Wireless NUM: Rate and Reliability Tradeoffs in Random Environments

Daniel O'Neill* Boon Sim Thian* Andrea Goldsmith* and Stephen Boyd*
*Department of Electrical Engineering,
Stanford University
Stanford CA 94305
Email: {dconeill, bsthian, andrea}@stanford.edu

***Abstract*— We describe Wireless Network Utility Maximization, WNUM, and compare its performance to NUM for wireless networks of interfering links under random time varying channel conditions. WNUM is shown to simultaneously offer greater rate and reliability performance in simulations operating under Rayleigh fading. A general method for finding adaptive network control policies is presented that is sample-based and converges to the optimal control policies for the network.**

I. Introduction

Managing and controlling mobile ad-hoc networks (MANET's) presents many technical challenges. Considerable recent research has investigated using the Network Utility Maximization (NUM) framework to optimize MANET's [1], [2], [3]. While the NUM framework has shown to be a useful tool in wireline networks, recent simulation results have shown NUM performance to be disappointing when applied to MANET's [4].

We theorize that this disappointing performance results from several fundamental limitations of the canonical NUM framework when applied to the dynamic lossy environments of MANET's. In particular, NUM does not capture the effects of lossy randomly varying wireless channel conditions on system performance nor does it explicitly include reliability mechanisms to address this fundamental issue. Further, canonical NUM only measures the information-carrying rate of a system in evaluating its performance and does not capture the inherent rate-reliability tradeoff of wireless systems. Recent work [5] has extended the canonical NUM approach to take into account reliability in addition to information rate in evaluating system performance, but it does not capture the effect of random channel variations, a significant source of transmission error. To address these limitations, we extend NUM to dynamic wireless environments by explicitly incorporating assumptions about the physical channel such as time varying fading, mobility, link reliability, etc. We term this framework Wireless NUM (WNUM).

In this paper we specialize WNUM to manage the rate-reliability tradeoff in wireless networks with randomly time-varying channels. We extend [5] and our work in [6] to find optimal adaptive policies to manage self-interfering networks under power rate and reliability constraints. Our results show that WNUM offers significant performance gains over the NUM formulation, simultaneously improving both information rate and information reliability in our simulations. We present a method for finding optimal control policies that sample the channel and make no parametric assumptions about the distribution of channel states. The approach converges to the system parameters necessary to optimally control the system. The convergence of WNUM is briefly analyzed as a Stochastic Approximation problem.

The remainder of the paper is organized as follows. In Section II we describe our wireless system model. In Section III we describe NUM and discuss its performance characteristics in fading channels. In Section IV we describe WNUM and formulate the associated network problem. In Section V we describe a method for finding optimal control policies and in Section VI interpret the generated Lagrange multipliers. In Section VII we investigate the performance of our approach and compare it to the NUM formulation in [5]. Section VIII describes our conclusions.

II. System Model

There are M logical source/destination pairs and L links in the network. Each source and destination is associated with an upper layer protocol stack. The flow of information over the network from a logical source to a logical destination, possibly over multiple links, is termed an information flow. Flows from different sources m may traverse the same link l. The routing of information flows over links is described by the routing matrix A, where $A_{lm} = 1$ if information on flow m traverses link l and is otherwise zero.

A single link is shown in Figure 1. One or more sources and their associated upper layer protocol stacks send packets into the link encoder at information rates r_m. In the figure a single source and its associated upper layer protocol stack is shown for simplicity. Packets are encoded and are injected into the link buffer. The encoder uses block convolutional codes, which add additional bits to the information flow to enhance error detection and recovery. The ratio of the total number of useful information bits to the total number of bits exiting the encoder per unit time is termed the code rate $0 < \theta_l \leq 1$. Encoded bits are removed from the link buffer and transmitted by the wireless link at rate R_l. The rate at which useful information is transmitted across the link is $\theta_l R_l$.

