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## An Econometric Model of the Term Structure of Interest-Rate Swap Yields

DARRELL DUFFIE and KENNETH J. SINGLETON\*

### ABSTRACT

This article develops a multi-factor econometric model of the term structure of interest-rate swap yields. The model accommodates the possibility of counterparty default, and any differences in the liquidities of the Treasury and Swap markets. By parameterizing a model of swap rates directly, we are able to compute model-based estimates of the defaultable zero-coupon bond rates implicit in the swap market without having to specify a priori the dependence of these rates on default hazard or recovery rates. The time series analysis of spreads between zero-coupon swap and treasury yields reveals that both credit and liquidity factors were important sources of variation in swap spreads over the past decade.

ALTHOUGH *PLAIN VANILLA* FIXED-for-floating interest-rate swaps comprise a major segment of the fixed-income derivative market, notably few econometric models for pricing swaps have been developed in the literature. Perhaps the primary reasons for this are: (i) swap contracts embody default risk and hence equilibrium or arbitrage-free term structure models developed for default-free government bond markets are not directly applicable to the swap market; (ii) empirical modeling of the default event underlying credit spreads on defaultable bonds and swaps has met with limited success at explaining the time-series properties of spreads; and (iii) swap spreads are likely to depend on other factors such as liquidity that are not directly related to default events. Also, until recently, data have not been widely available. In this article we develop a multifactor econometric model of the term structure of U.S. fixed-for-floating interest-rate swap yields that accommodates many of the institutional features of swap markets. Specifically, using results in Duffie and Singleton (1996), we show that the fixed payment rate of a swap, assuming that the floating rate is London Interbank Offering Rate (LIBOR), can be expressed in terms of present values of net cash flows of the swap contract

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discounted by a default and liquidity-adjusted instantaneous short rate. In other words, there is an adjusted short rate process that allows us to develop a term structure model for the swap market in much the same way that models have been developed for government yield curves. Default and liquidity risks are “collapsed” into a risk-adjusted short rate for computing the present values of future risky cash flows.

Our formulation provides a model-based alternative to swap valuation models used by many investment and commercial banks. Many financial institutions use forward interest rates obtained by interpolating between fixed maturity points on the swap yield curve to derive discount rates for future net cash payments between the swap counterparties. Instead, we develop an arbitrage-free multifactor term structure model of swap yields that leads to an implied, model-based zero curve that can be used in valuing off-the-run swaps, caps, floors, and swaptions (options to enter into a swap contract at some future date). Moreover, unlike interpolation schemes, our model provides a framework for dynamically hedging swaps or derivatives, such as caps and swaptions.

The literature on swap rates has typically focused on the spreads between swap rates and the corresponding point on the U.S. Treasury (default-free) term structure. Sun, Sundaresan, and Wang (1993) examine the average swap spreads to Treasuries and the bid/ask spreads by the credit class of the swap counterparties. Brown, Harlow, and Smith (1994) regress swap spreads on various contemporaneous measures of credit risk and the hedging costs of market makers. They find that both sets of variables are correlated with swap spreads. Neither study develops a dynamic swap-pricing model.

Grinblatt (1995) argues that liquidity risk is a more plausible explanation for swap spreads than credit risk. Liquidity enters his model as a convenience yield to Treasuries associated with their relative liquidity and potential to go “on special” in markets for repurchase agreements (repo). Holders of Treasury bonds that go on special can effectively receive a special dividend by borrowing at below market rates, using Treasuries as collateral. (See also Duffie (1996) and Jordan and Jordan (1997).) Using the one-month LIBOR-Treasury bill spread as a proxy for the convenience yield of holding government securities, Grinblatt calibrates a Vasicek (1977) representation of riskless and convenience-yield processes. He finds that this model explains about 35 to 40 percent of the variation in swap spreads for maturities of two through ten years.

Instead of focusing on swap spreads to the default-free term structure, we focus on swap yields directly. Our goal is to develop a model of the swap market without needing to be precise a priori about the economic mechanisms that generate swap spreads. Subsequent to fitting the model, we can study the properties of the defaultable zero-coupon yields implied by the swap market with the goal of a better understanding of the economic factors that determine their spreads to Treasury zero-coupon yields. In Section I we show that, under the assumption that the counterparties have symmetric probabilities of default, a swap is “priced” by the present value of its cash flows discounted by a risk- and liquidity-adjusted short-rate process. Once one adopts a parameter-

ization of this short-rate process, the parameters of the model can be estimated without having to specify a priori the functional forms for default probabilities, liquidity premiums, and so forth. Virtually any of the models examined previously for government yield curves can be used to model the default-adjusted short-rate process, including affine processes for riskless rates and Heath-Jarrow-Morton (1992) type models of forward rates.<sup>1</sup>

In Section II, we discuss the econometric models of defaultable swap yields. The econometric model of swaps studied in Section III presumes that the liquidity- and default-adjusted short-rate process is the sum of two independent square-root diffusions (a *2-factor* model). Assuming that the model exactly prices swaps at two points along the swap yield curve, and using the fact that the conditional distribution of discretely sampled data from a square-root diffusion has a noncentral chi-square density (e.g., Cox, Ingersoll, and Ross (1985) (CIR)), we estimate our model using the joint likelihood of swap yields at several maturities.<sup>2</sup> The distributions of the fitted swap yields, evaluated at the maximum likelihood (ML) estimates, have sample moments that are similar to those of the corresponding actual yields. Moreover, deviations between the actual and fitted swap yields are on average zero, with standard deviations between four and seven basis points for seven years of weekly data. (By comparison, the bid/ask spreads in the swap market averaged about four basis points during our sample period.)

In light of the close fit of the model to swap yields, we proceed to compute the implied risky zero-coupon bond yields, evaluated at the ML estimates, and compute their spreads to the corresponding U.S. Treasury zero-coupon yields. In Section IV the properties of these swap zero-coupon yield spreads are studied in the context of a multivariate vector autoregression (VAR) in an attempt to shed some light on the relative importance of liquidity and credit factors in the determination of swap spreads. Included in the VAR, along with swap zero spreads, are variables proxying for corporate credit risk and liquidity in the U.S. Treasury market. The results suggest that both liquidity and credit factors affect the temporal behavior of swap zero spreads, but that the responses of swap spreads to changes in these factors follow very different time paths. Liquidity effects are short-lived, whereas responses to credit shocks are weak initially and then increase in importance over a horizon of several months. Concluding remarks are presented in Section V.

## I. Valuation of Swaps

Consider a set of  $M$  plain vanilla fixed-for-floating swaps. The  $m$ th swap has  $\tau_m$  years to maturity. The floating side is reset semi-annually to the six-month LIBOR rate from six months prior. The fixed side pays a coupon  $c^m$  at the reset

<sup>1</sup> See Duffie and Singleton (1996) for a discussion of alternative formulations of risky discount rate processes for valuing defaultable bonds.

<sup>2</sup> This likelihood function has the same form as that studied by Chen and Scott (1993) and Pearson and Sun (1994) in their studies of the U.S. Treasury market.

dates. Let  $r_t^L$  denote the LIBOR rate set at date  $t$  for loans maturing six months in the future and  $PV(t, t + \tau_m)$  denote the present value of the promised net cash flows between the parties to the swap agreement. We assume that, at the inception date of the swap,

$$0 = PV(t, t + \tau_m) = \sum_{j=1}^{2\tau_m} E_Q \left[ \exp \left( - \int_t^{t+0.5j} R_s ds \right) (c_t^m - r_{t+0.5(j-1)}^L) \middle| \mathcal{F}_t \right], \quad (1)$$

where  $E_Q$  denotes expectation under an equivalent martingale measure for the information sets  $\{\mathcal{F}_t : t \geq 0\}$  commonly available to investors, and where  $R$  is an instantaneous discount-rate process defined below. In other words, the fixed-side coupon  $c_t^m$  of the swap is set at date  $t$  so that the present value of the net cash flows exchanged by the counterparties at the reset dates is zero at the inception of the swap, as in equation (1).

We let

$$B_t^\tau = E_Q \left[ \exp \left( - \int_t^{t+\tau} R_s ds \right) \middle| \mathcal{F}_t \right], \quad (2)$$

be the discount factor at time  $t$  for maturity  $\tau$  associated with the short term discount-rate process  $R$ . We further assume that risky zero-coupon bonds are priced at the appropriate LIBOR rate in the interbank lending market, or

$$B_t^{0.5} = (1 + r_t^L)^{-1}. \quad (3)$$

Using equations (1) to (3), the present value of the floating rate payments is  $1 - B_t^{\tau_m}$ . Combining these observations, the coupon rate  $c_t^m$  on the fixed side of the swap can be expressed as

$$c_t^m = \frac{1 - B_t^{\tau_m}}{\sum_{j=1}^{2\tau_m} B_t^{0.5j}}. \quad (4)$$

In deriving the valuation model (1)–(4), we have implicitly made several important assumptions. Since the assumptions needed for equation (4) to hold are significantly weaker than those typically made (see, for example, Litzenberger (1992) for a review of the literature), we briefly examine in more depth the key underlying assumptions.

The discounting in equations (1) and (2) takes the same form as the discounting of default-free cash flows by the riskless, instantaneous interest rate in risk-neutral representations of standard term-structure models. However, models for pricing default-free cash flows are not directly applicable to swaps and LIBOR contracts, because they are defaultable instruments. Nevertheless, using results in Duffie and Singleton (1996), we can interpret  $R$  as a default-adjusted discount rate and, under this interpretation plus additional assumptions outlined below, equation (1) correctly prices a defaultable swap.

More precisely, we define a defaultable claim to be a pair  $((X, T), (X', T'))$  of contingent claims. The underlying claim  $(X, T)$  is the obligation of the issuer. The secondary claim  $(X', T')$  defines the stopping time  $T'$  at which the issuer defaults, and the payment  $X'$  to be received at default. This means that the actual claim  $(Z, \tau)$  generated by a defaultable claim  $((X, T), (X', T'))$  is defined by

$$\tau = \min(T, T'); \quad Z = \begin{cases} X & \text{if } T < T', \\ X' & \text{if } T \geq T', \end{cases} \tag{5}$$

where  $\tau$  is the stopping time at which  $Z$  is actually paid.

The ex-dividend price process  $V$  of any given contingent claim  $(Z, r)$  is defined by  $V_t = 0$  for  $t \geq \tau$  and

$$V_t = \mathbb{E}_Q \left[ \exp \left( - \int_t^\tau r_u \, du \right) Z | \mathcal{F}_t \right], \quad t < \tau. \tag{6}$$

Evaluation of the pricing formula (6) is complicated in practice by the possibility of default reflected in the payoff  $Z$  in equation (5) and that the probability of default will in general be correlated with the short-rate process,  $r$ . To circumvent these difficulties, we follow Duffie and Singleton (1996), and assume that the secondary claim  $(X', T')$  is defined by a hazard rate process  $h$  and a fractional default loss process  $\lambda$ . The hazard rate  $h$  can be thought of as the arrival intensity of a Poisson process whose first jump occurs at default. The state dependent process  $\lambda$  defines the fraction of *market value* of the claim that is lost upon default.

Under mild technical regularity conditions given in Duffie and Singleton (1996), the valuation of this defaultable claim can proceed as if the promised payoff  $X$  is default-free; however, with discounting at a default-adjusted discount rate  $R$  instead of the riskless rate. Consequently, under those conditions,

$$V_t = \mathbb{E}_Q \left[ \exp \left( - \int_t^T R_s \, ds \right) X | \mathcal{F}_t \right], \quad t < T', \tag{7}$$

where

$$R_t = r_t + h_t \lambda_t. \tag{8}$$

We interpret each promised cash flow of a swap contract as one of the promised payments  $X$  described above. By repeatedly applying the same logic used to derive equation (7) to the sequence of promised cash flows of a swap, we get equation (1). This approach to swap valuation avoids a more complicated structural model based on knowledge of the asset-liability structure of the swap counterparties, in the style of Merton (1974), and as applied to the swaps

market by Rendleman (1992). The development of a structural model seems impractical given our data on generic market swap rates.

Beyond default, a factor that may affect swap spreads to Treasuries is the relative liquidities of the two markets (Grinblatt (1995), Bansal and Coleman (1996)). Therefore, we also include a convenience yield  $l_t$  that allows for the effect of differences in liquidity and repo specialness between the Treasury and swap markets. With this modification, the swap rate  $c^m$  at inception solves equation (1) with the adjusted discount rate,

$$R_t = r_t - l_t + h_t \lambda_t. \quad (9)$$

The processes  $r$ ,  $l$ ,  $h$ , and  $\lambda$  are assumed to be adapted to the investors' information sets  $\{\mathcal{F}_t : t \geq 0\}$  and thus, may be state dependent and mutually correlated.<sup>3</sup>

For the case of swaps, modeling the default time as an inaccessible stopping time, such as a Poisson arrival, seems reasonable because default events, when they do occur, are rarely fully anticipated even a short time before the default. Changing expectations concerning the likelihood of default are captured by the stochastic properties of the hazard rate process  $h$ . Indeed, under the risk-neutral measure  $Q$ , the conditional probability at time  $t$  of default over the next "instant" of time of length  $\Delta t$  is approximately  $h_t \Delta t$ .

