Optimal Windows for Aggregating Ratings in Electronic Marketplaces

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A seller in an online marketplace with an effective reputation mechanism should expect that dishonest behavior results in higher payments now, while honest behavior results in higher reputation—and thus higher payments—in the future. We study the Window Aggregation Mechanism, a widely used class of mechanisms that shows the average value of the seller's ratings within some fixed window of past transactions. We suggest approaches for choosing the window size that maximizes the range of parameters for which it is optimal for the seller to be truthful. We show that mechanisms that use information from a larger number of past transactions tend to provide incentives for patient sellers to be more truthful, but for higher quality sellers to be less truthful.

Key words: reputation mechanisms, ratings, online markets

1. Introduction

In online trading communities, sellers have a temptation to dishonesty, because potential buyers have to decide how much to pay for an item without being able to observe it firsthand. In particular, the buyer typically cannot know in advance whether the seller is describing the item honestly, and hence may be afraid that he might be exploited if he trusts the seller. This effect is exacerbated because a buyer will often interact with sellers with whom he has never interacted before and may only seldom interact in the future. The absence of trust created by this information asymmetry may result in market failure (Akerlof 1970).

If there is a *reputation mechanism* in place, dishonesty involves a greater immediate payoff at the expense of a lower long-term payoff. In such a mechanism, after a transaction the buyer can rate the seller; all past ratings are aggregated into the seller's score. When a seller posts a description for an item, potential buyers also observe the seller's score. Using this information, we postulate that buyers employ simple heuristics (e.g., Tversky and Kahneman 1974), Bayesian techniques or some combination of the two, in order to decide how much to bid (Resnick et al. 2006). In our model the expected payment to the seller is thus a function of his description and his score, and may also depend on the mechanism by which ratings are aggregated. Empirical studies have shown that sellers with high scores enjoy a *price premium*: on average they sell at higher prices than less reputable sellers; see Resnick et al. (2006) for a survey. Motivated by these studies, we assume that the expected payment to the seller is increasing in his score.

A natural goal for a reputation mechanism is to encourage sellers to describe their items truthfully. While incentivizing truthfulness is intuitively appealing, it is also connected to *efficient trade*. Trade is efficient if the item is allocated to the agent that values it most. The buyers' valuations depend on the item's characteristics, which only the seller knows before an online sale. Thus, it is important that the seller gives an accurate description to potential buyers.

We study the *Window Aggregation Mechanism*, a widely used mechanism in which the seller's score is the average value of his T most recent ratings. With the goal of incentivizing the seller to describe the item truthfully, we address the *design question* of choosing the right window size T. We define an optimal window size as one which maximizes the range of parameters for which it is optimal for the seller to be always truthful, and we study the dependence of the optimal window size on the parameters of the model. Our main result is an interesting qualitative tradeoff: informally, *increasing the window size is more likely to make patient sellers truthful, while it is less likely to make high quality sellers truthful.* In Section 3 we introduce our design framework, which we use in subsequent sections to show the tradeoff between incentivizing patient and high quality sellers for a range of settings. First, we consider the case of *perfect monitoring* (Section 4), where the buyer rates the seller accurately after every transaction. Then, we consider two types of *imperfect monitoring*: the seller may not receive ratings after some transactions (Section 5), and the rating that the seller receives may not accurately reflect his action (Section 6).

In our model, the objective of the aggregation mechanism is to induce sellers to behave in a way that benefits the market. This approach has also been taken by Dellarocas (2005) and Fan et al. (2005). Dellarocas (2005) studies a setting where the seller has two possible effort levels which buyers observe imperfectly. He shows that there is no equilibrium where the seller always exerts high effort, and that eBay's simple mechanism is capable of inducing the maximum theoretical efficiency. In this paper we take a non-equilibrium approach. We consider the best response of the seller to a *fixed* behavior of the buyers; that is, a fixed payment function which only depends on the information available to buyers (i.e., the window size, the seller's score, and the description of the item). We believe this nonequilibrium approach may be reasonable in practice because the large and dynamic set of participants in the major online markets makes the rationality, knowledge, and coordination required for equilibrium difficult to ensure. At the very least, it seems reasonable that short run behavior in these markets may not be an equilibrium. Along these lines, Fan et al. (2005) also take a non-equilibrium approach and assume that the seller has a belief over the average bidder behavior. They propose exponential smoothing as a good way to aggregate ratings, and evaluate it through simulations. By contrast, we study the Window Aggregation Mechanism, which is widely used in electronic marketplaces, and prove results on how the optimal window size is affected by the seller's attributes.

2. Model

We consider a single seller who is a long-lived player with discount factor δ. The seller interacts with short-lived potential buyers, i.e., buyers who are interested in the seller's item for exactly one round and then depart. We do not explicitly model the buyers' behavior and the market mechanism at each time period; instead, we abstract the aggregate behavior of all the buyers in a single time period via a single premium function, described further below.

In every period the seller has an item for sale whose value is a random variable X which takes values in $[v_L, v_H]$, where $0 \leq v_L < v_H \leq 1$. In particular, we denote by v_L (resp., v_H) the minimum (resp., maximum) value of an item that the seller may have for sale. We assume that the values of items in different periods are independent and identically distributed, and define $q \equiv EX$ to be the expected value of the items for sale by a particular seller. The seller observes the value of the item at the beginning of a period and decides what description to post. Potential buyers observe the seller's description and the average value of the ratings that the seller received in the last T periods. The expected payment to the seller is a function of this aggregate information and the seller's description.