The channel is modeled by a channel state (gain) matrix $G \in \mathbf{R}^{L\times L}$, where G_{ij} is the power gain from the transmitter on link j to the receiver on link i. The vector of transmitter powers is given by $S \in \mathbf{R}^{L}$. Each transmitter has an average power budget $\bar{S}$. For concreteness the link rate function is assumed to be of the form

$$R_l(S,G) = \log\left(1 + \frac{KG_{ii}S_l}{\sum_{j\neq l} G_{lj}S_j + N}\right) \quad l = 1,\ldots,L \tag{1}$$

where K is fixed and scales the received power [7] and N is receiver noise. The distribution of $G \sim p(G)$ is stationary and ergodic and is unknown to the network. We assume the channel state is estimated without error and is known at the set of transmitters. Since the channel is randomly varying, the link rates can also vary, resulting in link congestion and queuing delay at the link buffers.

The error probability of bits flowing over the link is defined as $E(\theta)$ and is assumed to be an increasing function of the code rate θ. Explicit expressions for $E(\theta)$ are difficult to find and we use the upper bound

$$E(\theta) = \frac{1}{2}2^{-N(R_0-\theta)} \tag{2}$$

where N is the code block length used by the encoder and R_0 is the cutoff rate [8]. The reliability of an information flow m is defined by ϕ_m

$$\phi \leq 1 - A^T E(\theta) \tag{3}$$

where $A^T E(\theta)$ is the sum of the error rates on the links traversed by the flow.

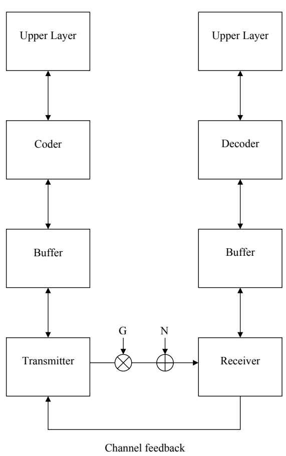

Fig. 1. System Model

The performance of upper layer protocols are modeled as utility functions. Each source m has a utility function $U(r_m,\phi_m)$. Utility functions are strictly concave increasing functions of the information rate and information reliability. We consider the following parameterized family of utility functions, which are extensions of utility functions [9], [1] often used in the literature:

$$U(r,\phi) = \begin{cases} \beta\frac{r^{1-\alpha}}{1-\alpha} + (1-\beta)\frac{\phi^{1-\alpha}}{1-\alpha}, & \alpha > 0 \quad \alpha \neq 1 \\ \beta\ln(r) + (1-\beta)\ln(\phi), & \alpha = 1. \end{cases} \tag{4}$$

where $0 \leq \beta \leq 1$ weights the relative importance of information rate and reliability.

The system can adapt to changing channel conditions by estimating G and adapting parameters such as transmit power $S = S(G)$, transmitter link rate $R = R(S(G),G)$, the upper layer information rate $r = r(G)$, code rate $\theta(G)$ and information reliability $\phi(G)$. The reliability policy can be thought of as a command to the wireless network to supply a required level of reliability.

III. NUM PERFORMANCE

In this section we briefly describe NUM. The canonical NUM problem is to find the optimal information rates r that maximize overall network utility. Information flows across a network of links. The links are assumed to have fixed, error free link rates, $\bar{R}$ and to have a link buffer of unlimited capacity. Formally the NUM problem can be expressed as

$$\begin{array}{ll} \underset{r\geq 0}{\text{maximize}} & \sum_m U_m(r_m) \\ \text{subject to} & \\ & Ar \leq \bar{R} \end{array} \tag{5}$$

where A describes the fixed topology of the network. The operation of the network is described as an optimization algorithm seeking to solve this problem.

In many practical systems the information rate and information reliability of a link can be changed by adjusting the link code rate. In [5] the authors extend the basic NUM formulation to capture this tradeoff as

$$\begin{array}{ll} \underset{r\geq 0,\theta,\phi}{\text{maximize}} & \sum_m U_m(r_m,\phi_m) \\ \text{subject to} & \\ & Ar \leq \text{Diag}(\theta)\bar{R} \\ & \phi \leq 1 - A^T E(\theta) \\ & 0 \leq \theta \leq 1 \\ & 0 \leq \phi \leq 1. \end{array} \tag{6}$$

This approach has the benefit of explicitly modeling information reliability and information rate, but completely ignores the effects of randomly time varying wireless channels. In particular, it takes the link rate $\bar{R}$ as a fixed constant. In a flat fading context, this implies that the fading can be perfectly offset via power control (implying unconstrained transmitter power) or that the network will experience outages when channel conditions fall below those necessary to support this fixed transmission rate. Such outages reduce the effective information throughput and increase average delay [10], since the link is unable to transmit successfully during the outage.