The cash payment upon default of a swap contract is based on the market value of the remaining obligations under the terms of the swap and negotiations between the counterparties. If  $PV(t_d, t_d + \tau)$  denotes the value of a  $\tau$ -year swap just prior to the default time  $t_d$ , then  $(1 - \lambda_{t_d})PV(t_d, t_d + \tau)$  is the present value at  $t_d$  of the cash flows ultimately generated by the negotiated settlement. The liability, if any, according to standard International Swap Dealers Association (ISDA) swap contracts, is based on midmarket quotes for similar swaps.

One implication of these observations is that the assumption of no default made by Smith, Smithson, and Wakeman (1988) and Sun, Sundaresan, and Wang (1993), among others, in justifying expressions like equation (4) is unnecessarily strong. The market yields on A-rated LIBOR issues are occasionally at large spreads to U.S. Treasuries, and these spreads fluctuate substantially over time, as we show in Section III. The pricing relation (4) implicitly allows this spread to arise from a spread between  $R$  and the riskless rate  $r$ .

There are still, however, some important implicit assumptions remaining in the valuation model (1)–(4).

#### A. Exogenous Default Risk

The valuation model (1), for a given "credit spread" process  $\lambda h$ , implicitly presumes that revaluation of the swap is not in itself a significant determinant

<sup>3</sup> These processes are also assumed to be jointly measurable in state and time and to satisfy mild integrability conditions described in Duffie and Singleton (1996).

of default likelihood or losses on default. (This endogeneity would exist, for example, with a swap that constitutes the major part of the liabilities of one of the counterparties.) If we were to compensate by allowing  $\lambda h$  to depend endogenously on the market value of the swap, the linear valuation model (1) would be replaced with a nonlinear model described by Duffie and Singleton (1996). This would significantly complicate the econometric model. The fact that we are using generic market quotes for swap rates, and do not depend on the valuation of a swap between two particular counterparties, presumably mitigates the impact of any endogeneity of the default spread  $\lambda_i h_i$ . Moreover, based on the numerical results noted below, the impact of moderate dependence of  $\lambda h$  on the market value of the swap is minimal.

*B. “Refreshed” A-Quality Counterparties*

Discounting the net cash flows at all maturities with the same risky discount rate  $R$  in equation (1) presumes that the counterparties maintain the credit rating underlying generic (say, A-rated) swaps *for the life of the swap contract*. More precisely, at the inception of the swap, the counterparties are presumed to have a credit rating of A. Subsequently, there is the possibility that either counterparty could be upgraded or downgraded (or suffer a change in liquidity) during the life of the swap. This itself is not a problem for pricing in terms of equation (1), since the potential of a change in quality can be captured in the stochastic process for the discount rate  $R$ . However, we do not have data on the yields for specific swaps as they mature, but rather on new A-rated swaps with constant maturity. Thus, we are valuing a hypothetical swap priced by dealers who presume that the counterparties will maintain the quality of newly issued A-rated debt over the life of the swap. This problem is not unique to our model of swaps, but rather is inherent in any model of new-issue rates on defaultable debt from a fixed credit class. Based on the numerical calculations of Duffie and Huang (1996) and Li (1995), described below, there is a minimal impact on swap rates of reasonably anticipated variation in the credit quality of the counterparties over the life of the swap.<sup>4</sup>

*C. Symmetric Credit-Quality*

Related to the point above is the possibility that the two counterparties to a given swap may have different credit quality, either at the inception of the swap, or subsequently. Extending our model to allow for asymmetric credit qualities of the counterparties would add substantial complexity to the pricing model. For example, even if the credit-spreads  $\lambda_A h_A$  of counterparty A and  $\lambda_B h_B$  of counterparty B are each given exogenously, and do not depend on the market value of the swap, in keeping with the exogenous default risk assumption above, there is nevertheless the effect of an endogenous dependence of the

<sup>4</sup> For alternative models of the impact of default risk on swap rates, see Abken (1993), Cooper and Mello (1991), Hull and White (1992), Jarrow and Turnbull (1995), Li (1995), Rendleman (1992), Solnik (1990), Sorensen and Bollier (1994), and Sundaresan (1991).



credit spread on market value if  $\lambda_A h_A \neq \lambda_B h_B$ . This arises from the fact that the current market value of the swap determines the firm whose default risk currently "matters," based on netting provisions and the implications of one-way or two-way ("no fault") payment schemes that are usually adopted in standard swap agreements. (If the market value of the swap to counterparty A is positive at a given point in time, then it is the default risk of counterparty B that "matters" at that moment.) Therefore, the relevant credit spread would become dependent on the market value of the swap, and the nonlinear valuation model discussed in Duffie and Singleton (1996) would apply. Duffie and Huang (1996) and Li (1995) explore theoretically and numerically this aspect of swap contracts, and find that the degree of asymmetry in credit quality is a relatively minor determinant of swap rates for typical interest-rate swaps. (For example, with less than a 10-year maturity and under typical parameters, there is a correction of roughly 1 basis point or less in swap yields for a credit risk asymmetry generating a 100 basis point difference in bond yields.) Thus, there is likely to be negligible misspecification error from proceeding under the assumption that swaps are priced as if there is symmetric counterparty risk.

#### *D. Homogeneous LIBOR-Swap Market Credit Quality*

There is no reason in theory that LIBOR corporate bond rates, including the floating-rate swap payments  $r_t^L$ , should be determined by discounting at the same liquidity and credit adjusted short-rate  $R$  used for discounting net swap payments in equation (1). The default scenarios may be different in the two markets, recovery rates may differ, and the liquidities of the two markets are also typically different. That is, it is a nontrivial assumption that equation (3) holds.<sup>5</sup>

Although these remarks have focused on the default component of swap spreads, there is no presumption in equation (9) that default is the only or even the primary determinant of swap spreads. Empirically,  $l_t$ , which represents other factors that determine effective carrying costs such as liquidity, may be as (or more) important a source of variation in  $R_t - r_t$ .

## **II. Econometric Models of Defaultable Swap Yields**

In this section we discuss two alternative formulations of the adjusted short-rate  $R$  that may lead to econometrically identified models of the term structure of swap rates. One approach is to focus directly on  $R$  and assume that  $R_t = \rho(Y_t)$ , where  $Y$  is a Markov state vector determining the default-adjusted short-rate. Under this approach, no attempt is made to distinguish between

<sup>5</sup> There are many alternative variable-rate payments streams which, in place of LIBOR floating rate payments, would justify equation (4). It is the fact that the LIBOR floating rate payments are made in arrears that forces us to make equation (3) as an assumption. Specifically, if equations (1), (2), and (4) are to hold at all maturities, and the floating rate payment at time  $t$  is known at time  $t - 0.5$ , then one can show by induction, using the law of iterated expectations, that the floating rate payment at time  $t + 0.5$  must be  $(B_t^{0.5})^{-1} - 1$ .

the contributions of the riskless rate  $r$  and the premium  $z_t \equiv -l_t + h_t\lambda_t$ . The second approach parameterizes both  $r$  and the mean loss rate  $z$ .<sup>6</sup>

A. Formulations of the Adjusted Short-rate R

Consider again a generic, defaultable, contingent claim with a promised payoff of  $X$  at maturity date  $T$ . Suppose that there is a Markov (under the equivalent martingale measure  $Q$ ) state-variable process  $Y$ , such that the promised contingent claim is of the form  $X = g(Y_T)$ , for some function  $g$ , and the default and liquidity adjusted short-rate process  $R$  is of the form  $\rho(Y_t)$ , for some function  $\rho(\cdot)$ . Then equation (7) implies that the claim to payment of  $g(Y_T)$  at time  $T$  has a price at time  $t$ , assuming that the claim has not defaulted by time  $t$ , of

$$V(Y_t, t) = E_Q \left[ \exp \left( \int_t^T -\rho(Y_s) ds \right) g(Y_T) \middle| Y_t \right]. \tag{10}$$

We emphasize that the riskless short-rate  $r_t$  and mean loss rate  $z_t = h_t\lambda_t - l_t$  do not enter directly into this pricing model (10), but rather enter implicitly through the default-adjusted short-rate  $R_t = \rho(Y_t)$ . For a given defaultable bond market, parameterizing  $R$  directly, as opposed to separate parameterization of  $r$  and  $z$ , provides less information concerning the mean-loss-rate process  $z$ . On the other hand, this formulation permits empirical characterizations of the adjusted short-rate process  $R$  without a need to commit to a formulation of the credit spread. One reason that this may be attractive is that there may be nondefault factors that determine spreads between bonds of various classes. Many of these nondefault factors can be accommodated in our model by appropriate reinterpretation of the adjusted short-rate process. This robustness is an attractive feature of this econometric modeling strategy when one is most concerned with characterizing the distribution of  $R$ , say for the purpose of computing a zero-coupon yield curve implied by a defaultable yield curve. On the other hand, the robustness highlights the fact that nothing can be learned directly about the default processes  $h$  and  $\lambda$  from this approach.

Pursuing this approach further, suppose  $Y_t = (Y_{1t}, \dots, Y_{nt})'$ , for some  $n$ , solves a stochastic differential equation of the form

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t, \tag{11}$$

where  $B$  is a standard Brownian motion in  $\mathbb{R}^n$  under  $Q$ , and where  $\mu$  and  $\sigma$  are well behaved functions on  $\mathbb{R}^n$  into  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times n}$ , respectively. Then we know from the ‘‘Feynman–Kac formula’’ that, under technical conditions (Friedman

<sup>6</sup> Even if liquidity effects are absent, it is in general not possible to separately identify the components  $h_t$  and  $\lambda_t$  of  $z_t$  using swap data (or corporate bond data) alone. See Duffie and Singleton (1996) for a more in-depth discussion of this point and a demonstration that the hazard and recovery rates can be identified from price information on credit derivatives.

(1975), Krylov (1980)), equation (11) implies that  $V$  solves the backward Kolmogorov partial differential equation

$$\mathcal{D}^{\mu, \sigma} V(y, t) + \rho(y)V(y, t) = 0, \quad (y, t) \in \mathbb{R}^n \times [0, T], \quad (12)$$

with the boundary condition

$$V(y, T) = g(y), \quad y \in \mathbb{R}^n, \quad (13)$$

where

$$\begin{aligned} \mathcal{D}^{\mu, \sigma} V(y, t) = & V_t(y, t) + V_y(y, t)\mu(y) \\ & + (1/2) \text{trace}[V_{yy}(y, t)\sigma(y, t)\sigma(y, t)']. \end{aligned} \quad (14)$$

The identification problem for this modeling strategy is identical to that of standard term-structure problems for default-free yield curves. All of the models for the short-rate process  $r$  that have been successfully studied for default-free term structures are also identified, at least in principle, in the case of defaultable bonds, simply by replacing  $r$  with  $R$ . For instance, consider the special case  $Y_t = (Y_{1t}, \dots, Y_{nt})'$ , where  $Y_1, Y_2, \dots, Y_n$  are independent (under risk-neutral probabilities) square-root processes. That is, equation (11) applies with  $\mu_i(y) = \kappa_i(\theta_i - y_i)$ ,  $\sigma_{ii}(y) = \bar{\sigma}_i \sqrt{y_i}$ , for positive constants  $\kappa_i$ ,  $\theta_i$ , and  $\bar{\sigma}_i$ , and  $\sigma_{ij}(y) = 0$ , for  $i \neq j$ . By taking  $\rho$  to be affine, one can then apply the CIR solution to equations (13) and (14) with  $g(y) \equiv 1$ , allowing defaultable zero-coupon bond prices to be computed in closed form. This is the approach taken in Section III.

The same valuation model (12)–(13) applies, under mild technical conditions given in Duffie and Singleton (1996), when the underlying state-variable process  $Y$  is a jump-diffusion, or more general continuous-time Markov process. One merely replaces  $\mathcal{D}^{\mu, \sigma}$  with the infinitesimal generator of  $Y$ . Allowing for “jumps” in credit quality may be useful if one has in mind the potential for unusually large credit-quality events.

### B. Modeling the Mean Loss Rate Process

A second modeling strategy is to parameterize the behavior of the joint process  $(r, z)'$  for the default-free short-rate and the risk-neutral mean-loss-rate processes, respectively.<sup>7</sup> Perhaps the simplest example of this strategy would be a case in which one studies the joint distribution of the returns on a defaultable bond and the associated reference Treasury bond used in pricing the defaultable bond. To be concrete, suppose that  $w = (r, z)'$  follows an “affine” diffusion

$$dw_t = (\alpha_0 + \alpha_1 w_t)dt + \beta(w_t)dB_t, \quad (15)$$

<sup>7</sup> In this section we abstract from the nondefault factors discussed in Section V. If one has observable proxies for liquidity, for example, then the following discussion is extendable to accommodate a liquidity premium in  $R$ .

where

- $B$  is a standard Brownian motion in  $\mathbb{R}^2$  under an equivalent martingale measure  $Q$ ;
- $\alpha_0 \in \mathbb{R}^2$ ;  $\alpha_1$  is a  $2 \times 2$  matrix;
- For each  $i$  and  $j$ , there is a fixed constant  $\gamma_{0ij}$  and fixed vector  $\gamma_{1ij}$  in  $\mathbb{R}^2$  such that  $[\beta(w)\beta(w)']_{ij} = \gamma_{0ij} + \gamma_{1ij} \cdot w$ ; and,
- $\alpha_0$ ,  $\alpha_1$ , and  $\beta$  satisfy joint restrictions<sup>8</sup> for the existence of solutions to equation (15).