After purchase, the buyer rates the seller. Let r_i be the value of the rating that the seller received i periods ago, and let $\vec{r} = (r_1, r_2, ...)$ be the vector of ratings that the seller has received up to now.

As mentioned above, we assume that potential buyers have access to the mean value of the ratings that the seller received in the last T periods, for some $T \geq 1$. In particular, potential buyers see

$$\frac{1}{T}\sum_{i=1}^{T} r_i,$$

which we call the seller's *score*. To compute this score, the aggregation mechanism needs to keep information on the T most recent ratings of the seller. Both the seller and the mechanism have access to $\vec{r}$, but potential buyers only observe the seller's score. We call this mechanism the *Window Aggregation Mechanism*, and we refer to T as the *window size*.

The Window Aggregation Mechanism is widely used in online marketplaces, such as eBay and the Amazon Marketplace. In both marketplaces the average value of the seller's ratings in the last twelve months is shown in a prominent position: next to the description of the item on eBay and below the seller's name on Amazon. Additional information, such as the seller's scores for other window sizes,[1] is available in subsequent pages. Cabral and Hortacsu (2009) conduct an empirical study that shows that the information that is prominently shown to buyers has a larger effect on the expected payment to the seller. This motivates us to study the use of a single window size.

It is worth noting that online markets use a fixed interval of *time* rather than a fixed number of transactions. To avoid this potential complexity, for the moment we simply assume that the seller has an item for sale in every period. More generally, we can consider a model where the seller has multiple items for sale in each period; using such a model, we further discuss the mapping between a fixed number of transactions and a fixed interval of time in Section 7.

We assume that the expected payment to the seller is $v_d \cdot b_T(s)$, where s is his score and v_d the description he posted. The expected payment to the seller is thus increasing in the description he posts, as is reasonable to assume in practice. We note that if $v_L > 0$, then the seller receives a premium even for the lowest possible value items. We call $b_T(s)$ the *premium function*. This function measures how much buyers are willing to pay to transact with sellers that have high scores. Motivated by empirical studies (e.g., Ghose et al. 2005, Cabral and Hortacsu 2009), we assume that the premium function is increasing in the seller's score. We use the subscript T on the premium function to denote that the buyers may react to the window size that is being used.

The seller chooses a policy that is a best response to the premium function $b_T(\cdot)$. In our model, we emphasize that the seller is not intrinsically honest or dishonest; he is rational and chooses the description that maximizes his payoff. We assume that the seller chooses a description in $[v_L, v_H]$; however, our results hold even if the maximum possible description is a constant that is greater than v_H.

3. Design Framework

This paper considers the problem of *designing* a good aggregation mechanism; in particular, we consider a setting where the system designer wishes to choose the optimal window size in a Window Aggregation Mechanism. We assume that the mechanism designer's goal is to *maximize the range of seller parameters q and δ for which truthfulness can be guaranteed.* Ultimately, our analysis of this design problem lends qualitative insight into the design of aggregation mechanisms; in particular, we find that averaging over a longer past history of ratings is *more likely* to incentivize patient sellers to be truthful, but *less likely* to incentivize high quality sellers to be truthful. In this paper we show this result for the Window Aggregation Mechanism; however, it also holds (under a related

[1] The Amazon Marketplace shows the mean value of ratings out of five stars in the last twelve months, as well as the percentage of positive, neutral and negative feedback that the seller received within the last 30, 90, and 365 days. On the other hand, eBay shows the absolute numbers of positive, neutral and negative feedback for the past 1, 6, and 12 months.

but different set of assumptions) for a more general class of aggregation mechanisms (Aperjis and Johari 2008). In this section we introduce a general design framework, which we use in subsequent sections to show the tradeoff between incentivizing patient and high quality sellers for a range of settings.

Throughout the paper we make the following assumption.

ASSUMPTION 1. *There exists some $T \geq 1$, and $q, \delta < 1$ such that a seller with attributes q, δ is always incentivized to be truthful under a Window Aggregation Mechanism with window size T.*

Assumption 1 says that the premium function is such that there exists a T at which some sellers (with sufficiently large q and δ) are incentivized to be always truthful, so that the optimization problem of the mechanism designer is meaningful. In the case of perfect monitoring, this assumption holds for a large class of premium functions, which includes strictly convex functions. It also holds for concave functions as long as the minimum value that the seller may have for sale, i.e., v_L, is sufficiently greater than 0 (Aperjis and Johari 2008). When monitoring is imperfect, Assumption 1 still holds for a large class of premium functions as long as inaccurate and missing ratings are not too frequent.

We now discuss the general optimization framework that we use. We provide sufficient conditions to ensure that (1) the window size that maximizes the range of q for which the seller is always truthful is *increasing in* δ, and (2) the window size that maximizes the range of δ for which the seller is always truthful is *decreasing in* q. In the following sections we verify these sufficient conditions for perfect (Section 4) and imperfect (Sections 5, 6) monitoring. Thus, we conclude that the optimal window is increasing in δ and decreasing in q for a wide range of settings.

For each setting that we study we identify conditions under which there exists some function $F(T, \delta)$ such that *the seller is always truthful if and only if q, δ, and T jointly satisfy the following constraint:*

$$q \cdot F(T, \delta) \geq 1. \tag{1}$$

Moreover, for all settings of interest we show that *the function $F(T, \delta)$ is increasing in δ.* Define the set

$$T^*(\delta) = \arg\max_T F(T, \delta) \tag{2}$$

for each δ. For all settings of interest we show $T^*(\delta)$ *is increasing in* δ in the following sense: for $\delta \geq \delta'$,

(i) $\max\{T : T \in T^*(\delta)\} \geq \max\{T : T \in T^*(\delta')\}$; and

(ii) $\min\{T : T \in T^*(\delta)\} \geq \min\{T : T \in T^*(\delta')\}$.