Figure 2 shows the effect of outages on NUM information throughput as a function of average SNR and under Rayleigh fading. For clarity we model a single link using (6). For networks with independent link fading and infinite link buffers the results are similar. We compare two cases each with the same average SNR. In the first case, we find the information throughput of the system at 10% outage when the channel is randomly varying. In the second or NUM case, we find the information rate of the system with a fixed SNR, corresponding to $\bar{R}$. The ratio of information throughput for the time varying case to the static case is the effective information throughput. At 10% outage probability and an average SNR of 20 dB, the effective throughput is only 55% of what is anticipated. As can be seen at moderate to low SNR's the effective information throughput is further reduced.

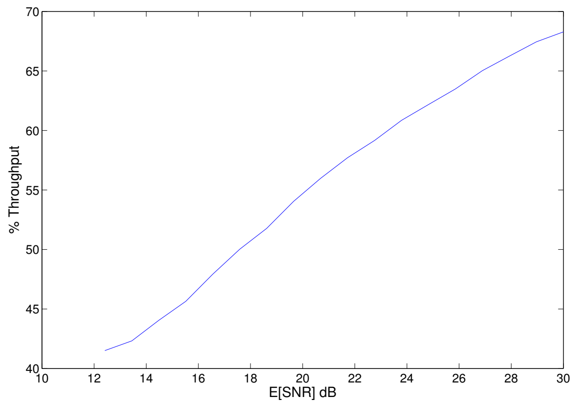

Fig. 2. NUM Effective Rate vs. Ave SNR

IV. WNUM: Optimal Rate and Reliability Policies

In this section we extend (6) to wireless time varying channels by reformulating it as a WNUM problem and describe a method of solution. In any particular wireless application the distribution of channel states is likely to be unknown, and so our method of solution is sample based and does not assume the system has prior knowledge about $p(G)$

WNUM formally introduces random channel (or other network component) variations and focuses on performance metrics and constraints represented as averages. The idea is to find adaptive rate, reliability and power policies that maximize the average utility of the network, under constraints on information rates, link rates, reliability and average power transmitted. By policies we mean functions that optimally adapt to changes in the channel state, and we write $S(G)$, $r(G)$, $R(S(G),G)$, $\phi(G)$, $\theta(G)$ for the respective transmitter power, information rate, link rate, code rate and reliability policies.

The formal WNUM problem is

$$\begin{aligned} &\text{maximize} \quad \mathbf{E}[\textstyle\sum_m U_m(r_m(G), \phi_m(G))] \\ &\text{subject to} \\ &\qquad \mathbf{E}[S_l(G)] \leq \bar{S}_l, \; l = 1, \ldots, L \\ &\qquad \mathbf{E}[Ar] \leq \mathbf{E}[\text{Diag}(\theta(G))R(S(G),G)] \\ &\qquad \mathbf{E}[\phi(G)] \leq 1 - \mathbf{E}[A^T E(\theta(G))] \\ &\qquad 0 < \theta(G) < 1 \\ &\qquad 0 < \phi(G) < 1 \end{aligned} \tag{7}$$

where $\mathbf{E}$ is the expectation operator. The optimization is over the policies $r(G), S(G), R(G), \phi(G), \theta(G)$ and indirectly the link rate $R(S(G),G)$. The first constraint limits the *average* transmitter power $\mathbf{E}[S_l(G)]$ to be less than or equal to $\bar{S}$. At a given channel state G, $S(G)$ can be greater than or less than $\bar{S}$. The second constraint requires that the *average* information flow into a link buffer must be less than the average transmission rate of information across the link. For poor channel states, the information flow into the buffer may exceed the information flow across the link, increasing congestion and conversely for good channel states. The third constraint requires that on *average* the reliability of the network exceed that commanded by the upper layer protocols through $\phi(G)$. The remaining constraints simply reflect the relevant ranges of these policies, but require that they be met for every channel state.