Let  $G_t^n$  and  $C_t^n$  denote the prices of default-free and defaultable zero-coupon bonds of maturity  $n$ , respectively. The identification of  $z$  as the instantaneous default premium comes from the simultaneous estimation of the implied pricing equations for  $G_t^n$  and  $C_t^n$ , and the imposition of identifying restrictions on  $\alpha_1$  and  $\beta(\cdot)$ . The latter are necessary because, as shown in Dai and Singleton (1996), there will in general be a family of observationally equivalent affine models in the absence of normalizations in  $\alpha_1$  and  $\beta$ .

As an illustrative example of how information about  $z$  can be inferred from data on  $G_t^n$  and  $C_t^n$ , consider the special case (15) in which

$$dr_t = (\alpha_{0r} + \alpha_{1r}r_t)dt + \sqrt{\gamma_{0rr} + \gamma_{1rr}r_t} dB_t^{(1)} \tag{16}$$

$$dz_t = (\alpha_{0z} + \alpha_{1z}z_t)dt + \sigma_{21}\sqrt{\gamma_{0rr} + \gamma_{1rr}r_t} dB_t^{(1)} + \sqrt{\nu'_{1zz}w_t} dB_t^{(2)}. \tag{17}$$

The default-free term structure is driven by CIR-style factor  $r$  in equation (16). Thus, the parameters  $\alpha_{0r}$ ,  $\alpha_{1r}$ ,  $\gamma_{0rr}$ , and  $\gamma_{1rr}$  are identifiable from information on one or more government bond prices  $G_t^n$ . The parameters of  $r$  and the default process  $z$  are therefore identifiable from information on default-free and defaultable bond prices (see Dai and Singleton (1996) and Duffie and Singleton (1996)). Note that the Brownian motions associated with  $r_t$  and  $z_t$  may be correlated, though  $\sigma_{21}$  and  $\nu_{1zz}$  must satisfy certain existence conditions.

Two special cases of equation (16) and (17) are: (i)  $R_t = Br_t + \varepsilon_t$ , with  $\varepsilon_t$  and  $r_t$  being independent diffusions, and (ii) either  $h_t$  or  $\lambda_t$  is specified a priori at a fixed value with the other assumed to follow a given affine diffusion. In the first case, if the issuer is exposed to interest rate risk, then this could be captured by the assumption that  $B > 1$ . In the second case, the strategy of modeling  $h$  and  $\lambda$  simplifies to this case of modeling  $z$ . One example of the latter simplification is the model of Nielsen and Ronn (1995), who assume that  $\lambda_t$  is fixed at either 0.0 or 0.5, the drift of  $z$  is independent of  $r$ , and the diffusion coefficient for  $z$  is constant over time. The affine example (17) illustrates how these special cases can easily be extended to allow the mean-loss-rate process  $z$  to depend on the riskless interest rate. This seems like a potentially important extension in light of the documented cyclicity in credit spreads.

<sup>8</sup> These restrictions are basically that the state vector does not enter a region where the square root of a negative number would have to be taken, as for example with a negative drift at zero in the Cox-Ingersoll-Ross (1985) model. See Duffie and Kan (1996) for details.

More generally, credit spreads are related to business-cycle variables, as shown by Bernanke (1990), Friedman and Kuttner (1993), Jaffee (1975), and Stock and Watson (1989). Among the more prominent variables are general stock market returns, measures of output growth, panel data on consumer sentiment, and levels of capital investment. There may be value in pursuing empirical models in which the risk-neutral expected loss rate process  $z$  is linked with such macro-economic variables. The state vector in equation (15) could be expanded to introduce macro information, although the observation frequency of many macro series might limit the applications of such models.

These examples lead to closed or nearly closed-form expressions for the defaultable zero-coupon bond prices in terms of  $r_t$  and  $z_t$ . However, there is no need to restrict attention to analytic solutions for zero-coupon prices. As long as it is computationally feasible to compute the zero-coupon bond prices numerically at the same time that the objective function defining the estimator of the unknown parameters governing the process  $(r, z)$  is being optimized, our preceding comments continue to apply. Of course, the correlation structure of  $r_t$  and  $z_t$  must be such that the parameters are identified.

### III. An Econometric Model of Swap Yields

In this section we implement the first estimation strategy under the assumption that the default and liquidity adjusted short-rate process  $R$  is a linear combination of independent square-root diffusion models. To be concrete, let  $Y_t$  be a Markov state vector that determines the current risk-adjusted short-rate  $R$ . Let  $\mathcal{B}(Y_t, \beta_o)$  denote a vector of  $M_1$  prices of defaultable zero-coupon bonds and  $\mathcal{C}(Y_t, \beta_o)$  denote a vector of  $M_2$  yields on newly issued swaps ( $M = M_1 + M_2$ ) implied by the term-structure model. The parameter vector governing the probability model of the state process  $Y$  is  $\beta_o$ . Also, let  $\varepsilon_t$  denote an  $M$ -vector of measurement errors contaminating the observed counterparts of the zero prices and yields,  $\mathbf{B}_t$  and  $\mathbf{C}_t$ , that are independent of the state vector  $Y$ . Then the econometric model takes the form

$$\begin{pmatrix} \mathbf{B}_t \\ \mathbf{C}_t \end{pmatrix} = \begin{pmatrix} \mathcal{B}(Y_t, \beta_o) \\ \mathcal{C}(Y_t, \beta_o) \end{pmatrix} + \varepsilon_t(\beta_o), \quad (18)$$

where, for ease of notation, we combine the parameter vectors governing the probability laws of  $Y$  and  $\varepsilon$  into  $\beta_o$ .

The special case of equation (18) that we will study is the multifactor square root model, with

$$dY_t^i = \kappa_i(\theta_i - Y_t^i)dt + \sigma_i \sqrt{Y_t^i} dW_t^i, \quad i = 1, 2, \quad (19)$$

where  $(W^1, W^2)$  is a standard Brownian motion in  $\mathcal{R}^2$ . The positive scalars  $\kappa_i$ ,  $\theta_i$ , and  $\sigma_i$  have interpretations in terms of mean-reversion, steady-state mean, and volatility, respectively, that have been developed by Feller (1951) and, in the context of term-structure models, by CIR. In our empirical analysis we take

$R_t \equiv Y_t^1 + Y_t^2$ , in which case equation (2) and the independence of  $\{Y^1\}$  and  $\{Y^2\}$  under an equivalent martingale measure  $Q$  leave us with

$$B_t^\tau = \prod_{i=1}^2 p_i(Y_t^i, \tau), \tag{20}$$

where  $p_i(y, \tau) = a_i(\tau)e^{-b_i(\tau)y}$ , for

$$a_i(\tau) = \left[ \frac{2\gamma_i \exp[(\gamma_i + \kappa_i + \lambda_i)\tau/2]}{(\gamma_i + \kappa_i + \lambda_i)[\exp(\gamma_i\tau) - 1] + 2\gamma_i} \right]^{\alpha(i)}, \tag{21}$$

and

$$b_i(\tau) = \frac{2[\exp(\gamma_i\tau) - 1]}{(\gamma_i + \kappa_i + \lambda_i)[\exp(\gamma_i\tau) - 1] + 2\gamma_i}. \tag{22}$$

$\alpha(i) = 2\kappa_i\theta_i/\sigma_i^2$ ,  $\gamma_i = ((\kappa_i + \lambda_i)^2 + 2\sigma_i^2)^{1/2}$ , and  $\lambda_i$  denotes a risk-premium coefficient explained by CIR. That is, each  $p_i$  is the form of a zero-coupon bond price in a univariate square-root diffusion model, and the price of a defaultable zero-coupon bond in our setting is the product of the univariate bond price formulas.

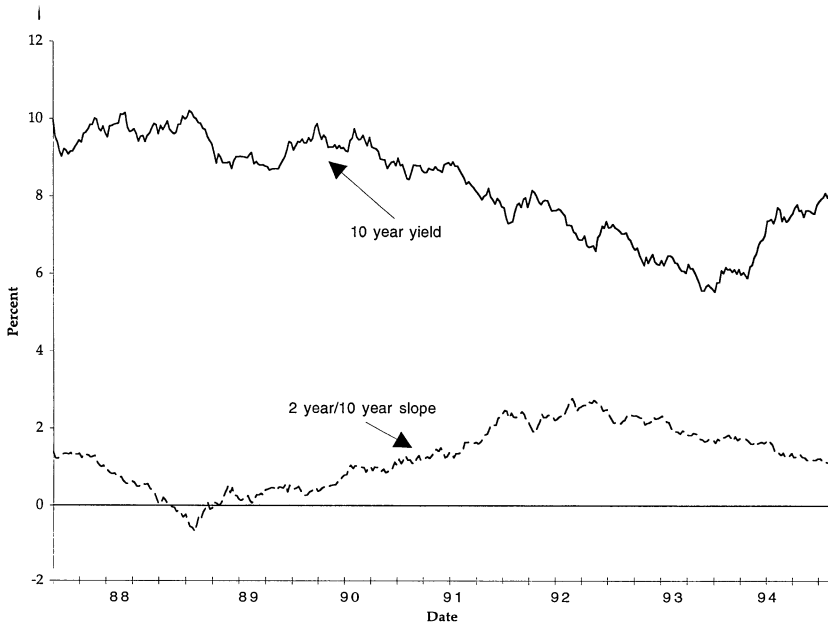
The state vector  $Y$  is unobservable. Therefore we proceed, as do Chen and Scott (1993), under the assumption that two elements of the yield vector  $\{\mathcal{B}, \mathcal{C}\}$  (corresponding to the number of state variables) are measured without error. The pricing model can then be inverted to express the state variables as functions of these swap and LIBOR yields. Alternatively, we could allow all of the swap and LIBOR yields to be measured with error and then estimate the model using the simulated method of moments (Duffie and Singleton (1993)). While this strategy would permit us to obtain consistent estimates of the parameters, the state variables would be unknown. In principle, the latent state variables could be estimated using filtering methods. However, the approach taken here has several potential advantages for pricing, including having direct observations of the state variables and forcing the model to fit a subset of the swap yields exactly. These are important considerations for valuing derivative claims based on swap or LIBOR yields. The model we examine assumes that the two- and ten-year swap rates satisfy

$$c_t^\tau = \frac{1 - B_t^\tau}{\sum_{j=1}^{2\tau} B_t^{0.5j}}, \tag{23}$$

for  $\tau = 2, 10$ , with no measurement errors.<sup>9</sup> Since equation (23) is a nonlinear function of the state variables, inferring  $(Y_t^1, Y_t^2)$  from  $(c_t^2, c_t^{10})$  must be done numerically for each observation.

<sup>9</sup> We also examine a model in which the price of a six-month LIBOR instrument satisfies

$$-\ln B_t^{0.5} = \ln a_1(0.5) - b_1(0.5)Y_t^1 + \ln a_2(0.5) - b_2(0.5)Y_t^2,$$



**Figure 1. Level and Slope of Swap Yield Curve, January 8, 1988 to October 28, 1994.**

We make one additional modification to the model. The square-root parameterization precludes negative state variables. As can be seen from Figure 1, the swap data exhibit substantial changes in the level of rates (as measured by the 10-year yield) and in the magnitude and sign of the slope of the swap curve (as measured by the 2-to-10-year yield spread). We are unable to find admissible parameters for equations (19) and (20) for which the implied state variables  $Y^1$  and  $Y^2$  are positive for the entire sample period. Therefore, we modify the discount process to be

$$R_t = Y_t^1 + Y_t^2 - \bar{y}, \quad (24)$$

where  $\bar{y}$  is a positive constant.<sup>10</sup> With this modification, the zero-coupon bond price defined by equation (20) is replaced with

$$B_t^\tau = \prod_{i=1}^2 p_i(Y_t^i, \tau) e^{-\bar{y}\tau}. \quad (25)$$

and in which equation (23) holds for  $\tau = 10$ . We will comment briefly in the next section on the comparative fit of this alternative model.

<sup>10</sup> While the inclusion of  $\bar{y}$  solves the problem of negative state variables, the adjusted discount rate  $R$  may be negative. In this respect, the modified model differs from the original square-root diffusion specification. Pearson and Sun (1994) study the same specification of a two-factor model of the riskless rate  $r$  underlying U.S. Treasury yields (their "extended model").

The conditional densities of the state variables  $\{Y^1, Y^2\}$  are well known to be non-central chi-square (e.g., CIR). Through a change of variables, these conditional densities can be rewritten as functions of observed swap yields multiplied by the Jacobian of the transformations (23). The Jacobian is nonlinear and time dependent.

