin{aligned}
\mathrm{FtD}^{\mathrm{Intensity}}(0,T;T) &= P(0,T)\left[\mathbb{Q}(\tau_1 \le T) + \mathbb{Q}(\tau_2 \le T) - \mathbb{Q}(\tau_1 \le T, \tau_2 \le T)\right] \\
&= P(0,T)\left[1 - \exp(-\Gamma_1(T)) + 1 - \exp(-\Gamma_1(T)) \right.\\
&\quad \left. -C_\theta\left(1 - e^{-\Gamma_1(T)}, 1 - e^{-\Gamma_2(T)}\right)\right] \\
&= P(0,T)\left[2 - \exp(-\Gamma_1(T)) - \exp(-\Gamma_2(T))\right]. \\
&\quad - \int_{-\infty}^{\Phi^{-1}(1-u)}\int_{-\infty}^{\Phi^{-1}(1-v)} \frac{1}{2\pi\sqrt{1-\theta^2}} \exp\left(-\frac{x^2 - 2\theta xy + y^2}{2(1-\theta^2)}\right) dxdt
\end{aligned}$$

Therefore the parameter θ satisfies

$$\theta = \arg\min\left[g\left(\mathrm{FtD}^{\mathrm{Intensity}}(0,T;T), \mathrm{FtD}^{\mathrm{Struct}}(0,T;T)\right)\right]$$

where g is a distance function. In our numerical examples, we set $g(x,y) = |x-y|$.

But since both models were calibrated to CDS quotes, i.e. they have the same marginal distributions, then computing θ amounts to compute the difference between the joint default probabilities, i.e. :

$$\theta^* = \arg\min_{\theta\in[0,1]} \left[g\left(h_R\left(\Gamma_1,\Gamma_2,T,\theta\right), h_S(b_1,b_2,\sigma_1,\sigma_2,T,\rho)\right]\right.$$

where $h_R\left(\Gamma_1,\Gamma_2,T,\theta\right) = \int_{-\infty}^{\Phi^{-1}\left(1-e^{-\Gamma_1(T)}\right)}\int_{-\infty}^{\Phi^{-1}\left(1-e^{-\Gamma_2(T)}\right)} \frac{1}{2\pi\sqrt{1-\theta^2}}\exp\left(-\frac{x^2-2\theta xy+y^2}{2(1-\theta^2)}\right)dxdy$ and $h_S\left(b_1,b_2,\sigma_1,\sigma_2,T,\rho\right) = 4\int_{b_1}^{\infty}\int_{b_2}^{\infty}\frac{1}{2\pi\sqrt{\left(\bar{\sigma}_T^1\bar{\sigma}_T^2\right)^2-\rho^2\int_0^T\sigma_s^1\sigma_s^2ds}}\exp\left(-\frac{\left(\bar{\sigma}_T^1\right)^2x^2-2xy\rho\int_0^T\sigma_s^1\sigma_s^2ds+\left(\bar{\sigma}_T^2\right)^2y^2}{2\left(\left(\bar{\sigma}_T^1\bar{\sigma}_T^2\right)^2-\rho^2\int_0^T\sigma_s^1\sigma_s^2ds\right)}\right)dxdy$ and $\rho\in[0,1]$. Note that we are only interested with positive correlations, and for this reason we have $\rho\in[0,1]$.

Proposition 4. *Let θ be the gaussian copula parameter and ρ be the equity correlation parameter. For a fixed time T, let f be the correspondance defined on $[0,1]\times[0,1]$ mapping ρ to θ, then f is a bijection from $[0,1]$ to $[0,1]$.*

Proof. The proof of the proposition needs to be done through 2 steps. In the first step we show that both h_S and h_R are strictly increasing in ρ and θ respectively. This follows immediately from a result shown by Sungur (1990) who noted the property that

$$\frac{dF\left(x_1,x_2;\rho\right)}{d\rho} = f\left(x_1,x_2;\rho\right) > 0$$

where $f\left(x_1,x_2;\rho\right)$ is the standard bivariate normal density function and $F\left(x_1,x_2;\rho\right)$ is the corresponding joint cumulative distribution.