Equation (7) is not a convex problem and globally optimal policies may not exist. It can be transformed into a convex problem through a transformation of the vector variables $S(G)$ and $r(G)$ and the introduction of the auxillary vector variable $\pi(G)$. In particular, if we define $\hat{S}(G) = \log(S(G))$ and $\hat{r}(G) = \log(r(G))$, then (7) can be rewritten in convex form

$$\begin{aligned} &\text{maximize} \quad \mathbf{E}[\textstyle\sum_m \hat{U}_m(\hat{r}_m(G), \phi_m(G))] \\ &\text{subject to} \\ &\qquad \mathbf{E}[\exp(\hat{S}_l(G))] \leq \bar{S}_l, \; l = 1, \ldots, L \\ &\qquad \mathbf{E}[\pi(G)] \leq \mathbf{E}[\text{Diag}(\theta(G))R(S(G),G)] \\ &\qquad \mathbf{E}[\phi(G)] \leq 1 - \mathbf{E}[A^T E(\theta(G))] \\ &\qquad A\hat{r}(G) \leq \log(\theta(G)) + \log(\pi(G)) \\ &\qquad 0 < \theta(G) < 1 \\ &\qquad 0 < \phi(G) < 1. \end{aligned} \tag{8}$$

The variables $\hat{S}(G)$ and $\hat{r}(G)$ can be interpreted as proportional to power and code rate in dB. The constraint $A\hat{r}(G) \leq \log(\theta(G)) + \log(\pi(G))$ is equivalent to

$$\frac{a_l^T r(G)}{\theta_l(G)} \leq \pi_l(G) \quad l = 1, \ldots, L \tag{9}$$

where $a_l^T r(G)$ is the information arriving at the link, $\theta_l(G)$ is the code rate and $\pi_l(G)$ is the rate at which coded bits enter the link queue. Equation (9) has the natural interpretation that the rate at which bits enter the link queue waiting to be transmitted is equal to the information rate of bits arriving at the encoder divided by the code rate. This relationship holds for every channel state G. Bits in the link queue are removed and transmitted at rate $R(S(G),G)$. The constraint

$$\mathbf{E}[\pi(G)] \leq \mathbf{E}[\text{Diag}(\theta(G))R(S(G),G)] \tag{10}$$

requires the link buffer to be serviced at an average rate greater than or equal to the average arrival rate of bits to the link buffer.

V. Method of Solution

Full Recourse Optimization with Expected Constraints, FROEC, is used to solve (7). FROEC is an online discrete time approach to optimization. It takes as input the sequence of channel states seen by the network and produces as its output estimates of the optimal policy values. As a by-product FROEC produces the optimal Lagrange multipliers associated with (7). The time index is k, and we indicate the estimates of optimal Lagrange multiplier λ^* by λ^k. Policy values are denoted by $r^k = r(G^k, \lambda^k)$, $S^k = S(G^k, \lambda^k)$, and $R^k = R((G^k, \lambda^k), G^k)$, etc. For example S^k is the value of the power policy at channel state G^k and λ^k. FROEC does not assume knowledge of $p(G)$ and under suitable conditions adjusts to changes in the channel's empirical distribution.

FROEC solves the dual problem to (7). The dual function is defined as

$$g(\lambda) = \underset{\Omega}{\operatorname{argmax}} L(r(G), S(G), \phi(G), \theta(G), \lambda) \tag{11}$$

where $\Omega = \{r(G) \geq 0, S(G) \geq 0, 0 < \theta(G) < 1, 0 < \phi(G) < 1\}$ and

$$\begin{aligned} L(.) = \quad & \mathbf{E}[U(r(G)) \\ & -\lambda_q(r(G) - R(S(G), G)) \\ & -\lambda_s(S(G) - \bar{S})) \\ & -\lambda_\phi[\phi(G) - (1 - \mathbf{E}[A^T E(\theta(G))]]) \end{aligned} \tag{12}$$

We define $\lambda = [\lambda_q^T, \lambda_s^T, \lambda_\phi^T]^T$ as the vector of Lagrange multipliers.