Measurement errors are included in virtually all econometric models of the term structure because the state vector  $Y$  usually has low dimension (say, three or less) relative to the number of yields  $M$ .<sup>11</sup> Thus, without additional sources of uncertainty, the models would imply deterministic relations among prices or yields that are clearly violated in the data. In constructing the likelihood function, we assume that the (nonzero) measurement errors  $\{\varepsilon_t\}$  for the swap yields follow univariate AR(1) processes, with innovations that are normally distributed and that may have nonzero correlation. The log-likelihood function of the swap yield data is the sum of the log-density of the noncentral chi squares of the square-root processes, adjusted for  $\bar{y}$ , and the log-density of the multivariate normal associated with the measurement errors.<sup>12</sup> We estimate the swap-pricing model using weekly data from January 4, 1988 through October 28, 1994. The swap yields are constructed as follows. Weekly data on constant-maturity U.S. Treasury bond yields are constructed by concatenating yields on the current, on-the-run Treasury bonds for maturities two, three, five, seven, and ten years. The average of the quoted bid/ask swap spreads are then added to these Treasury series to obtain the swap yield data.<sup>13</sup> The short-term, six-month rate is taken to be the dollar LIBOR.

Initially, several one-factor models are fitted with models indexed by the point on the swap curve used to extract the single state variable  $Y^1$ . In all of these models, the yield curves evaluated at the ML estimates tended to be too flat on average compared to the actual yield curves. Additionally, the one-factor models are unable to fit simultaneously the volatilities of changes in yields at the long and short ends of the swap curve. Finally, the deviations between actual and model-implied slopes of the swap curve exceeds 50 basis points for several extended periods. These poor results motivate our focus on a two-factor model. The state variables for the two-factor model are extracted from the two- and ten-year swap rates. The rates assumed to be measured with error are those on the three-, five-, and seven-year swaps. LIBOR is excluded from the econometric model, but is used subsequently in assessing the fit of the

<sup>11</sup> Pearson and Sun (1994) proceed instead by forming portfolios of bond yields and using as many portfolios as there are unobserved state variables. Thus, their model is presumed to fit the portfolio yields exactly. We have chosen to work directly with a cross-section of maturities of swap yields in order to assess the fit at specific points along the swap yield curve, and because a cross-section of swap yields embodies more information about the yield curve than portfolios do.

<sup>12</sup> See Chen and Scott (1993) and Pearson and Sun (1994) for further discussion of the likelihood function for square-root diffusions. Our likelihood function differs from that used by Chen and Scott because of the nonlinear Jacobian for swap yields. We are grateful to Qiang Dai for developing the approximation to a noncentral chi square distribution used in our numerical routines.

<sup>13</sup> The swap data are taken from the Telerate brokers screens and represent average bid and ask rates quoted by several large dealers.



**Table I**  
**Estimates of the 2-Factor Model Weekly Data, January 8, 1988 to**  
**October 28, 1994**

The parameters  $\kappa_i$ ,  $\theta_i$ , and  $\sigma_i$  govern the diffusion for the  $i$ th state variable  $Y^i$ ,

$$dY^i = \kappa_i(\theta_i - Y^i)dt + \sigma_i \sqrt{Y^i} dB^i,$$

and  $\lambda_i$  is the associated risk premium. The  $\rho$ s are the autocorrelations of the measurement errors. Two sets of standard errors are given in parentheses below each estimate: the first is based on the outer product of the score and the second is based on the usual Hessian of the likelihood function.

Panel A: Parameters of Diffusion $Y^1$			
$\kappa_1$	$\theta_1$	$\sigma_1$	$\lambda_1$
0.544	0.374	0.023	-0.036
(0.052)	(0.444)	(0.014)	(0.046)
(0.029)	(0.017)	(0.0005)	(0.012)
Panel B: Parameters of Diffusion $Y^2$			
$\kappa_2$	$\theta_2$	$\sigma_2$	$\lambda_2$
0.003	0.258	0.019	-0.004
(0.016)	(0.874)	(0.014)	(0.012)
(0.0004)	(0.010)	(0.001)	(0.0006)
Panel C: Autocorrelations of Measurement Errors			
$\rho_{3yr}$	$\rho_{5yr}$	$\rho_{7yr}$	
0.777	0.836	0.871	
(0.025)	(0.017)	(0.013)	
(0.007)	(0.006)	(0.005)	

two-factor model.<sup>14</sup> Thus, the model has twelve parameters: four each for the two state variables (including the risk premia), the adjustment parameter  $\bar{y}$ , and the autocorrelations of the measurement errors for the three-, five-, and seven-year swap rates. The parameter estimates are displayed in Table I, with two sets of standard errors. The first is computed from the outer product of the score vector, and the second is based on the usual Hessian matrix of the log-likelihood function. As is commonly the case in small samples, the two estimates differ.

The estimate of the adjustment factor  $\bar{y}$  is 0.58, with estimated standard errors of 0.58 and 0.003. Together with the estimates of the  $\alpha_i$ , this implies an

<sup>14</sup> Initially, we keep LIBOR in the econometric model as a defaultable zero rate measured with error, but find that the model does not fit well relative to the version with LIBOR omitted altogether. Therefore, we report estimates for the version excluding LIBOR, and then examine the fit of the implied six-month swap rate to LIBOR. The previous draft of this article also presents estimates for a two-factor model in which the state variables are extracted from LIBOR and the ten-year swap rate. The fitting errors for the various swap rates are large and often over 60 basis points.

estimated long-run mean  $\bar{R} \equiv \theta_1 + \theta_2 - \bar{y}$  of  $R$ , of 5.2 percent.<sup>15</sup> The risk-premium coefficients  $\lambda_1$  and  $\lambda_2$  are both negative, which implies that the term premiums are, on average, positive as maturity increases. In the context of pricing defaultable bonds, excess returns over the instantaneous rate  $R$  may reflect term premiums in the underlying riskless term structure and/or an increasing term structure of average credit spreads. The term structure of credit spreads for the zero-coupon bond yields implied by our swap-pricing model are subsequently examined in more depth.

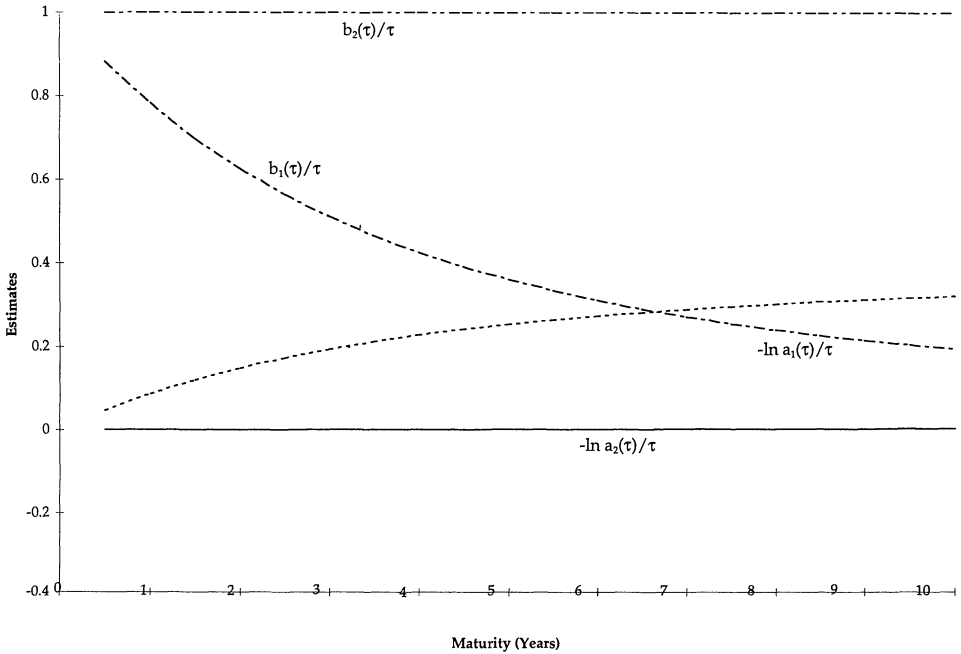
The estimates of the mean-reversion parameters  $\kappa_1$  and  $\kappa_1 + \lambda_1$  are much larger than the corresponding estimates for the second factor. Indeed,  $\kappa_2 + \lambda_2$  is very close to zero, suggesting that there is at most weak mean reversion in the second state variable. To interpret these findings, it is informative to examine the relations between the state variables extracted from the swap yields ( $\hat{Y}^1$  and  $\hat{Y}^2$ ) and the swap yield curve.<sup>16</sup> The sample correlation of  $\Delta\hat{Y}^1$  with changes in the slope of the swap curve (10 year–2 year) is approximately  $-0.99$ , whereas the correlation of  $\Delta\hat{Y}^1$  with changes in the ten-year swap yield is  $-0.02$ . The correlations between  $\Delta\hat{Y}^2$  and changes in the two-, five-, and ten-year swap yields are 0.60, 0.78, and 0.93, respectively. Thus, the first factor behaves like the negative of the slope of the swap curve, and the second factor behaves like the level of the ten-year swap yield.

Further insights come from examination of the coefficients of the zero-coupon bond yields,  $-\ln B_i^r/\tau$  (see Fig. 2). From equation (20), these coefficients are  $-\ln a_i(\tau)/\tau$  and  $b_i(\tau)/\tau$  for the two state variables and  $\bar{y} = 0.580$  for the scale factor in equation (25). Across the maturity  $\tau$ , the coefficients  $b_2(\tau)/\tau$  are nearly constant at unity and the coefficients  $-\ln a_2(\tau)/\tau$  are essentially zero. It follows that the second factor represents a parallel shift in the entire zero-coupon yield curve induced by changes in  $Y_t^2$ . In light of the correlations of swap yields and state variables, this parallel-shift factor is well proxied empirically by the ten-year swap yield. In particular, the likelihood function is not maximized by selecting a short-term swap rate as the “level” risk factor. Figure 2 also shows that  $b_1(\tau)/\tau$  declines with maturity  $\tau$  such that positive shifts in  $Y_t^1$  induce flattenings in the slope of the swap curve. The findings that  $\Delta Y_t^1$  and slope of the swap curve were nearly perfectly negatively correlated, and that  $b_1(\tau)/\tau$  is large for small values of  $\tau$ , suggest that slope changes were associated with greater variation in short-maturity yields during the sample period. The rapid decline in  $b_1(\tau)/\tau$  as  $\tau$  increases is induced in the square-root model largely by a fast rate of mean reversion of the first factor (large  $\kappa_1$ ).

The terms  $-\ln a_1(\tau)/\tau$  and  $-\ln a_2(\tau)/\tau$  contribute to an inherent upward slope in the zero-coupon yield curve, as the former is upward sloping, while the latter

<sup>15</sup> We are grateful to Mark Fisher for pointing out that the long-run mean of the  $R$  was implausible in an earlier version of this article. Upon exploring the likelihood frontier further, we found a higher value of the likelihood function with the parameter estimates reported in Table 1 and, in particular, a plausible value for  $\bar{R}$ . The time-series properties of the fitted swap rates are unchanged compared to the previous results.

<sup>16</sup> More precisely, time series on the two state variables are computed by inverting the pricing model evaluated at the ML estimates for each date of the sample.



**Figure 2. Coefficients of Zero Coupon Bond Yields.** The price of a zero-coupon bond with maturity  $\tau$  is

$$B_t^\tau = a_1(\tau)a_2(\tau)\exp\{b_1(\tau)Y^1(t) + b_2(\tau)Y^2(t)\},$$

where  $Y^1(t)$  and  $Y^2(t)$  are the two state variables. Therefore, the yield on a  $\tau$ -year zero is

$$-\ln B_t^\tau/\tau = -\ln a_1(\tau) - \ln a_2(\tau) + (b_1(\tau)/\tau)Y^1(t) + (b_2(\tau)/\tau)Y^2(t).$$

This figure displays these coefficients of zero yields as functions of maturity  $\tau$ .

is essentially zero for all maturities. Thus,  $Y^2$  represents nearly a pure parallel shift. On the other hand, at  $(Y^1 = 0, Y^2 = 0)$ , the yield curve is upward sloping due to the contribution of  $-\ln a_1(\tau)/\tau$ . It follows that the slope of the yield curve is matched by adjusting  $Y^1$ , and that matching an inverted swap curve may require a large value of  $Y^1$  to offset the upward slope induced by  $-\ln a_1(\tau)/\tau$ . In such cases, the resultant large value of  $Y_t^1 b_1(\tau)/\tau$  (the mean of  $\hat{Y}_t^1$  is 0.37) would clearly overstate the level of zero-coupon yields, especially for short maturities. Thus, in order to simultaneously explain the temporal behavior of the slope of the swap curve and fit the average level of the curve, an adjustment is necessary. This explains the economically significant value of the constant adjustment of  $\bar{y}$ .

Descriptive statistics of the swap yields implied by the models are presented in Table II. Panel A displays the sample means of the levels and first differences of the historical swap yields (“Sample”), as well as the swap yields implied by the two-factor model. The model matches the average three-, five-,

**Table II**  
**Descriptive Statistics from the 2-Factor Model**

Panel A displays the sample means of the historical (Sample =  $c^\tau$ ) and model-implied (Model =  $\hat{c}^\tau$ ) rates on the London Interbank Offering Rate (LIBOR) and  $n$ -year swap contracts for  $n = 3, 5,$  and  $7$ . Results for the level of rates ( $c^\tau$ ) and changes in rates ( $\Delta c^\tau$ ) are presented. Panel B presents the analogous results for the sample standard deviations of swap yields. Panel C restricts attention to changes in swap rates ( $\Delta c^\tau$ ) and displays sample historical and model-implied skewness and kurtosis statistics. Panel D displays the standard deviations of the differences between the actual and model-implied swap rates ( $c^\tau - \hat{c}^\tau$ ), and of ( $\Delta c^\tau - \Delta \hat{c}^\tau$ ).