For tractability, we assume that for all δ, the set $T^*(\delta)$ is nonempty.

These technical insights illuminate the dependence of the optimal window choice on the available information regarding δ and q, as we discuss below.

1. *Both δ and q are known by the mechanism designer.* In this case, the goal is to use a window size T such that (1) holds. Whether or not this will be possible depends on whether q and δ are large enough, given the premium functions b_T. In particular, we can ensure the seller is always truthful if and only if $\max_T F(T, \delta) \geq 1/q$; in this case any choice in $T^*(\delta)$ is optimal.

2. *The mechanism designer knows δ, but not q.* A reasonable choice of T is one which maximizes the range of values of q for which the seller will be always truthful. From (1), this implies we should maximize $F(T, \delta)$, i.e., any $T \in T^*(\delta)$ is an optimal choice. We conclude that if $T^*(\delta)$ is increasing, then *the set of optimal windows is increasing in δ.*

3. *The mechanism designer knows q, but not δ.* A reasonable choice of T is one which maximizes the range of values of δ for which the seller will be always truthful; thus, given q, we solve:

$$\begin{aligned} &\text{minimize } \delta \\ &\text{subject to } q \cdot F(T, \delta) \geq 1 \end{aligned}$$

Let $\delta^*(q)$ denote the optimal value of the preceding problem; this is the smallest value of δ such that a seller with quality q and discount factor δ can be guaranteed to be truthful under *some* window size. It then follows that any $T \in T^*(\delta^*(q))$ is an optimal choice. Observe that since the constraint is increasing in q and δ, it follows that $\delta^*(q)$ is decreasing in q. We conclude that if $T^*(\delta)$ is increasing, then *the set of optimal windows is decreasing in* q. In words, as q increases, it is possible to make sellers with smaller δ truthful, and for such sellers a smaller window size is more appropriate.

A critical theme emerges from the preceding discussion: *regardless* of the information available to the mechanism designer, solving (2) is an important step in finding the best window size. Moreover, if $F(T,\delta)$ is increasing in δ and $T^*(\delta)$ is increasing in δ (as we show in the following sections), then the optimal window is increasing in δ and decreasing in q. This is summarized in the following theorem.

THEOREM 1. *Suppose that: (1) it is optimal for the seller to be always truthful if and only if* $q \cdot F(T,\delta) \geq 1$*; (2)* $F(T,\delta)$ *is increasing in* δ*; and (3) the set* $T^*(\delta)$ *is increasing in* δ*.*

(i) If δ *is known and the goal is to maximize the range of* q *for which the seller is always truthful, then the set of optimal windows is increasing in* δ*.*

(ii) If q *is known and the goal is to maximize the range of* δ *for which the seller is always truthful, then the set of optimal windows is decreasing in* q*.*

We can now see the insight discussed at the beginning of the section. First, increasing the window size is more likely to make patient sellers (those with high δ) truthful. This is an intuitive result, since sellers with larger δ are more patient, and thus an aggregation mechanism with longer memory can successfully couple current behavior with distant future payoffs. On the other hand, a larger window is less likely to make high quality sellers (those with high q) truthful. When q is high and the window is large, the seller is likely to have a high score regardless of what actions he takes when he receives a low value item, because most items are high quality. This makes a smaller window more desirable, because it magnifies the impact of the seller's actions in those periods where he has a low value item for sale.

By choosing a window size T, the system designer determines *how much* and for *how long* the seller's future scores decrease if he does not describe his current item truthfully. In particular, if in the current period the seller receives a rating that is smaller than the maximum possible rating by d, then his score will decrease by d/T in each of the next T periods (relative to receiving the maximum possible rating). We observe the following tradeoff between the intensity and the duration of this score reduction: the intensity is decreasing in T, while the duration is increasing in T. The optimal value of T will depend on the available information on the seller's attributes and the premium function. As δ increases the duration effect becomes more important, so the optimal window size increases. On the other hand, as q increases the intensity effect becomes more important, so the optimal window size decreases.

Crucially, we observe that this tradeoff is also faced by a mechanism designer who knows *neither* δ nor q: a choice must be made regarding the incentives provided to patient sellers and those provided to high quality sellers.

As suggested by the discussion above, our analysis in each of the three following sections consists of the following steps.

1. First, we identify assumptions under which it is optimal for the seller to be always truthful if and only if $q \cdot F(T,\delta) \geq 1$ for some function $F(T,\delta)$ that is increasing in δ.
2. Second, we show that the set $T^*(\delta) = \arg\max_T \{F(T,\delta)\}$ is increasing in δ.
3. Finally, we use Theorem 1 to conclude that the optimal window is increasing in δ and decreasing in q.

The preceding program rests heavily on Step 1: in particular, we must find assumptions under which a suitable function $F(T,\delta)$ exists. As we will show in the subsequent sections, the main assumption that we make is that the premium function is *logarithmically concave*; a function is logarithmically concave if its logarithm is concave. Before proceeding, we comment on this assumption.