In the second step we need to show that h_R and h_S coincide on the boundary of their domain, i.e. we need to show that $h_R\left(\Gamma_1,\Gamma_2,T,1\right) = h_S\left(b_1,b_2,\sigma_1,\sigma_2,T,1\right)$ and $h_R\left(\Gamma_1,\Gamma_2,T,0\right) = h_S\left(b_1,b_2,\sigma_1,\sigma_2,T,0\right)$.

$$\begin{aligned}
h_R\left(\Gamma_1,\Gamma_2,T,1\right) &= \mathbb{Q}(\tau_1\le T,\tau_2\le T)\\
&= \mathbb{Q}(U\ge e^{-\Gamma_1(T)}, V\ge e^{-\Gamma_2(T)})\\
&= \mathbb{Q}(U\ge e^{-\Gamma_1(T)}, U\ge e^{-\Gamma_2(T)})\\
&= \min\left\{\mathbb{Q}(\tau_1\le T),\mathbb{Q}(\tau_2\le T)\right\}
\end{aligned}$$

Note that this corresponds to the upper Fréchet bound.

$$\begin{aligned}
h_S\left(b_1,b_2,\sigma_1,\sigma_2,T,1\right) &= \mathbb{Q}(M_t^1\ge b_1, M_t^2\ge b_2) \text{ with } \rho\left(M_t^1,M_t^2\right)=1.\\
&= \min\left\{\mathbb{Q}(M_t^1\ge b_1),\mathbb{Q}(M_t^2\ge b_2)\right\}\\
&= \min\left\{\mathbb{Q}(\tau_1\le T),\mathbb{Q}(\tau_2\le T)\right\}
\end{aligned}$$

Now,

$$\begin{aligned}
h_R\left(\Gamma_1,\Gamma_2,T,0\right) &= \mathbb{Q}(U\ge e^{-\Gamma_1(T)}, V\ge e^{-\Gamma_2(T)})\\
&= \mathbb{Q}(\tau_1\le T)\times\mathbb{Q}(\tau_2\le T)
\end{aligned}$$

$$\begin{aligned}
h_S\left(b_1,b_2,\sigma_1,\sigma_2,T,0\right) &= \mathbb{Q}(M_t^1\ge b_1)\times\mathbb{Q}(M_t^2\ge b_2)\\
&= \mathbb{Q}(\tau_1\le T)\times\mathbb{Q}(\tau_2\le T)
\end{aligned}$$

□

Since we cannot invert the Gaussian cumulative distribution, it is not possible to find the analytical expression of the bijection f. However, numerical examples are presented in section (6)

Before proceeding to the numerical results, it is intersting to see what are the conditions that we need to have so that using the asset correlation ρ as the copula correlation parameter.

Proposition 5. *If $t \to \sigma_t$ is a constant function then the bijection f is an identity function.*

Proof. If σ_1 and σ_2 is constant then

$$\begin{aligned}
\mathbb{Q}(M_t^1 &\geq b_1, M_t^2 \geq b_2) = \mathbb{Q}(\max_{0\leq u\leq t} B_u \geq \frac{b_1}{\sigma_1}, \max_{0\leq u\leq t} B_u \geq \frac{b_2}{\sigma_2}) \\
&= \mathbb{Q}\left(U \leq \Psi\left(\frac{b_1}{\sigma_1}, t\right), V \leq \Psi\left(\frac{b_1}{\sigma_1}, t\right)\right) \text{ where } \Psi(x,t) = 2\mathbb{Q}(B_t \geq x) \\
&= \mathbb{Q}\left(U \leq 1 - e^{-\Gamma_1(t)}, V \leq 1 - e^{-\Gamma_2(t)}\right) \text{ since } 1 - e^{-\Gamma_i(t)} = \mathbb{Q}\left(M_t^i \geq b_i\right) \ \forall i \in \{1,2\}
\end{aligned}$$

Hence both models have the same correlation parameter. □

However to calibrate the CDS term structure we need to have σ_t as function of time. This allow us to conclude that we can not use the asset correlation as an input in the copula function while keeping the function $t \to \sigma_t$ flexible to capture the risk spread term structure.

6 Numerical Results

6.1 Single name calibration

As explained above, we start by calibrating the structural and intensity single name models to CDS data. The data used is CDS quotes on 03/10/2004 for FIAT and FORD as displayed in table (1) and table (2).

Tables (3) to (7) display the single name calibration for the structural and intensity models.