Panel A: Means of Swap Yields					
	$\tau =$	LIBOR	3 yr	5 yr	7 yr
Sample	$c^\tau$	6.303	7.308	7.756	8.028
	$\Delta c^\tau$	-0.005	-0.004	-0.005	-0.005
Model	$\hat{c}^\tau$	6.131	7.309	7.754	8.018
	$\Delta \hat{c}^\tau$	-0.004	-0.004	-0.005	-0.005

Panel B: Standard Deviations of Swap Yields					
	$\tau =$	LIBOR	3 yr	5 yr	7 yr
Sample	$c^\tau$	2.262	1.772	1.526	1.403
	$\Delta c^\tau$	0.143	0.148	0.141	0.136
Model	$\hat{c}^\tau$	2.385	1.756	1.512	1.379
	$\Delta \hat{c}^\tau$	0.183	0.143	0.134	0.132

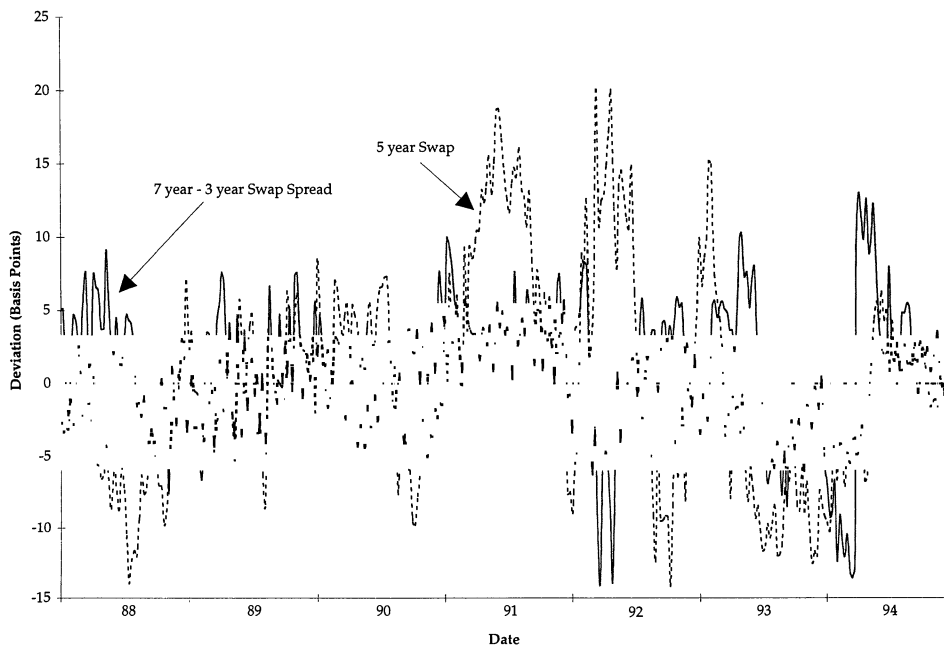
Panel C: Skewness/Kurtosis for Changes in Swap Yields					
		LIBOR	3 yr	5 yr	7 yr
Sample	Skewness	-0.316	0.156	0.218	0.216
	Kurtosis	4.357	3.001	3.071	3.148
Model	Skewness	0.052	0.141	0.152	0.145
	Kurtosis	3.096	2.875	2.878	2.931

Panel D: Standard Deviations of Fitting Errors (Basis Points)					
		LIBOR	3 yr	5 yr	7 yr
	$c^\tau - \hat{c}^\tau$	32.34	4.88	7.16	6.21
	$\Delta c^\tau - \Delta \hat{c}^\tau$	11.35	2.83	3.26	2.71

and seven-year swap yields, as well as the standard deviations of yield changes (Panel B), to the third decimal place. The skewness and kurtosis statistics of the changes in implied swap yields displayed in Panel C are slightly smaller than the corresponding statistics for their sample counterparts.

The two- and ten-year swap yields are fit exactly by construction. As such, perhaps the most challenging rate to fit is the five-year rate, because changes in the curvature of the swap curve may result in movements of the five-year swap rate that are independent of the longer and shorter ends of the swap curve. Figure 3 shows that the model fits the five-year swap yield to within 20 basis points (bp) over the seven-year sample period. The mean error is only 0.3 basis points. Moreover, the sample standard deviation of the measurement error for the level of the five-year swap rate is 7.2 bp (see Table II, Panel D), which was approximately the bid/ask spread during this sample period.



**Figure 3. Deviation from Fit of Swap Yields.** The dashed line is the deviation, in basis points, between the actual and fitted five-year swap rates. The solid line is the corresponding deviation for the seven-year minus three-year swap yield spread.

Figure 3 shows that there is also a close correspondence between movements in the slope of the swap yield curve (7 year–3 year) and the implied slope from the model. The maximal deviation is less than 16 basis points over the sample period, with an average fitting error of 1.1 basis points. The most challenging period to fit seems to have been the trough in swap rates during late 1993 and early 1994.

As additional evidence on goodness-of-fit, we report in Table III, Panel A, the results from regressing the changes in the actual swap yields against their fitted counterparts. If the two-factor model describes the data well, then we would expect that the estimated intercept ( $\hat{\alpha}$ ) and slope ( $\hat{\beta}$ ) to satisfy  $\hat{\beta} \cong 1$  and  $\hat{\alpha} \cong 0$ , and that the  $R^2$ s would be close to one. Regressions are run with differenced yields, because of the high degree of persistence in levels and our desire to evaluate the model's explanatory power for changes. Only the results for LIBOR differ markedly from those predicted by the model. Indeed, it is striking that the model explains approximately 95 percent of the variation of the *changes* in individual swap rates over the seven year sample period. Table II expresses this finding in terms of basis points: The standard deviations of the fitting errors for the swap yields are between 4.48 bp and 7.16 bp.

**Table III**  
**Correlations of Actual and Fitted Yields**

Panel A displays the results from regressing the changes in actual swap rates ( $\Delta c_t^\tau$ ) on the changes in fitted rates from the model ( $\Delta \hat{c}_t^\tau$ ). Standard errors of the estimated intercept ( $\hat{\alpha}$ ) and slope ( $\hat{\beta}$ ) are given in parentheses.  $R^2$  is the coefficient of determination and s.e.e. is the standard error of the residual. Results are reported for the London Interbank Offering Rate (LIBOR) rate and three year, five year, and seven year swap rates. Panel B displays the corresponding results for a model fit with Treasury bond yields instead of swap rates (i.e.,  $c_t^\tau$  is the yield on a  $\tau$ -year Treasury bond).

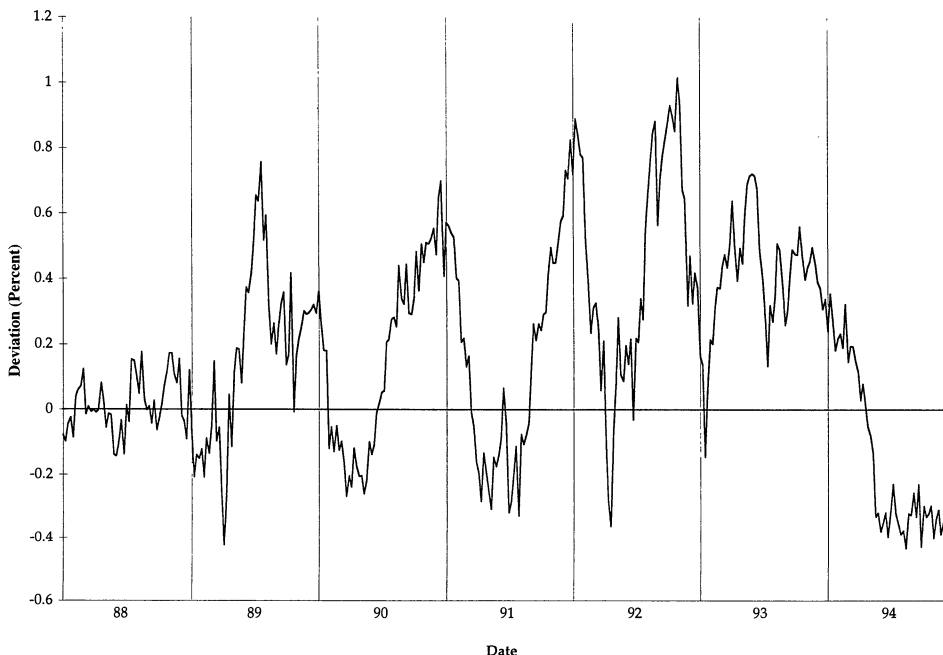
Maturity	$\Delta c_t^\tau = \alpha + \beta \Delta \hat{c}_t^\tau + \varepsilon_t$			
	$\hat{\alpha}$	$\hat{\beta}$	$R^2$	s.e.e.
Panel A: Actual Against Fitted Swap Yield Regressions				
LIBOR	-0.0024 (0.005)	0.615 (0.026)	0.62	0.089
3 Year	0.0002 (0.002)	1.014 (0.010)	0.96	0.028
5 Year	0.0002 (0.002)	1.018 (0.013)	0.95	0.033
7 Year	-0.0000 (0.001)	1.014 (0.011)	0.96	0.027
Panel B: Actual Against Fitted U.S. Treasury Yield Regressions				
3 Year	0.0045 (0.001)	1.039 (0.013)	0.95	0.032
5 Year	0.0006 (0.001)	1.050 (0.012)	0.97	0.137
7 Year	0.0001 (0.001)	1.035 (0.012)	0.97	0.023

For comparison, we reestimate our two-factor square-root model using constant-maturity U.S. Treasury yields for the same sample period.<sup>17</sup> As with swaps, the state variables are extracted from the two- and ten-year maturity instruments. The results from regressing actual on fitted changes in Treasury yields are displayed in Table III, Panel B. The Treasury model also fits reasonably well during this sample period, although the  $\hat{\beta}$  coefficients are significantly different from one at conventional significance levels.<sup>18</sup> Also, the standard error of the estimate (s.e.e.) for the five-year rate is larger for the Treasury model than for the swap model.

Although LIBOR is not used in estimation, the six-month rate implied by the two-factor model can be computed at the ML estimates and compared to LIBOR. The results in Table II show that the fitted LIBOR is about 18 basis

<sup>17</sup> The constant maturity Treasury data are not exactly what one would want in order to fit a model for the U.S. Treasury curve, since the maturities of, say, the ten-year note change slightly and the on-the-run issue is not always at par between auctions. However, for our purposes of providing a benchmark for assessing fit, the data seem adequate.

<sup>18</sup> Standard errors are computed using Hansen's (1982) correction for serial correlation and heteroskedasticity with five noncontemporaneous terms in the asymptotic covariance matrix.



**Figure 4. Deviation from Fit to LIBOR.** This figure displays the deviation in percent between the actual and fitted six-month LIBOR rates.

points too small on average and is more volatile than the actual LIBOR. Moreover, the standard deviation of the fitting error is over 32 bp. The sample skewness of changes in LIBOR is negative and its kurtosis is larger than three, whereas the sample estimates of skewness for the changes in swap rates are positive and the sample estimates of their kurtoses are close to three. Not surprisingly, the model does not explain the negative skewness and excess kurtosis of LIBOR. Together, these findings suggest that the six-month LIBOR series has distinctive characteristics that are not shared by the longer end of the swap yield curve. One interpretation of these results is that LIBOR loans are of a somewhat lower average perceived quality than multi-year swaps of the same credit rating. This is consistent with the evidence presented in Sun, Sunderson, and Wang (1993). Alternatively, it may be that additional non-credit factors are needed to describe simultaneously the distributions of the long and short ends of the swap curve.

The poor fit for LIBOR is displayed in Figure 4. The deviations between actual and implied LIBOR fluctuate substantially and often exceed 50 basis points in absolute value. Moreover, there is evidently a seasonal pattern to the spreads between actual and fitted LIBOR. In every year, there are large changes in the deviations around the calendar year end. (The vertical lines in Figure 4 occur on the last Friday of each calendar year that markets were open.) For end-of-year 1989, 1990, and 1991, the actual-fitted LIBOR differences fell substantially. On the other hand, for end-of-year 1992 and 1993,

there is evidence of an increase, although in the latter case the increase was quickly followed by a steady decline in the actual-fitted LIBOR deviations. These calendar effects may well be related to balance-sheet adjustments by financial institutions concerned about capital requirements or the risk profile of their securities positions at year end. Whatever the source, these results suggest that using a short-term rate as the state variable describing the entire yield curve may lead to misleading conclusions about the shapes of the distributions of long-term swap yields.

How can the poor fit for LIBOR and relatively good fit for the swap rates be reconciled with the assumption of homogeneous LIBOR-swap market credit quality? One reconciliation comes from noting that equations (4) and (18) can be used to fit the model-based discount factors  $B_t^\tau$  implicit in the swap market without reference to the LIBOR market. If, as seems common in practice, swap traders also extract estimates  $\hat{B}_t^\tau$  for pricing swaps from swap rates (and Eurodollar futures) without reference to LIBOR markets, then our econometric analysis provides a model-based construction of the traders' implied discount factors,  $\hat{B}_t^\tau$ , and pricing rules for the longer end of the swap curve. At the short end of the swap curve, additional factors are evidently necessary for our model to generate consistent pricing of Eurodollar futures and LIBOR contracts, that is, to reproduce the discount factor at the short end of the swap zero-coupon curve.