We note that various empirical studies suggest that the expected premium is logarithmically concave in the mean value of ratings of the seller (e.g., Cabral and Hortacsu 2009, Lucking-Reiley et al. 2007, Ghose et al. 2005). These studies regress the payment to the seller or its logarithm against some function of the average rating of the seller or some other function of the number of positive and negative ratings (in the case of eBay). Even though these studies consider all the ratings that the seller has received in his lifetime as a seller or in the last twelve months, we can get some insight in the dependence of the payment on the percentage of positive ratings in the last T transactions by fixing the total number of transactions to T. Cabral and Hortacsu (2009) use a data set from eBay and regress the logarithm of the price against the percentage of positive ratings of the seller. Since the logarithm of the price is a linear function of the average rating, the expected payment is logarithmically concave in the seller's score (by definition). Ghose et al. (2005) use a data set from the Amazon Marketplace and regress the logarithm of the premium against the mean rating of the seller in the last twelve months. We note that the Amazon Marketplace asks buyers to rate sellers out of five stars. Again, this corresponds to a logarithmically concave function. Lucking-Reiley et al. (2007) use a data set from eBay and regress the price against the logarithm of the number of positive ratings and the logarithm of the number of negative ratings. Setting T equal to the sum of the number of positive and negative ratings, we observe that according to this regression the price is a logarithmically concave function of the seller's score.

4. Perfect Monitoring

In this section we apply the framework of Section 3 to the case of perfect monitoring. In particular, we assume that the seller receives a rating that accurately reflects his action after every transaction.

We assume that the rating that the seller receives depends on the difference between the description that the seller posted (v_d), and the true value of the item (v). In particular, we assume that the rating is equal to $\alpha - \beta(v_d - v)^+$, where $x^+ \equiv \max\{x, 0\}$ is the positive part of x, and $0 \le \beta(v_H - v_L) \le \alpha$. (Recall that $0 \le v_L \le v_H \le 1$.) That is, the seller receives a low rating if he exaggerated the value of the item in his description, but not if he understated it. The seller receives the best possible rating (i.e., α) if he did not exaggerate the value of the item in his description. This is a reasonable assumption; however, it is not essential: for instance, our results also hold if the rating is equal to $\alpha - \beta|v_d - v|$. *For the remainder of the section, we assume without loss of generality that* $\alpha = \beta = 1$; the results continue to hold for any values of α and β (as long as $0 \le \beta(v_H - v_L) \le \alpha$).

For this section we assume that the value of the item X takes values in $[v_L, v_H]$ according to a distribution with continuous density f with support $[v_L, v_H]$. The extension to discrete probability distributions is straightforward.

Let $V(\vec{r})$ be the maximum infinite horizon discounted payoff of the seller when his current vector of ratings is $\vec{r}$. Since the expected payment to the seller is increasing in his description, it is never optimal for the seller to understate the value of an item.

The seller's optimal policy is given by the following dynamic program.

$$V(\vec{r}) = \int_{v_L}^{v_H} \max_{y\in[x,v_H]} \left\{ y \cdot b_T \left(\sum_{i=1}^{T} r_i/T \right) + \delta \cdot V(1-(y-x)^+, \vec{r}) \right\} f(x)dx \tag{3}$$

In particular, if the value of the item is x, the seller chooses a description that maximizes his infinite horizon discounted payoff. If the seller describes the item truthfully, his payoff is $x \cdot b_T(\sum_{i=1}^{T} r_i/T) +$

$\delta V(1,\vec{r})$, since he receives $x \cdot b_T(\sum_{i=1}^T r_i/T)$ and his ratings "increase" to $(1,\vec{r})$. If he describes it as an item of value $y > x$, his payoff is $y \cdot b_T(\sum_{i=1}^T r_i/T) + \delta V(1-(y-x),\vec{r})$, since he receives $y \cdot b_T(\sum_{i=1}^T r_i/T)$ now, but his ratings "decrease" to $(1-(y-x),\vec{r})$. The seller will choose the description with the maximum payoff.

We say that the seller is *truthful at* $\vec{r}$ if it is optimal for him to describe an item of value x truthfully when his rating vector is $\vec{r}$ for all $x \in [v_L, v_H]$ with $f(x) > 0$. By (3), it is optimal for the seller to be truthful at $\vec{r}$ if and only if

$$(y-x) \cdot b\left(\sum_{i=1}^{T} r_i/T\right) \leq \delta(V(1,\vec{r}) - V(1-(y-x),\vec{r}) \tag{4}$$

for all $y \in [x, v_H]$. In particular, if the seller is untruthful and posts a description $y > x$, then his current payoff will increase by $(y-x) \cdot b\left(\sum_{i=1}^T r_i/T\right)$ but his expected payoff starting from the next period will decrease by $V(1,\vec{r}) - V(1-(y-x),\vec{r})$ (relative to being truthful).

Let $D \equiv v_H - v_L$ and

$$F_p(T,\delta) = \left(\min_{d\in(0,D]} \left\{\frac{b_T(1) - b_T(1-d/T)}{d \cdot b_T(1)}\right\}\right) \sum_{i=1}^{T} \delta^i.$$

We use the subscript p to denote that we are considering perfect monitoring, i.e., that the seller receives a rating that accurately reflects his action after every transaction. Note that $F_p(T,\delta)$ is increasing in δ for any fixed T.

LEMMA 1. *If b_T is logarithmically concave, then it is optimal for the seller to be truthful at all $\vec{r}$ if and only if q, δ, and T jointly satisfy the following constraint:*

$$q \cdot F_p(T,\delta) \geq 1.$$

Lemma 1 reduces the problem of checking whether it is optimal for the seller to be always truthful to checking whether the following inequality holds for all $d \in [0, D]$.

$$d \cdot b_T(1) \leq (b_T(1) - b_T(1-d/T)) \sum_{i=1}^{T} \delta^i$$

This condition ensures that the seller does not deviate from being truthful when his score is $s = 1$. This is the case if expected future gains for being truthful are greater than current gains for being untruthful. In particular, if the seller exaggerates his description by d, then his current payment increases by $d \cdot b_T(1)$, but future payments decrease by $b_T(1) - b_T(1-1/T)$ in each of the T next periods.