The dual problem is

$$\underset{\lambda \geq 0}{\text{minimize}} \quad g(\lambda). \tag{13}$$

The FROEC approach samples the channel and generates a sequence of stochastic subgradients to $g(\lambda)$. These in turn are used to optimize (13). The FROEC algorithm has three steps. In the first step, the channel is estimated at time k and policy values calculated:

$$[r^k, S^k, \theta^k, \phi^k] = \underset{\Omega}{\operatorname{argmax}} \left[U(r) - \lambda_q^k(r - R(S, G^k)) - \lambda_s^k(S - \bar{S}))\right]. \tag{14}$$

The second step calculates stochastic subgradients:

$$\delta g = -\left[\begin{array}{c} \left(r^k - \theta R^k\right) \\ \left(S^k - \bar{S}\right) \\ \left(\phi(G) - (1 - [A^T E(\theta(G)))\right) \end{array}\right] \tag{15}$$

which is a vector composed of the "slack" in the constraints evaluated at the current policy estimates. In the third step, the λ^k are updated using the subgradient recursion

$$\lambda^{k+1} = \left[\lambda^k + \Delta_k \delta g\right]^+ \tag{16}$$

where $[]^+$ is the positivity operator and the step size $\Delta_k = \frac{\gamma}{k}$ is a sequence of decreasing positive constants with $\gamma \leq 1$.

A. Convergence

Our method of solution can be rewritten as a Stochastic Approximation problem [11] when the distribution of G^k is iid. A stochastic approximation problem is of the form

$$\lambda^{k+1} = \lambda^k + \Delta_k(h(\lambda^k) + M^{k+1}) \tag{17}$$

where M^k are uncorrelated and zero mean. By making the substitution $h(\lambda) = \mathbf{E}[\delta g(\lambda, G)]$ and

$$M^{k+1} = \delta g(\lambda, G) - \mathbf{E}[\delta g(\lambda, G)] \tag{18}$$

equation (16) can be put into this form. Under suitable conditions it is shown in [12] that the estimated Lagrange multiplier converges to an optimal value $\lambda^k \to \lambda^*$ with probability one, and moreover that the λ^k remain bounded with probability one: $\sup ||\lambda^k|| < \infty$.

B. Convergence Behavior Numerical Example

In this Section we present a simple numerical example to illustrate the convergence properties of our approach. For clarity we consider a single link. In Section VII we evaluate the performance of a multi-link interfering network after the network has converged.

The simulation is over 2000 discrete time periods, with $\bar{S} = 10$, $N = 1$, and G iid Rayleigh distribution with $E[G] = 1$. The utility function has parameter, $\alpha = 0.5$.

Figure 3 shows the values of λ versus the sample number. As can be seen, the λ's converge after about 500 iterations. In Figure 4, the running average of power is shown. Running averages are used to approximate the expectation operation in (7). The vertical axis is measured in terms of the percentage of target average power $\overline{S}$. As anticipated the average transmitter power approaches 100% of the target average power. Figure 5 shows the running average information and link throughput. Both the information and link throughput improve as the link is optimized. Figure 6 shows the code rate and the reliability of the link. As the number of samples increases, the code rate stabilizes at a value of 0.7 and the reliability stabilizes at a value of 0.83.

VI. Interpretation of Optimal Lambdas

The post convergence Lagrange multipliers, λ_q^*, λ_s^* and λ_ϕ^*, have several interpretations. From (12) we can see the Lagrange multipliers are constants of proportionality between the constraints and the average network utility function. A small change in a constraint [13] results in a proportionate change in the network utility. For example, a change $\Delta\bar{S}$ in the average power constraint yields a change

$$\mathbf{E}[\Delta U] = \lambda_S^* \Delta\bar{S} \tag{19}$$

in average utility. Similarly a small exogenous change in the link rates, say by a reduction of thermal noise at the receiver, will result in

$$\mathbf{E}[\Delta U] = \lambda_q^* \Delta R \tag{20}$$

improvement in performance. Note that $\lambda_q^*, \lambda_S^* \geq 0$ matching the intuitive idea that an increase in link power budget or an exogenous improvement in receiver performance cannot decrease overall network performance in an interfering network. Similar interpretations can be made for the reliability constraint.