Regressing the deviation between actual and fitted LIBOR (DEVLIB) on the first and second lagged values of itself gives<sup>19</sup>

$$\text{DEVLIB}_t = 0.002 + 0.787 \text{ DEVLIB}_{t-1} + 0.173 \text{ DEVLIB}_{t-2},$$

(0.006)      (0.05)                      (0.05)

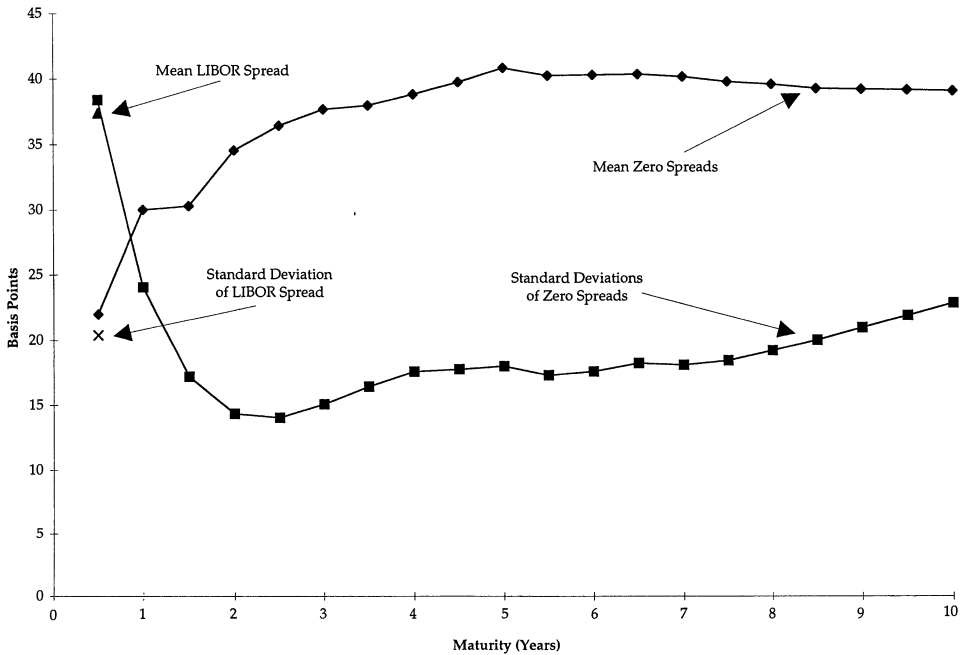
with an  $R^2$  of 0.90 and with a standard error of the residual of 11.5 basis points. The finding that a low-order autoregression explains most of the within-sample variation in the misfitting of LIBOR suggests that the two-factor model could be modified to include a third factor to accommodate the dynamics of the entire swap-LIBOR yield curve. Evidence consistent with this conjecture is presented in Dai and Singleton (1996).

#### IV. Analysis of Implied Swap Zero-coupon Yields

We turn next to an examination of the properties of the zero-coupon bond yields implied by our swap-pricing model. Specifically, we compute the yields  $(-\log B_t^\tau/\tau) \times 100$  (i.e., continuously compounded yields) with  $B_t^\tau$ , given by equation (20), evaluated at the ML estimates. Our focus is on the spreads of these zero-coupon bond yields to their counterparts in the U.S. Treasury market. In light of the small residuals from fitting swap yields, the defaultable zero-coupon yields implied by our model should serve as reasonable proxies for

<sup>19</sup> While supported by a different model and different data, our characterization is largely consistent in this regard with independent work by Brown and Schaefer (1993), who study U.S. Treasury yields.





**Figure 5. Means and Standard Deviations of Swap Zero Spreads, Sample Period: January 15, 1988 to October 2, 1994.** The squares and diamonds are the standard deviations and means, respectively, of the spreads between zero rates implied by the swap and Treasury yield curves. The mean and standard deviation of the actual LIBOR spread to the six-month Treasury bill rate are displayed as a triangle and  $\times$ , respectively.

the defaultable zero yields implicit in the swap curve. For the U.S. Treasury zero-coupon (continuously compounded) yields, we use the discount function implied by a statistical spline model.<sup>20</sup> Whereas the swap curve is sparsely reported and a model-based interpolation scheme seems desirable, the substantial data on coupon U.S. Treasury yields and strip prices are used to compute Treasury zero rates.

Figure 5 displays the term structure of sample means and standard deviations of spreads between the swap and U.S. Treasury zero-coupon bond yields. The term structure of zero-coupon default spreads is, on average, upward sloping during this sample period from about 20 basis points for the six-month spread, up to about 40 basis points for the five-year spread. Beyond five years to maturity, the spreads are nearly constant at about 38 to 40 basis points. The large difference between the means of the fitted and actual spreads of LIBOR to the six-month Treasury rate is consistent with the presence of money-market effects noted earlier that are not captured by the two-factor swap model.

<sup>20</sup> The U.S. Treasury zero-coupon yields, provided to us by Goldman Sachs, are computed from the coupon yields and strip prices.

Spread volatilities tend to be increasing beyond two years to maturity, although volatilities are roughly flat between four and seven years. (The volatilities of the zero-coupon bond yields decline with maturity for both the U.S. Treasury and swap markets.) As maturity declines below two years, the spread volatilities increase. The actual six-month spread volatility in the money market is much lower than the implied spread volatility from the swap model.

To assess the relative importance of the liquidity ( $l_t$ ) and credit ( $\lambda_t h_t$ ) components of  $R_t$  in determining swap zero spreads, vector autoregressions (VAR) are estimated for zero spreads, and for proxies for credit and liquidity. For square-root diffusions, the zero yields are linear in the state variables  $Y^1$  and  $Y^2$ . If the U.S. Treasury curve can similarly be described by a discount function that is the sum of square-root diffusions (Table III suggests that this is approximately so during our sample), then these VARs can be interpreted as linear projections of differences in linear combinations of the state variables driving the risk-adjusted and Treasury interest rate processes onto the variables in the VAR.<sup>21</sup> The variables included in the VARs are the six-month Treasury zero yield (TB6), the spread between the generic three-month repo rate for the ten-year Treasury note and the repo rate of the current on-the-run Treasury note (REPOSP), the spread between rates on BAA- and AAA-rated commercial paper (CPS), and the spread between the ten-year zero rates implied by the swap and Treasury markets (ZEROSP10).<sup>22</sup> All VARs are fit with eight lags.<sup>23</sup>

The inclusion of TB6 is intended to capture the effect of the level of riskless interest rates on spreads for defaultable zero-coupon bonds. CPS captures default risk. Liquidity differences may also contribute to a spread between BAA- and AAA-rated corporate paper, but we expect these differences to be small compared to the effects of the different credit rating.

REPOSP captures the specialness of the Treasury coupon note with the same maturity as the zero being studied. Of course, there is not a simple linear mapping between the specialness of a coupon bond and the spread in the underlying zero markets. However, the specialness in the coupon market is the only set of repo data available, and it should shed some light on the importance of repo effects on swap spreads. An increase in REPOSP (the repo rate on the

<sup>21</sup> Although, under the assumptions of our model, the innovations in the VARs are not normal, the model does imply linear expectations conditional on past  $Y$ s. Therefore, we expect that the VAR analysis will provide suggestive descriptive evidence about the contributions of liquidity and credit factors to swap spreads.

<sup>22</sup> We also fit VARs including the spread between Moody's corporate bond yield indices for BAA- and AAA-rated credits as an additional conditioning variable to proxy for the long-term corporate spreads. However, this spread is not a significant explanatory variable for zero swap spreads, and the proportion of variance of ZEROSP10 explained by the Moody's series is essentially zero.

<sup>23</sup> A Bayesian prior on the lag distribution, which has the own first lag of each variable equal to unity and all of the other coefficients equal to zero, is also imposed. Specifically, in the notation of the manual of the RATS statistical computer package, we use a symmetric prior with TIGHT = 0.15, and 0.5 as the weight on the other variables in the equation. The results are not substantially different with and without the prior.

**Table IV**  
**F-Tests of Exclusion Restrictions in VARs**

This table displays the values of the  $F$ -statistics for testing the hypothesis that all of the coefficients on lagged values of the variable indicated under "Variable Excluded" are zero in the equation with the dependent variable given in the left-hand column. Marginal significance levels of the statistics are given in parentheses. Two VARs are estimated: VAR4 includes the 6-month Treasury bill rate (TB6), the spread between generic and on-the-run repo rates for ten-year Treasury bonds (REOSP), the spread between BAA- and AAA-rated commercial paper (CPS), and the spread between the ten-year zero rates implied by the swap and Treasury markets (ZEROSP10); VAR3 includes TB6, REOSP, and ZEROSP. The \* indicates exclusion of CPS from the VAR3 model.

Dependent Variable	Variable Excluded							
	TB6		REOSP		CPS		ZEROSP10	
	VAR4	VAR3	VAR4	VAR3	VAR4	VAR3	VAR4	VAR3
TB6	476.5 (0.000)	580.53 (0.000)	2.26 (0.024)	1.54 (0.143)	3.71 (0.000)	*	1.06 (0.395)	.573 (0.800)
REOSP	0.54 (0.827)	0.53 (0.830)	72.22 (0.000)	70.37 (0.000)	0.97 (0.459)	*	1.18 (0.352)	1.08 (0.374)
CPS	0.80 (0.602)	*	1.62 (0.121)	*	79.51 (0.000)	*	0.65 (0.738)	*
ZEROSP10	1.42 (0.187)	1.68 (0.104)	2.93 (0.004)	3.06 (0.003)	0.65 (0.739)	*	96.92 (0.000)	101.7 (0.000)

current bond falls relative to the generic repo rate) implies that holders of the current-issue bond receive an extra "dividend." Therefore, as argued by Grinblatt (1995), when the on-the-run Treasury notes go on special, Treasury prices tend to rise and swap spreads tend to increase.<sup>24</sup>

Table IV presents the  $F$ -tests and their marginal significance levels for the exclusion of all eight lags for each of the explanatory variables in the VAR. Two sets of results are displayed: those for a four-variable VAR (VAR4) and those for a three-variable VAR (VAR3) that excludes the commercial paper spread CPS. The test statistics for the ZEROSP10 equation, presented in the last row of Table IV, suggest that the histories of ZEROSP10 and REOSP have significant predictive power for ZEROSP10. The  $F$ -statistic for TB6 is smaller, but significant at the 10 percent level in VAR3. The corporate spread CPS has little explanatory power for ZEROSP10 in VAR4, which is why we also examine VAR3. Interestingly, REOSP is significant or borderline significant at conventional significance levels in all equations of both VARs.

<sup>24</sup> Alternatively, Evans and Parente-Bales (1991) and Brown, Harlow, and Smith (1994) argue that the costs to dealers of hedging net swap exposure is a key factor underlying changes in swap spreads. Brown, Harlow, and Smith (1994) use the repo rate as a proxy for a dealer's costs of hedging a swap exposure with Treasury bonds. A net exposure to the pay fixed side of a swap would be hedged by taking a long position in government bonds, which are financed in the repo market. Thus, they argue that a decrease in the repo rate, by lowering hedging costs, *reduces* swap spreads. The sign of this effect depends, however, on whether in the aggregate dealers are net pay- or receive-fixed counterparties in the market, so this hedging effect may change in sign over time.