Our analysis depends on analyzing the set $T_p^*(\delta)$ defined as follows for each δ:

$$T_p^*(\delta) = \arg\max_{T\geq 1} F_p(T,\delta).$$

The following proposition characterizes the behavior of $T_p^*(\delta)$.

PROPOSITION 1. *$T_p^*(\delta)$ is increasing in δ in the following sense: for $\delta \geq \delta'$,*
(i) $\max\{T : T \in T_p^*(\delta)\} \geq \max\{T : T \in T_p^*(\delta')\}$*; and*
(ii) $\min\{T : T \in T_p^*(\delta)\} \geq \min\{T : T \in T_p^*(\delta')\}$.

Surprisingly, note that this result holds *regardless* of the dependence of b_T on T.

From Lemma 1, Proposition 1 and Theorem 1, we conclude in the case of perfect monitoring and logarithmically concave premia: (1) if δ is known and the goal is to maximize the range of q for which the seller is always truthful, then the set of optimal windows is increasing in δ; (2) if q is known and the goal is to maximize the range of δ for which the seller is always truthful, then the set of optimal windows is decreasing in q.

5. Missing Feedback

The previous section assumed perfect monitoring, i.e., that the seller receives a rating which accurately reflects his action after every transaction. However, various studies have shown that monitoring may be imperfect in practice (e.g., Dellarocas and Wood 2008, Chwelos and Dhar 2008, Bolton et al. 2009). In this section and the following section we relax the assumption on perfect monitoring in two ways. First, in this section, we consider the case of missing feedback, where after some transactions the buyer does not rate the seller. In the next section, we consider the case where ratings may not always reflect the seller's action. In this section we show that in the presence of missing feedback, the optimal window size is increasing in δ and decreasing in q.

For simplicity, we focus on a binary setting: we assume that in every period the seller has an item for sale which has value v_H with probability q_H and value v_L with probability $1-q_H$, where $0 \le v_L < v_H$. As before, let $q \equiv q_H v_H + (1-q_H)v_L$ be the expected value of the item that the seller has for sale. In the beginning of every period, the seller observes the item he has for sale and chooses a description of v_L or v_H.

Let p_{av} be the probability that the seller receives no rating when his action is $a \in \{t,u\}$ for being truthful and untruthful respectively, and the true value of the item is $v \in \{v_H, v_L\}$. If the seller receives a rating, we assume that it accurately reflects his action: he receives a good rating (of value 1) for describing his item truthfully, and a bad rating (of value 0) for exaggerating the value of a low value item in his description. We assume that p_{tH} is not significantly larger than p_{uH}, so that it is optimal for the seller to describe a high value item truthfully.

Let $V(\vec{r})$ be the maximum infinite horizon discounted value when the current vector of ratings is $\vec{r}$; then:

$$\begin{aligned} V(\vec{r}) =& q_H \left(v_H \cdot b_T \left(\sum_{i=1}^{T} r_i/T \right) + \delta((1-p_{tH})V(1,\vec{r}) + p_{tH}V(\vec{r})) \right) \\ &+ (1-q_H) \max \left\{ v_H \cdot b_T \left(\sum_{i=1}^{T} r_i/T \right) + \delta((1-p_{uL})V(0,\vec{r}) + p_{uL}V(\vec{r})), \right. \\ &\left. v_L \cdot b_T \left(\sum_{i=1}^{T} r_i/T \right) + \delta((1-p_{tL})V(1,\vec{r}) + p_{tL}V(\vec{r})) \right\} \end{aligned} \quad (5)$$

In particular, with probability q_H the seller has a high value item for sale, which he describes truthfully. The immediate payment he receives is $v_H \cdot b_T(\sum_{i=1}^{T} r_i/T)$; with probability $1-p_{tH}$ he receives a good rating and with probability p_{tH}^n he receives no rating. With probability $1-q_H$ the seller has a low value item for sale. If he describes it as a high value item, his payoff is $v_H \cdot b_T(\sum_{i=1}^{T} r_i/T) + \delta((1-p_{uL})V(0,\vec{r}) + p_{uL}V(\vec{r}))$, since he receives $v_H \cdot b_T(\sum_{i=1}^{T} r_i/T)$ now, but his ratings "decrease" to $(0,\vec{r})$ with probability $1-p_{uL}$ and remain the same with probability p_{uL}. If he describes the item truthfully, he receives a lower payment now, but his ratings "increase" to $(1,\vec{r})$ with probability $1-p_{tL}$. The seller will choose the description that maximizes his payoff.

As in Section 3, our goal is to maximize the range of parameters for which is it optimal for the seller to be truthful. Let $p \equiv q_H \cdot p_{tH} + (1-q_H) \cdot p_{tL}$ be the ex ante probability (before the value of the item is known) that the seller receives no rating if he is truthful.

Let

$$F_m(T,\delta) = \frac{1-p_{uL}}{v_H - v_L} \frac{b_T(1) - b_T(1-1/T)}{b_T(1)} \sum_{i=0}^{\infty} \delta^{i+1} \sum_{j=0}^{\min(T-1,i)} \binom{i}{j} (1-p)^j p^{i-j}$$

We use the subscript m to denote that we are considering the possibility of missing feedback. The function F_m is increasing in δ.