The Lagrange multipliers have an economic interpretation as prices. Thus, λ_S^* is the marginal or incremental price of utility measured in utility per unit power and λ_q^* is the marginal or incremental price of utility measured in utility per unit of information rate. The ratio of Lagrange multipliers is the tradeoff between the constraints. For the lth link in a network the ratio

$$\frac{[\lambda_q^*]_l}{[\lambda_S^*]_l} \tag{21}$$

has units of Joules/bit and is the cost to transmit the next bit in the system on link l. Similarly the ratio

$$\frac{[\lambda_\phi^*]_l}{[\lambda_S^*]_l} \tag{22}$$

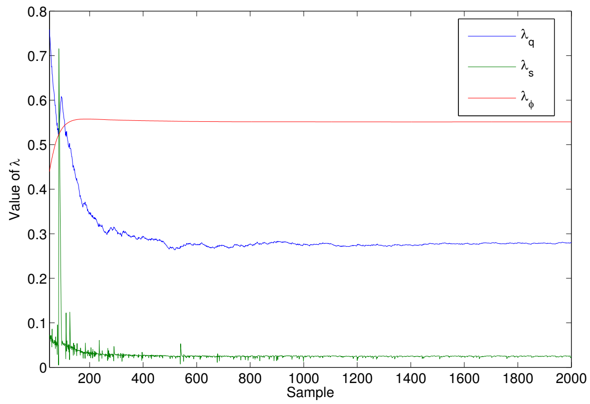

Fig. 3. Lambda Convergence

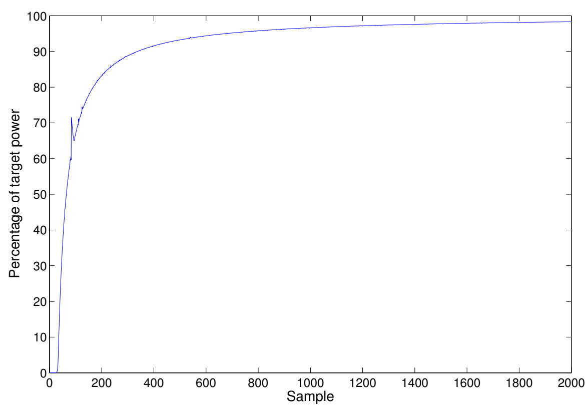

Fig. 4. Power Convergence

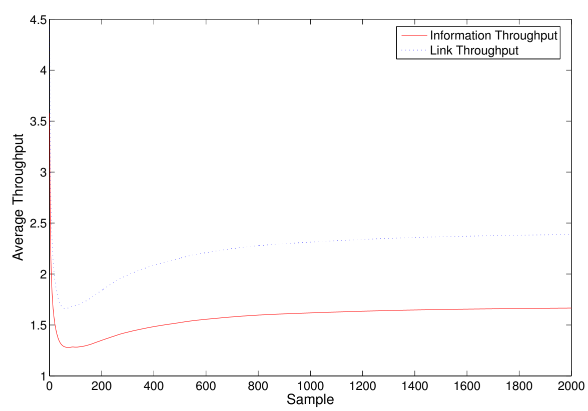

Fig. 5. Information and Link Throughput Convergence

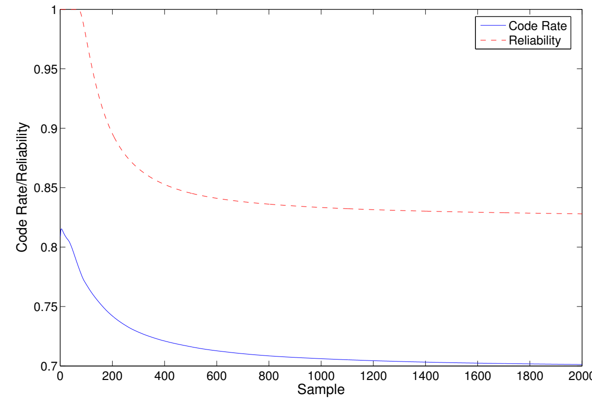

Fig. 6. Code Rate and Reliability Convergence

is the tradeoff between information reliability and transmitter power. The cross-link power ratio between two links l and z