Within a VAR system, the effects of, say, CPS on the future course of ZEROSP10 depends not only on the direct effects tested by the  $F$ -statistics in Table IV, but also on the indirect effects of CPS on the other variables in the VAR. CPS has significant predictive power, for example in the equation for TB6. In light of these feedback effects within the multivariate system, it is instructive to examine the impulse-response functions for the VARs. Each function traces out the effects of an innovation in one of the variables on the future course of ZEROSP10, taking into account the dynamic interactions among the variables. The shock is positive with magnitude equal to the estimated standard deviation of the variable's own residual in the VAR. The results for VAR4 are displayed in Figure 6.<sup>25</sup>

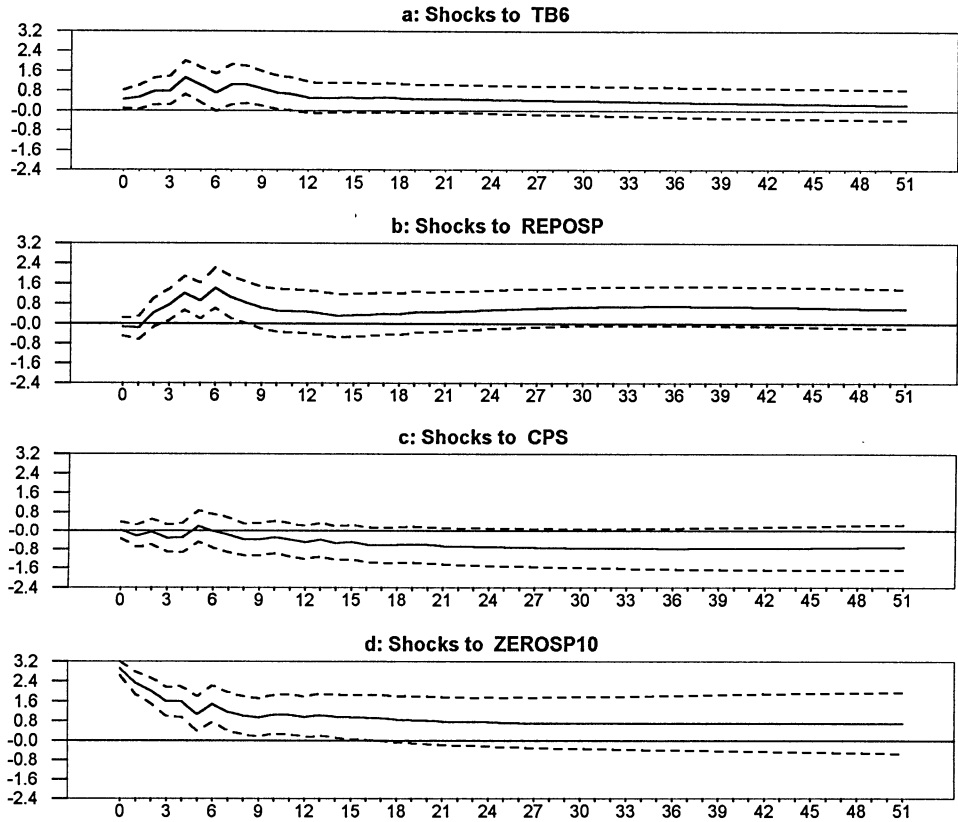
The impulse-response patterns are traced out over one year (52 weeks) subsequent to the impulse. In order to assess the significance of the responses, standard-error bands are computed using Monte Carlo methods. Specifically, assuming that residuals in the VARs are i.i.d. normal random variables, draws are made from the posterior distribution of the VAR coefficients and the impulse-response functions are computed based on these coefficients. These calculations are repeated 1000 times to derive upper and lower two-standard error bands for the responses over 52 weeks.<sup>26</sup>

The cumulative effects of these shocks are summarized in Table V, which presents the decomposition of variance. For a given VAR, each row must sum to one as the sum represents the total variation in ZEROSP10. For example, after 4 weeks, about 6.29 percent of the variation of the change in ZEROSP10 due to shocks in the four variables in VAR4 is due to TB6, about 2.74 percent is due to REPOSP, and so on.

The largest response of ZEROSP10 over the first weeks following the shocks is due to its own shock (Figure 6*d*). However, the effects of the own shocks die out relatively quickly. An increase in the level of interest rates (TB6) implies an increase in the ten-year zero spread (Figure 6*a*). This is consistent with the view that credit spreads widen during market sell-offs and narrow during rallies. Given the strong positive correlation between generic repo and TB6, this pattern is also consistent with an explanation based on dealer hedging costs under the presumption that dealers have more pay-fixed than receive-fixed swaps on their books. Interest rate "level" effects peak after about six

<sup>25</sup> The interpretation of the patterns is subject to the usual caveat that the VAR residuals are transformed to orthogonal shocks before computing the impulse responses. The variables are ordered in VAR4 as presented in Figure 6: TB6, REPOSP, CPS, and ZEROSP10. So contemporaneous correlation between the residual for TB6 and the residuals for the other three variables is attributed to variation in TB6, etc.

<sup>26</sup> In order to make determination of the posterior distribution tractable, we work directly with the unconstrained VAR and assume normally distributed innovations. So, in particular, the Bayesian priors imposed in calculating the results in Tables IV and V are not imposed here. As noted previously, the results are similar with and without the priors imposed. Of course, these are estimated standard-error bands based on normal innovations. Their use is therefore subject to the usual caveats that the sample may be small or the distribution for the Monte Carlo analysis may be misspecified.



**Figure 6. Impulse Responses for ZEROSP10 in VAR4.** A four-variable VAR is estimated with the six-month Treasury bill rate (*TB6*), the spread between generic and on-the-run repo rates for ten-year Treasury bonds (*REPOSP*), the spread between BAA- and AAA-rated commercial paper (*CPS*), the spread between the ten-year zero rates implied by the swap and Treasury markets (*ZEROSP10*). Then each variable is perturbed by a one-standard deviation shock in its (orthogonalized) innovation. This figure displays the responses of *ZEROSP10* to these shocks in basis points, over a period of 52 weeks following the impulses. The dashed lines represent plus and minus two-standard error bands around the impulse responses, estimated by Monte Carlo.

months (Table V, column *TB6*), and over a two-year horizon explain about 11 percent of the variation in *ZEROSP10*.

A shock to *REPOSP* initially has a small negative effect, but after the first couple of weeks the effect turns positive and statistically significant. This positive effect is predicted by Grinblatt's (1995) model of liquidity premiums, with *REPOSP* representing the convenience yield associated with treasury bonds. Over an eight-week horizon, *REPOSP* explains about 11.9 percent of the variation in *ZEROSP10*. Subsequently, variation in *ZEROSP10* due to *REPOSP* levels off at about 20 percent.

The immediate impact of an increase in *CPS* on the zero-coupon spreads is essentially zero. After a few weeks, the zero-coupon spreads tend to decrease (Figure 6c); a widening of the credit spread between BAA and AAA commercial

**Table V**

**Decompositions of Variance of Ten-Year Swap Zero Spreads**

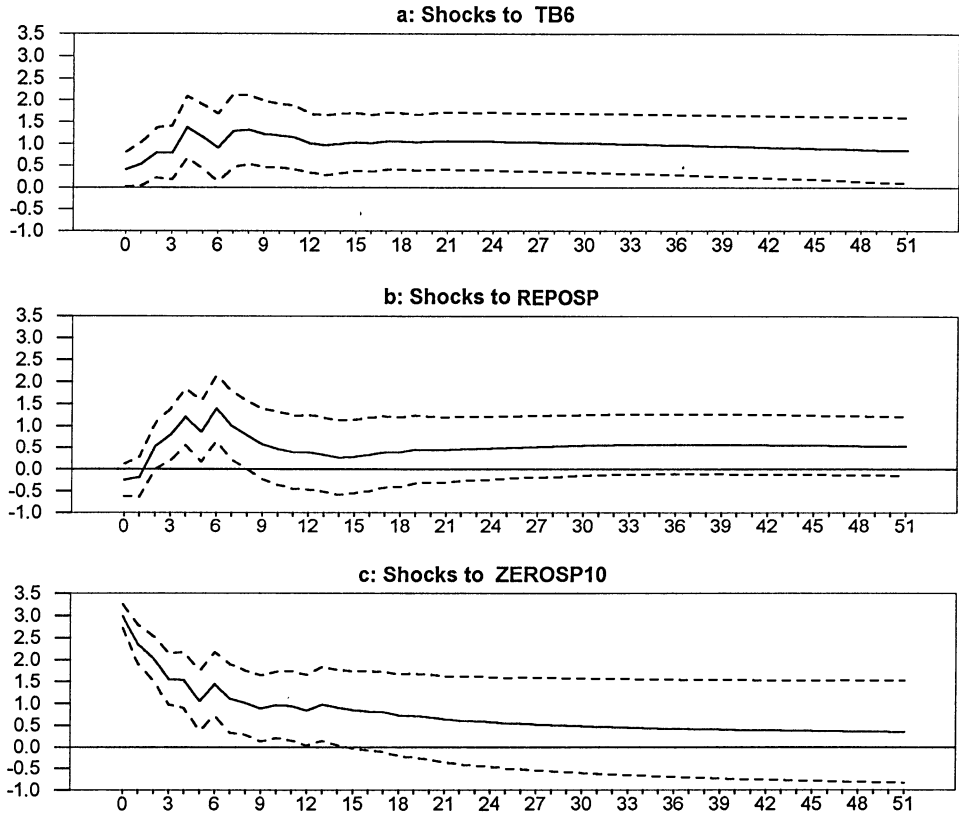
Each column displays the proportion of the variance of the error from forecasting ZEROSP10 *k* weeks ahead, due to the impulse in the variable indicated in the column heading. *k* ranges from 1 to 104 weeks (2 years). Two VARs are estimated: VAR4 includes the 6-month Treasury bill rate (TB6), the spread between generic and on-the-run repo rates for ten-year Treasury bonds (REPOSP), the spread between BAA- and AAA-rated commercial paper (CPS), and the spread between the ten-year zero rates implied by the swap and Treasury markets (ZEROSP10); VAR3 includes TB6, REPOSP, and ZEROSP10. The \* indicates exclusion of CPS from the VAR3 model.

Weeks Ahead	Percentage Due to Impulse From							
	TB6		REPOSP		CPS		ZEROSP10	
	VAR4	VAR3	VAR4	VAR3	VAR4	VAR3	VAR4	VAR3
1	2.44	2.01	0.69	0.90	0.12	*	96.75	97.09
4	6.29	6.07	2.74	2.85	0.13	*	90.85	91.08
8	12.00	14.29	11.90	11.37	0.16	*	75.95	74.39
26	15.26	32.10	12.88	11.60	6.98	*	64.88	56.30
52	12.84	41.91	18.60	15.90	16.75	*	51.81	42.19
78	11.55	45.09	19.88	17.94	19.46	*	49.11	36.98
104	11.12	46.47	20.13	18.80	20.29	*	48.46	34.73

paper leads to a narrowing of the spread between the swap zero and Treasury zero yields. A potential explanation for these patterns is that relative credit spreads (BAA – AAA) tend to widen during recessions when default probabilities increase, even though commercial paper spreads to Treasury bill rates tend to narrow. Thus, a widening of CPS might be expected to eventually lead to a narrowing of ZEROSP10 if the former reflects a weakening of the economy. That commercial paper spreads to Treasuries are informative leading indicators of the U.S. business cycle has been documented by Friedman and Kuttner (1993), among others. CPS might reasonably be expected to have a similar property.

Interpreting the effects of REPOSP and CPS on ZEROSP10 as liquidity and credit shocks, respectively, the patterns suggest that liquidity shocks are relatively important over short horizons of a few months. After about 9 weeks, the response to REPOSP shocks is insignificant out to about six months. Credit shocks have little impact on ZEROSP10 over the first couple of months. Over longer horizons the effect of CPS grows in importance to the point that after two years it accounts for the second largest percentage (20.3 percent) in the variance decomposition for ZEROSP10. The estimated standard error bands for CPS include zero for all of the horizons out to one year, however, so credit effects on swap spreads may be weaker than the numbers in Table V suggest.

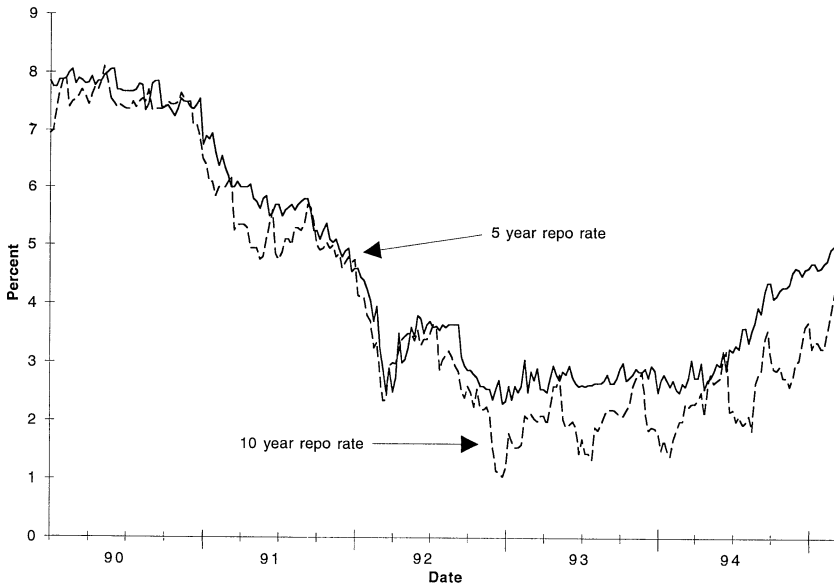
For comparison, we also estimate a three-variable VAR including TB6, REPOSP, and ZEROSP10, in that order. The impulse responses are displayed in Figure 7. The effect of TB6 is larger than in VAR4 (compare Figures 6a and 7a). This is reflected in the variance decompositions (Table V), where 42 percent of the variation in ZEROSP10 over a one-year horizon is attributable



**Figure 7. Impulse Responses for ZEROSP10 in VAR3.** A three-variable VAR is estimated with the six-month Treasury bill rate (TB6), the spread between generic and on-the-run repo rates for ten-year Treasury bonds (REPOSP), the spread between the ten-year zero rates implied by the swap and Treasury markets (ZEROSP10). Then each variable is perturbed by a one-standard deviation shock in its (orthogonalized) innovation. This figure displays the responses of ZEROSP10 to these shocks in basis points, over a period of 52 weeks following the impulses. The dashed lines represent plus and minus two-standard error bands around the impulse responses, estimated by Monte Carlo.

to shocks in TB6 in VAR3, compared with 12.8 percent in VAR4. Evidently, omission of CPS leads to the variation in ZEROSP10 attributed to CPS in VAR4 being attributed to TB6 in VAR3.<sup>27</sup> An explanation for these results can be gleaned from the results for CPS in VAR4. The  $F$ -statistics in Table IV suggest that there is statistically significant feedback from CPS to TB6. Furthermore, the impulse responses for TB6 show a negative effect of a shock to CPS on TB6: A widening of the commercial paper credit spread is associated

<sup>27</sup> Notice also that the variation in ZEROSP10 due to its own shock over a two-year horizon declines in VAR3, and this reduction is attributed largely to TB6. However, in light of the wide standard error bands around the impulse response function of ZEROSP10 over horizons beyond six months (Figure 7), this change may not be of much significance.