Lemma 2. *If b_T is logarithmically concave and $p_{uL} \ge p_{tL}$, then it is optimal for the seller to be always truthful if and only if*

$$q \cdot F_m(T,\delta) \ge 1. \quad (6)$$

Lemma 2 reduces the problem of finding whether it is optimal for the seller to be always truthful to checking whether the following inequality holds.

$$(v_H - v_L)b_T(1) \leq (1 - p_{uL})q(b_T(1) - b_T(1 - 1/T)) \sum_{i=0}^{\infty} \delta^{i+1} \sum_{j=0}^{\min(T-1,i)} \binom{i}{j} (1-p)^j p^{i-j}$$

This condition requires that the seller does not deviate from being truthful when his current score is equal to 1. The seller does not deviate from being truthful if the discounted future gains for being truthful are greater (in expectation) than the current gains for being untruthful. The current payment to the seller increases by $(v_H - v_L)b_T(1)$ if the seller deviates from being truthful. On the other hand, the future payments to the seller decrease by $b_T(1) - b_T(1 - 1/T)$ in every period that is affected by the current rating. This is the case until the seller receives T new ratings. The seller receives exactly j new ratings in i periods with probability $(1-p)^j p^{i-j}$. Therefore, the probability that the seller has not received T new ratings in i periods is $\sum_{j=0}^{\min(T-1,i)} \binom{i}{j}(1-p)^j p^{i-j}$.

Let $T_m^*(\delta) = \arg\max_T \{F_m(T, \delta)\}$.

Proposition 2. *$T_m^*(\delta)$ is increasing in δ in the following sense: for $\delta \geq \delta'$,*
(i) $\max\{T : T \in T_m^*(\delta)\} \geq \max\{T : T \in T_m^*(\delta')\}$*; and*
(ii) $\min\{T : T \in T_m^*(\delta)\} \geq \min\{T : T \in T_m^*(\delta')\}$*.*

We note that Proposition 2 holds for any logarithmically concave function. Using Lemma 2, Proposition 2 and Theorem 1, we conclude in the case of missing feedback (under the assumptions of Lemma 2) when p is fixed: (1) if δ is known and the goal is to maximize the range of q for which the seller is always truthful, then the set of optimal windows is increasing in δ; (2) if q is known and the goal is to maximize the range of δ for which the seller is always truthful, then the set of optimal windows is decreasing in q. We note that we are assuming that p remains fixed as δ and q change. Since $p = q_H \cdot p_{tH} + (1 - q_H) \cdot p_{tL}$ and $q = q_H \cdot v_H + (1 - q_H) \cdot v_L$, we can assume that q_H is fixed and q changes through v_H and v_L. Alternatively, we can assume that q_H is changing, but also p_{tH} and p_{tL} change accordingly so that p is fixed.

We conclude by discussing two cases that can be viewed as special cases of (5). First, consider the setting where the item that the seller has for sale is not always sold. If the probability that the item is not sold depends on the description that the seller posts, but not on his score, then we can use (5) and still interpret $b_T(s)$ as the expected premium to the seller.

Second, we note that our current model aggregates ratings from rounds in which the seller has a low value item in exactly the same way as ratings from rounds in which the seller has a high value item. This modeling decision was motivated by the way ratings are aggregated in online markets. However, the seller faces no moral hazard for describing a high value item, and it may be reasonable to assume that the seller is not given a rating for advertising a high value item truthfully. Then we can assume that $p_{tH} = 1$ in (5) and still conclude that the optimal window is increasing in δ and decreasing in q.

6. Inaccurate Ratings

The rating that the seller receives may not reflect his action, because the buyer may make a mistake. In this section we identify conditions under which in the presence of inaccurate ratings the optimal window is increasing in δ and decreasing in q.

As in Section 5 we consider a binary setting. We model inaccurate ratings by assuming that with probability p_{av} the seller receives the wrong rating when his action is $a \in \{t, u\}$ for being truthful and untruthful respectively, and the true value of the item is $v \in \{v_H, v_L\}$. We assume that the probability p_{tH} is sufficiently small so that the seller always describes a high value item truthfully.

Let $V(\vec{r})$ be the maximum infinite horizon discounted payoff when the current vector of ratings is $\vec{r}$. Then, the optimization problem of the seller is given by the following dynamic program.

$$\begin{aligned} V(\vec{r}) =& q_H \left(v_H \cdot b_T \left(\sum_{i=1}^{T} r_i/T \right) + \delta \left((1-p_{tH}) V(1,\vec{r}) + p_{tH} V(0,\vec{r}) \right) \right) \\ & + (1-q_H) \max \left\{ v_H \cdot b_T \left(\sum_{i=1}^{T} r_i/T \right) + \delta((1-p_{uL}) V(0,\vec{r}) + p_{uL} V(1,\vec{r})), \right. \\ & \left. v_L \cdot b_T \left(\sum_{i=1}^{T} r_i/T \right) + \delta((1-p_{tL}) V(1,\vec{r}) + p_{tL} V(0,\vec{r})) \right\} \end{aligned}$$

In particular, if the seller describes a low value item as a high value item, he receives a bad rating with probability $1-p_{uL}$ and a good rating with probability p_{uL}. On the other hand, if the seller has a low value item for sale and describes it truthfully, then he receives an immediate payment of $v_L \cdot b_T \left(\sum_{i=1}^{T} r_i/T \right)$ in expectation, and gets a good rating with probability $1-p_{tL}$; with probability p_{tL} he gets a bad rating despite the fact that he described the item truthfully.