Table 1: CDS FIAT quotes in %

Maturities	**Mid**	**Bid**	**Ask**
20-mar-05	**3,425**	**3,25**	**3,60**
20-mar-07	**3,900**	**3,80**	**4,00**
20-mar-09	**4,050**	**4,00**	**4,10**
20-mar-11	**4,150**	**4,00**	**4,30**
20-mar-14	**4,200**	**4,00**	**4,40**

Table 2: CDS Ford quotes in %

Maturities	**Mid**	**Bid**	**Ask**
20-mar-05	**1,105**	**1,11**	**1,10**
20-mar-07	**1,610**	**1,59**	**1,63**
20-mar-09	**2,045**	**2,04**	**2,05**
20-mar-11	**2,120**	**2,07**	**2,17**
20-mar-14	**2,260**	**2,21**	**2,31**

Table 3: Calibration of the Intensity Model to FIAT CDS quotes

Default Intensity Curve		
date	**intensity**	**survival pr**
10-mar-04	**4,880%**	**100,000%**
21-mar-05	**4,880%**	**95,031%**
20-mar-07	**5,945%**	**84,253%**
20-mar-09	**6,165%**	**74,339%**
21-mar-11	**6,408%**	**65,269%**
20-mar-14	**6,252%**	**53,965%**

Table 4: Calibration on the Structural Model to FIAT CDS quotes

Default Structural Curve		
date	**volatility**	**survival pr**
10-mar-04	**24,198%**	**100,000%**
21-mar-05	**24,198%**	**94,957%**
20-mar-07	**16,690%**	**84,173%**
20-mar-09	**17,842%**	**74,294%**
21-mar-11	**20,198%**	**65,263%**
20-mar-14	**23,113%**	**54,063%**

Table 5: Calibration of the Intensity Model to FORD CDS quotes

Default Intensity Curve		
date	**intensity**	**survival pr**
10-mar-04	**1,837%**	**100,000%**
21-mar-05	**1,837%**	**98,100%**
20-mar-07	**3,141%**	**92,055%**
20-mar-09	**4,694%**	**83,685%**
21-mar-11	**3,929%**	**77,268%**
20-mar-14	**4,593%**	**67,194%**

Table 6: Calibration on the Structural Model to FORD CDS quotes

Default Structural Curve		
date	**volatility**	**survival pr**
10-mar-04	**21,031%**	**100,000%**
21-mar-05	**21,031%**	**98,082%**
20-mar-07	**13,408%**	**92,013%**
20-mar-09	**15,367%**	**83,643%**
21-mar-11	**14,640%**	**77,237%**
20-mar-14	**17,257%**	**67,216%**

Table 7: The values of the initial barriers of the structural model

Company	**H**
Fiat	0,6164486
Ford	0,604486048

6.2 Multiname calibration

Now that the the marginal distribution have been calibrated to real data CDS quotes, we calibrate, as explained in section 4.1.1 and 4.2.1, the copula dependence parameters in the intensity model to the correlation parameter in the structural model. The "bridge" between the two models is built through the pricing of first to default protection leg calibration. Table (8) and (9) shows the difference between the default times correlation and the equity correlation. It is clear from these tables that the approach adopted by market participants, i.e. using the equity correlation parameter as an input the gaussian copula, is not the right thing to do.

Table 8: Equity correlation

	Ford	ATT	WMT	Liberty	CAT	DUK
Ford	1	0.233096	0.187524734	0.220291	0.442007	0.165346
ATT	0.233096	1	0.118501339	0.079341	0.144375	0.072367
WMT	0.187525	0.118501	1	0.134449	0.228967	0.194373
Liberty	0.220291	0.079341	0.134449215	1	0.300304	0.156763
CAT	0.442007	0.144375	0.22896701	0.300304	1	0.173284
DUK	0.165346	0.072367	0.194373495	0.156763	0.173284	1

Table 9: Default time correlation

	Ford	ATT	WMT	Liberty	CAT	DUK
Ford	1	0.423299	0.331825464	0.39758	0.835622	0.287451
ATT	0.423299	1	0.19442353	0.117735	0.245664	0.104204
WMT	0.331825	0.194424	1	0.22596	0.415005	0.345555
Liberty	0.39758	0.117735	0.225959927	1	0.557841	0.270326
CAT	0.835622	0.245664	0.415004743	0.557841	1	0.303316
DUK	0.287451	0.104204	0.345555468	0.270326	0.303316	1

7 Conlusion

In this paper we built the bridge between the structural models and intensity models based only on market information: Choose as a first possible bridge a first to default contract with protection payment at maturity. Then the protection leg price is essentially the survival copula (the complementary copula) in the intensity model above, whereas we may resort to multivariate barrier option formulas for the first to default in the structural model. Set then the equity correlation rho in the structural model to a value, compute the first to default price, and then invert the survival copula parameter theta in the intensity model so that the price obtained is the same attained with the structural model. This leads to two models that are calibrated to the same single name CDS data and that agree on the first to default price. In Brief, this is a consistent way to find the correlation parameter we need to put in the gaussian copula to price multi-name credit derivatives. For future work, we need to quantify the improvement obtained by using our approach.

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