**Figure 8. Repo Rates for On-the-Run Ten- and Five-Year U.S. Treasuries.**

with a subsequent decline in TB6. Associating widening relative credit spreads within the commercial paper market with a weakening economy, it appears that, as the economy weakens, Treasury rates tend to decline, relative commercial paper spreads tend to widen, and the ten-year swap zero spread tends to narrow. Thus, both changes in Treasury bill rates and relative commercial paper credit spreads reflect the weakness of the economy and any induced changes in the likelihood of default by swap counterparties. When CPS is omitted from the VAR, the predictive power of changing relative commercial paper credit spreads for ZEROSP10 is largely captured by TB6.

The effects of shocks in ten-year repo specialness, as proxied by REPOSP, are striking and much larger than when the corresponding repo spread for five-year Treasury bonds is used instead. Upon further exploration, we find a strong seasonal pattern in the on-the-run ten-year repo rate (actual rate level) that is not present in the five-year rate (see Figure 8). Especially since 1992, there is a strong quarterly seasonal effect, and the ten-year repo rate (REPO10) generally is below the five-year repo rate. This suggests that the liquidity premium (convenience yield) is typically larger in the ten-year sector and more seasonally variable.

Several factors contribute to the quarterly seasonality in REPO10. First, the peaks and troughs in REPO10 correspond approximately to the calendar end-of-quarter dates. Efforts by Treasury bond dealers to cover their short positions at the ends of quarters puts pressure on bond borrowing and, hence, downward pressure on repo rates (*Wall Street Journal*, September 29, 1992, page C-20). Short covering is encouraged by the treatment of short-sales as liabilities on dealers' quarterly balance sheet reports. Additionally, the Trea-



surety note futures contracts expire on the last date of the end of each calendar quarter. Borrowing bonds for delivery could contribute to this pressure in the repo market.<sup>28</sup>

However, the squeezes in the bond-borrowing market do not occur every quarter. Indeed, during 1992 and 1993, the largest dips in REPO10 were during March and September. The explanation for this pattern appears to be the reopening of ten-year note issues by the U.S. Treasury. Specifically, in May and November of 1992 and 1993 the U.S. Treasury reopened the bond from the previous auction instead of auctioning a new note. The presence of an outstanding stock of the reopened note on the reopening date implied that these issues did not go on special as much as the typical newly auctioned notes. Consequently, the repo rate for the on-the-run ten-year Treasury did not fall as much as for regular auction dates (February and August). The induced seasonal patterns in REPO10 appear to be associated with significant widenings of ten-year swap spreads after regular auctions. These Treasury "experiments" show a strong liquidity effect on swap spreads.

Within the four-variable VAR examined, nearly half of the variations of swap spreads over two-year horizons are explained by their own shocks. This suggests that an understanding of the time-series properties of swap zero spreads may require a deeper investigation of swap-specific market activity. For example, segmentation of supply or demand pressures by maturity sector due to institutional or accounting considerations may be important.

The behavior of swap spreads is also complicated by the state tax abatement on Treasuries, which would widen spreads, but in a manner whose dynamics are not clear or easily measured.

## V. Conclusion

We have shown that, under the assumption of symmetric counterparty credit risk, swaps can be priced using standard term-structure models based on a risk- and liquidity-adjusted discount rate. For the purpose of our econometric analysis, we assume that the adjusted discount rate can be expressed as the sum of two independent square-root diffusions. Upon maximizing the likelihood function for about seven years of weekly data, we find that this two-factor model fits many aspects of the swap term structure well. The primary exception is the very short end of the swap curve represented by six-month LIBOR. The deviations between the actual and the model-implied LIBORs evidence strong seasonality and tend to be particularly large near the calendar year-ends. The autocorrelation structure of the deviations suggests that a third factor might allow one to fit the short-term LIBOR curve and the swap curve simultaneously. This possibility, as well as the implications of using two versus three-factor models for valuing LIBOR-based derivatives, are left for future research.

<sup>28</sup> Other factors include the heavy use of ten-year treasury notes as hedges against positions in mortgage-related securities and the more frequent auction cycle of five- versus ten-year notes.

Using the estimated model, we then examine the dynamic properties of the spreads between the zero-coupon bond yields implicit in the swap curve and their Treasury counterparts. Three notable findings are as follows. First, the specialness of the on-the-run ten-year Treasury note has a positive effect on zero spreads that tends to peak in the first few weeks following the impulse, and over long horizons explains about 20 percent of the variation in ten-year zero spreads. These patterns suggest that liquidity advantages to trading in Treasury markets, and Treasury repo specials, have a substantial effect on swap spreads, consistent with Grinblatt's model, but that this liquidity-repo effect is not the only, or indeed the primary, determinant of swap-Treasury zero spreads. The same conclusion applies to the actual swap spreads as well.

This is not surprising given that the convenience yield  $l_t$  captures *relative* liquidity and repo effects in swap and treasury markets. The demands for the pay- and receive-fixed sides of swaps change over time and along the maturity spectrum. These demands may well have important effects on swap spreads. Indeed, the second conclusion that we draw from the VAR analysis is that, for the explanatory variables considered, between 35 percent and 48 percent of the variation in the ten-year zero spread over a two-year horizon is explained by its own shocks. That is, after accounting for the proxies of hedging costs, liquidity, and credit effects, a substantial fraction of the variation in swap spreads is left unexplained. Brown, Harlow, and Smith (1994) find weak evidence of an effect on the supply of corporate debt in the long end of the swap curve. Further exploration of these and other supply effects in the swap market, as well as the impact of asymmetric tax treatments, seems worthwhile.

Finally, interpreting the spreads between AAA and BAA commercial paper rates as a credit spread, we find that changes in the market's perception of credit risk have large effects on zero spreads at the ten-year maturity. The response of swap spreads to credit shocks is very different from the response to liquidity shocks. The credit effects are weak initially, then peak about six to seven months after the impulse. In terms of contribution to variation in zero spreads, the relative importance of credit shocks increases for over two years, reaching 20 percent.

However, the confidence intervals for the impulse responses for credit shocks are wide. Moreover, the channel by which CP credit spreads are correlated with swap spreads appears to be through the short-term riskless rate. The patterns are consistent with economic downturns leading to both a deterioration in credit quality (widening of relative credit spreads) and lower interest rates. Although an increasing hazard rate of default during economic downturns is one interpretation of these patterns, further analysis of the links between default hazard rates, the levels of riskless interest rates, and swap spreads seems necessary to conclude more definitively that business-cycle impacts on credit quality represent an economically significant credit component to variation in swap spreads.

These findings will prove useful in guiding a more complete model of swap spreads. Within our theoretical framework, the next step would be to parameterize the process generating the riskless rate ( $r_t$ ), the liquidity convenience

yield ( $l_t$ ), and the credit adjustment ( $\lambda_t h_t$ ). The VAR analysis suggests that the parameterizations adopted should be sufficiently flexible to allow very different dynamic structures for  $l_t$  and  $\lambda_t h_t$ , and allow for correlation between default hazard rates and the riskless interest rate. The VAR analysis may also provide potentially useful insights into the dynamic relations among our liquidity and credit proxies that may prove useful in parameterizing the correlations between the processes for  $r_t$ ,  $l_t$ , and  $\lambda_t h_t$ . As elaborated by Duffie and Singleton (1996), one is unable to disentangle the separate influences of  $\lambda_t$  and  $h_t$  using only yield data, without the benefit of default incidence or loss data, or of data on the prices of instruments that respond nonlinearly with the prices of the underlying bonds at default, such as credit-spread options.

#### REFERENCES

- Abken, Peter, 1993, Valuation of default-risky interest-rate swaps, *Advances In Futures And Options Research* 6, 93–116.
- Bansal, Ravi, and William Coleman II, 1996, A monetary explanation of the equity premium, term premium, and risk-free rate puzzles, *Journal of Political Economy* 104, 1135–1171.
- Bernanke, Ben S., 1990, Clearing and settlement during the crash, *Review of Financial Studies* 3, 133–151.
- Brown, Kevin, W. V. Harlow, and Donald J. Smith, 1994, An empirical analysis of interest-rate swap spreads, *Journal of Fixed Income* 3, 61–78.
- Brown, Roger, and Steven Schaefer, 1993, Interest rate volatility and the term structure of interest rates, *Philosophical Transactions of the Royal Society: Physics, Science, and Engineering* 347, 563–576.
- Chen, Ren-raw, and Louis Scott, 1993, Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates, *Journal of Fixed Income* 3, 14–31.
- Cooper, Ian A., and Antonio S. Mello, 1991, The default risk of swaps, *Journal of Finance* 46, 597–620.
- Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1985, A theory of the term structure of interest rates, *Econometrica* 53, 385–407.
- Dai, Qiang, and Kenneth J. Singleton, 1996, Specification analysis of affine term structure models, Working paper, Graduate School of Business, Stanford University.
- Duffie, Darrell, 1996, Special repo rates, *Journal of Finance* 51, 493–526.
- Duffie, Darrell, and Ming Huang, 1996, Swap rates and credit quality, *Journal of Finance* 51, 921–949.
- Duffie, Darrell, and Rui Kan, 1996, A yield-factor model of interest rates, *Mathematical Finance* 6, 379–406.
- Duffie, Darrell, and Kenneth J. Singleton, 1993, Simulated moments estimation of Markov models of asset prices, *Econometrica* 61, 929–952.
- Duffie, Darrell, and Kenneth J. Singleton, 1996, Modeling term structures of defaultable bonds, Working paper, Graduate School of Business, Stanford University.
- Evans, Ellen, and Gioia Parente-Bales, 1991, What drives interest rate swap spreads, in Carl R. Beidleman, Ed.: *Interest-Rate Swaps* (Business One Irwin Pubs., Homewood, Illinois).
- Feller, William, 1951, Two singular diffusion problems, *Annals of Mathematics* 54, 173–182.
- Friedman, Benjamin, and Kenneth Kuttner, 1993, Why does the paper-bill spread predict real economic activity? in *NBER Macro Annual*, 213–253.
- Grinblatt, Mark, 1995, An analytic solution for interest-rate swap spreads, Working paper, UCLA Anderson Graduate School of Management.
- Hansen, Lars, 1982, Large sample properties of generalized method of moments estimators, *Econometrica* 61, 929–335.
- Heath, David, Robert Jarrow, and Andrew Morton, 1994, Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation, *Econometrica* 60, 77–106.

- Hull, John, and Alan White, 1992, The impact of default risk on the prices of options and other derivative securities, *Journal of Banking and Finance* 19, 299–322.
- Jaffee, Dwight, 1975, Cyclical variations in the risk structure of interest rates, *Journal of Monetary Economics* 1, 309–325.
- Jarrow, Robert, and Steven Turnbull, 1995, Pricing options on financial securities subject to default risk, *Journal of Finance* 50, 53–86.
- Jordan, Brad, and Susan Jordan, 1997, Special repo rates—an empirical analysis, *Journal of Finance*, Forthcoming.
- Krylov, N., 1980, *Controlled Diffusion Processes* (Springer Verlag, New York).
- Li, Haitao, 1995, Pricing of swaps with default risk, Working paper, Yale University, School of Organization and Management.
- Litzenberger, Robert H., 1992, Swaps: Plain and fanciful, *Journal of Finance* 47, 831–850.
- Merton, Robert, 1974, On the pricing of corporate debt: The risk structure of interest rates, *Journal of Finance* 29, 449–470.
- Nielsen, S., and Ehud Ronn, 1995, The valuation of default risk, in *Corporate Bonds and Interest Rate Swaps* (University of Texas at Austin, Austin).
- Pearson, Neil, and Tong-Sheng Sun, 1994, An empirical examination of Cox, Ingersoll, and Ross model of the term structure of interest rates using the method of maximum likelihood, *Journal of Finance* 49, 929–359.
- Rendleman, Robert J., Jr., 1992, How risks are shared in interest rate-swaps, *Journal of Financial Services Research* 7, 5–34.
- Smith, Clifford W., Jr., Charles W. Smithson, and Lee MacDonald Wakeman, 1988, The market for interest rate swaps, *Financial Management* 17, 34–44.
- Solnik, Bruno, 1990, Swap-pricing and default risk: A note, *Journal of International Financial Management and Accounting* 2, 79–91.
- Sorensen, Eric H., and Thierry F. Bollier, 1994, Pricing swap default risk, *Financial Analysts Journal* 50, 23–33.
- Stock, James, and Mark Watson, 1989, New indexes of coincident and leading economic indicators, in *NBER Macro Annual*, 351–395.
- Sun, Tong-Sheng, Suresh Sundaresan, and Ching Wang, 1993, Interest rate swaps—an empirical investigation, *Journal of Financial Economics* 34, 77–99.
- Sundaresan, Suresh, 1991, Valuation of swaps, in *Recent Development in International Banking* (Elsevier, North Holland).
- Vasicek, Oldrich, 1977, An equilibrium characterization of the term structure, *Journal of Financial Economics* 5, 177–188.