Let $p \equiv q_H \cdot p_{tH} + (1-q_H) \cdot p_{tL}$ be the ex ante probability (before the value of the item is known) that the seller receives a negative rating if he is truthful. Let

$$F_w(T,\delta) = \frac{1-p_{uL}-p_{tL}}{v_H - v_L} \sum_{i=0}^{T-1} \delta^{i+1} \sum_{k=0}^{i} \binom{i}{k} p^k (1-p)^{i-k} \frac{b_T(1-k/T) - b_T(1-(k+1)/T)}{b_T(1)}$$

We use the subscript w to denote that we are considering that possibility of wrong feedback, i.e., that the rating may not accurately reflect the seller's action. We observe that $F_w(T,\delta)$ is increasing in δ.

LEMMA 3. *Suppose that one of the following conditions holds.*
(i) b_T is concave; or
(ii) b_T is logarithmically linear, i.e., $b_T(s) = e^{\alpha \cdot s + \beta}$ with $\alpha > 0$; or
(iii) b_T is strictly logarithmically concave and p is sufficiently small.
Then, it is optimal for the seller to be truthful at all $\vec{r}$ if and only if:

$$q \cdot F_w(T,\delta) \geq 1. \tag{7}$$

Lemma 3 reduces the problem of finding whether it is optimal for the seller to be always truthful to checking whether the following inequality holds.

$$(v_H - v_L) b_T(1) \leq (1-p_{uL}-p_{tL}) q \sum_{i=0}^{T-1} \delta^{i+1} \sum_{k=0}^{i} \binom{i}{k} p^k (1-p)^{i-k} \frac{b_T(1-k/T) - b_T(1-(k+1)/T)}{b_T(1)}$$

This inequality checks whether the seller would deviate from being truthful if his current score is equal to 1 and he has a low value item for sale. The seller does not deviate from being truthful if the discounted future gains for being truthful are greater (in expectation) than the current gains for being untruthful. If the seller deviates from being truthful by describing a low value item as a high value item, then his current payment increases by $(v_H - v_L) b_T(1)$. If the seller gets a good rating now and is truthful in future periods, then in i periods from now his score is $1-k/T$ with probability $\binom{i}{k} p^k (1-p)^{i-k}$. On the other hand, if the seller gets a bad rating now and is truthful in future periods, then in i periods from now his score is $1-(k+1)/T$ with probability $\binom{i}{k} p^k (1-p)^{i-k}$. The right hand side considers the difference in the expected payments in the next T periods, and discounts appropriately.

We conjecture that the condition that p must be sufficiently small in condition (iii) of Lemma 3 can be weakened. Numerical experiments with specific premium functions suggest that in the result of Lemma 3 holds for any value of p under condition (iii). However, technical verification of this fact in general remains an open problem.

According to Theorem 1, in order to conclude that the optimal window is increasing in δ and decreasing in q, we will derive conditions under which the set

$$T_w^*(\delta) = \arg\max_T \{F_w(T, \delta)\}$$

is increasing in δ. This is done in the following Proposition.

PROPOSITION 3. *Let*

$$h_{T,T'}(k) = \frac{b_{T'}(1 - k/T')}{b_{T'}(1)} - \frac{b_T(1 - k/T)}{b_T(1)}.$$

If for every $T' > T$ there exists a $k_0 \in \{0, ..., T\}$ such that $h_{T,T'}(k) \leq h_{T,T'}(k+1)$ for $k < k_0$ and $h_{T,T'}(k) \geq h_{T,T'}(k+1)$ for $k > k_0$, then $T_w^*(\delta)$ is increasing in δ in the following sense: for $\delta \geq \delta'$,

(i) $\max\{T : T \in T_w^*(\delta)\} \geq \max\{T : T \in T_w^*(\delta')\}$*; and*

(ii) $\min\{T : T \in T_w^*(\delta)\} \geq \min\{T : T \in T_w^*(\delta')\}$.

We use Theorem 1 to conclude that if the premium function satisfies one of the first two conditions of Lemma 3 and the condition of Proposition 3, then the optimal window is increasing in δ and decreasing in q (when p is fixed). Moreover, if condition (iii) of Lemma 3 and the condition of Proposition 3 hold, then for any $\bar{\delta} < 1$ and for sufficiently small p the optimal window is increasing in δ in the interval $[0, \bar{\delta}]$.[2] Similarly, if q is known and the goal is to maximize the range of δ for which the seller is always truthful, for every range of q and for sufficiently small p the optimal window is decreasing in q.

We have seen in Section 3 that requiring the premium function to be logarithmically concave is not a particularly strong condition. Moreover, the condition of Proposition 3 is satisfied by many premium functions. As an example, consider premium functions of the form $b_T(s) = \alpha(T) \cdot b(s) + \gamma(T)$, where $\alpha(T)$ is nondecreasing in T and $\gamma(T)$ is nonincreasing in T. This form captures the following intuition: *as the window size increases, buyers trust the information that is aggregated in the seller's score more.* For instance, if the seller has the maximum possible score ($s = 1$), we expect that the premium will increase as T increases; this is captured by the assumption that $\alpha(T)$ is increasing. On the other hand, if the seller has the minimum possible score ($s = 0$), we expect the premium to decrease as T increases; this is captured by the assumption that $\gamma(T)$ is nonincreasing. Simple calculations show that the condition of Proposition 3 is satisfied for various functions of this form; e.g., if $b(s) = s^n$ or $b(s) = e^{ns}$, and $\alpha(\cdot)$, $\gamma(\cdot)$ are arbitrary functions of T. We note that many empirical studies use regression forms that correspond to premia of the form $\alpha(T) \cdot b(s) + \gamma(T)$, where $b(s) = e^{ns}$ (e.g., Cabral and Hortacsu 2009, Lucking-Reiley et al. 2007, Ghose et al. 2005).