$$\frac{[\lambda_S^*]_l}{[\lambda_S^*]_z} \tag{23}$$

can be interpreted as the relative benefit of boosting power at the two links.

VII. WNUM - Multiple Links

In Section V-B we described the convergence properties of the FROEC algorithm. In this section, we consider the "operational" performance of the WNUM approach after it has converged to the optimal values of Lagrange multipliers λ_q^*, λ_s^* and λ_ϕ^*. We use the term operational to describe the steady state performance of WNUM optimal policies, since this is the operating regime of WNUM. We compare WNUM's operational performance to the operational performance of a NUM based approach described by equation (6) and investigated in [5]. The results indicate WNUM offers material performance advantages.

In both approaches we consider a wireless network with $L=6$ interfering links and use the utility function described by (4). We consider a range of different values of β, corresponding to different rate-reliability tradeoffs in the utility function metric. The channel state matrix G is drawn iid Rayleigh with the diagonal elements scaled to yield an average SINR of 30 dB over all links. The transmitter power limit is $\bar{S}=4$.

For $L\geq 2$, the rate link rate function $R(S(G),G)$ is not a convex function and global policies may not exist. However, in many practical situations, $SINR_l \geq 1$, $l=1,\dots,L$ and therefore we may consider the link rate function

$$R_l(S,G) = [\log\left(\frac{KG_{ii}S_l}{\sum_{j\neq l} G_{lj}S_j+N}\right)]^+ \quad l=1,\dots,L. \tag{24}$$

WNUM uses this rate function directly, adapting the link rate to changing channel conditions. Equation (6) however uses $\bar{R}$, a fixed link rate that is independent of channel states and transmitter power. As described in Section III a link transmitting at a fixed rate and power in Rayleigh fading will experience outages. To facilitate comparisons, we set $\bar{R}$ to correspond to a 10% probability of outage. Other outage probabilities will result in different results. A 1% outage probability will reduce $\bar{R}$ and network performance, and a 20% will increase $\bar{R}$ and network performance. To find $\bar{R}$ analytically is mathematically intractable and so Monte Carlo methods were used. The value of $\bar{R}$ was found by considering 20000 channel samples, calculating $R_l(S,G)$, and setting $\bar{R}$ such that fewer than 10% of the samples resulted in link rates less than $\bar{R}$.

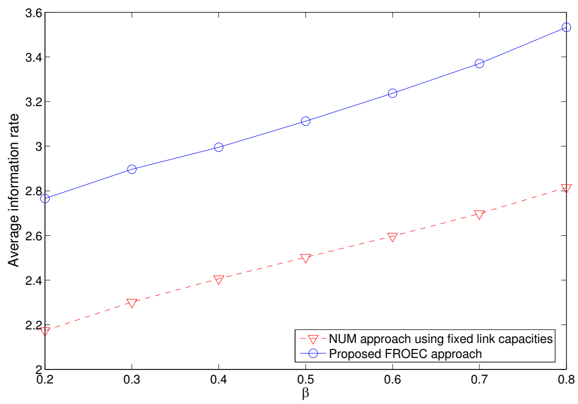

Fig. 7. Comparison of Information Rates

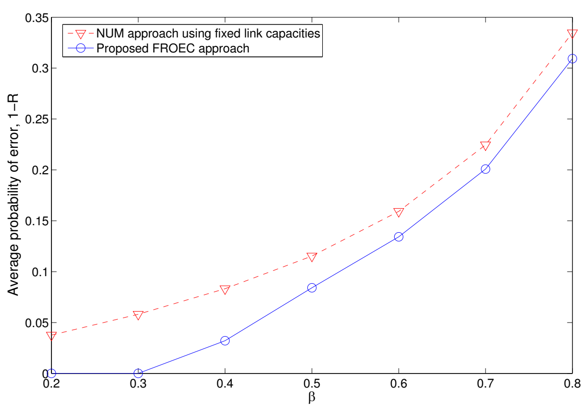

Fig. 8. Comparison of Reliability

Figures 7 and 8 show the performance of the two methods. As anticipated, both methods increase the network information throughput and decrease reliability with increasing β. Figure 7 compares the information rate averaged over the network's 6 links. WNUM outperforms NUM by approximately 30% over all values of β considered. Similarly, Figure 8 compares the reliability of the two methods, again averaged over the 6 links. The WNUM approach offers a lower probability of error for all β, with the greatest improvement occurring when information reliability is emphasized in the utility function metric.

VIII. Conclusions

We describe WNUM and compare its performance to NUM along the dimensions of information rate and information reliability under randomly time varying channel conditions. We consider networks of interfering links with routed traffic, under conditions of Rayleigh fading. At typical SINR's, simulations indicate a simultaneous 30% improvement in information throughput and a significant improvement in reliability for all choices of β. WNUM uses optimal policies to adapt to changing channel conditons by adjusting network resources. We consider explicit rate-reliability tradeoffs as measured by the utility function metric and describe a method for finding optimal control policies that meet constraints on average transmitter power, information flow and reliability. WNUM makes no parametric assumptions about the distribution of channel states, but rather samples the channel to estimate the Lagrange multipliers needed to compute optimal policies. The convergence properties of the FROEC-WNUM algorithm are briefly analyzed using Stochastic Approximation techniques.

Future work includes extending the formulation to include managing packet queuing delay.

References

[1] F. Kelly, A. Maulloo, and D. Tan, "Rate control in communications networks: Shadow prices, proportional fairness and stability," *Journal of Operations Research Society*, vol. 49, pp. 237–252, Nov. 1998.

[2] S. Low, L. Peterson, and L. Wang, "Understanding vegas: A duality model," *Journal of ACM*, vol. 49(2), pp. 207–235, Mar. 2002.

[3] M. Chiang, S. H. Low, A. R. Calderbank, and J. C. Doyle, "Layering as optimization decomposition: A mathematical theory of network architectures," *Proceedings of the IEEE*, vol. 95, pp. 255–312, January 2007.

[4] J. C. Ramming, "Control-based mobile ad-hoc networking (cbmanet) program," December 2007, unpublished.

[5] J.-W. Lee, M. Chiang, and A. R. Calderbank, "Price-based distributed algorithms for rate-reliability tradeoff in network utility maximization," *IEEE Journal on Selected Areas in Communications*, vol. 24, no. 5, 2006.

[6] D. ONeill, A. J. Goldsmith, and S. P. Boyd, "Cross-layer design with adaptive modulation: Delay, rate, energy tradeoffs," *Accepted to Globecom 2008*, 2008.

[7] G. J. Foschini and J. Salz, "Digital communications over fading radio channels," *Bell Syst. Tech. J.*, pp. 429–456, Feb. 1983.

[8] J. G. Proakis, Ed., *Digital Communications*. New York: McGraw-Hill, 1989.

[9] R. Srikant, Ed., *The Mathematics of Internet Congestion Control*. Boston: Birkhauser, 2003.

[10] D. O. ONeill, A. J. Goldsmith, and S. Boyd, "Wireless network utility maximization," *Submitted to Milcom 2008*, 2008.

[11] H. J. Kushner and G. G. Yin, Eds., *Stochastic Approximation and Recursive Algorithms and Applications*. New York: Springer, 1997.

[12] V. Borker and S. Meyn, "The o.d.e. method for convergence of stochastic approximation and reinforement learning," *SIAM Journal of Control Optimization*, vol. 38, 2000.

[13] S. Boyd and L. Vandenberghe, Eds., *Convex Optimization*. Cambridge, UK: Cambridge University Press, 2004.