The following Corollary restricts attention to premia that do not explicitly depend on the window size.

COROLLARY 1. *Suppose $b_T(\cdot) \equiv b(\cdot)$. If $b'(s)$ is logarithmically concave, then $T_w^*(\delta)$ is increasing.*

Examples of functions with a logarithmically concave derivative are $b(s) = s^n$, $b(s) = e^s$ and the logistic function $b(s) = 1/(1 + e^{a-bs})$ for $b > 0$. We note that the conclusion of Corollary 1 holds more generally if $b'(1-y) - yb''(1-y) < 0$ for some $y \in [0, 1]$ implies that $b'(1-z) - zb''(1-z) < 0$ for $z > y$.

[2] For condition (iii) of Lemma 3, the upper bound on p is some increasing function of T, say $u(T)$. Consider $\bar{\delta} < 1$. If b_T satisfies the condition of Proposition 3, then the optimal window is increasing for $\delta \in [0, \bar{\delta}]$ if $p \leq u(\max\{T_w^*(\bar{\delta})\})$.

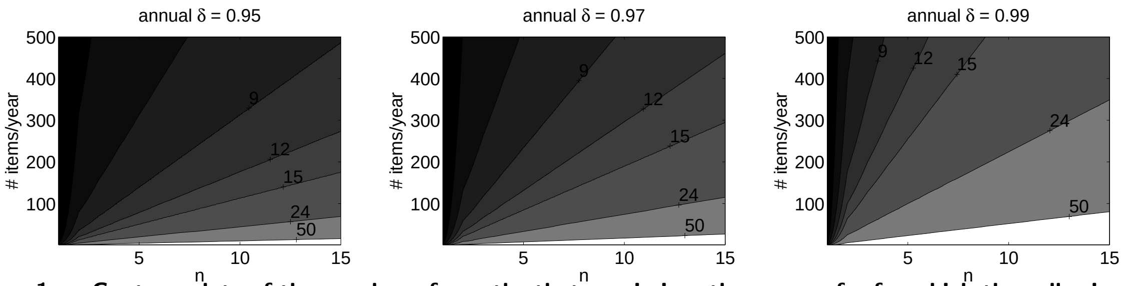

Figure 1 Contour plots of the number of months that maximizes the range of q for which the seller is always truthful. We assume that the premium function is $b(s) = s^n$. Each plot considers a different annual discount factor.

We conclude by summarizing the results of this section. In the case of inaccurate ratings, if any of the assumptions of Lemma 3 and the assumption of Proposition 3 hold, then assuming that p is fixed: (1) if δ is known and the goal is to maximize the range of q for which the seller is always truthful, then the set of optimal windows is increasing in δ; (2) if q is known and the goal is to maximize the range of δ for which the seller is always truthful, then the set of optimal windows is decreasing in q.

7. Conclusions

This paper considers the dependence of the optimal window on the seller's discount factor δ and the average value of the items for sale. We have shown that the optimal window is increasing in δ and decreasing in q for a range of settings. We conclude by briefly discussing the dependence of the optimal window on the rate at which the seller transacts and the steepness of the premium function.

Our analysis throughout the paper considers the optimal number of transactions that should be included in the seller's score. However, electronic marketplaces usually provide information for some fixed window of *real time*. As long as sellers transact at a fixed rate per unit time, the optimal window is increasing in δ and decreasing in q even when the goal is to choose the optimal window measured in real time (e.g., months or days).

We now investigate the dependence of the optimal window on the rate at which a seller transacts, and on the steepness of the premium function. We assume that the expected premium to the seller as a function of his score is $b(s) = s^n$, so that n characterizes the steepness of the premium function; we have observed similar qualitative behavior with other choices of premia. Moreover, we assume that all sellers have the same annual discount factor δ. Sellers may sell items at different frequencies. A seller that is selling f items per year discounts the payment he will receive from the next item that he will have for sale by $\delta^{1/f}$. Assuming that the annual discount factor is known, we wish to find the number of months that maximizes the range of q for which the seller is always truthful. (The same qualitative insights hold if q is fixed and the goal is to maximize the range of δ for which the seller is always truthful.)

Figure 1 shows contour plots of the optimal number of months for three values of the annual discount factor; recall that twelve months is the window size chosen by both eBay and the Amazon Marketplace. The horizontal axis represents the steepness of the premium function, and the vertical axis shows how many items the seller sells per year. The plots demonstrate that the optimal number of months increases as the discount factor increases (for a fixed point). Moreover, Figure 1 suggests that *the optimal number of months increases as the premium function becomes steeper, and as the number of items per year decreases.*

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Proofs of Statements

Appendix: Proofs

This appendix contains the proofs of the lemmas and propositions of the paper. We first show a result that is used in multiple proofs.

LEMMA EC.1. *Let $f:[0,1]\rightarrow R_+$ be an increasing and logarithmically concave function. Then*

$$(f(1)-f(1-t))\cdot f(k)\leq f(1)\cdot(f(k)-f(k-t)) \tag{EC.1}$$

for all $k<1$ and $t\in(0,k]$.

Proof of Lemma EC.1: Condition (EC.1) is equivalent to

$$f(1-t)\cdot f(k)\geq f(1)\cdot f(k-t).$$

Since f is nonnegative, this trivially holds if $f(k-t)=0$. Moreover, if either $f(1-t)=0$ or $f(k)=0$, then $f(k-t)=0$ (since f is increasing).

If $f(k-t)>0$, it suffices to show that $f(x)/f(x-t)$ is nondecreasing in x for $x\in[k,1]$. Since f is logarithmically concave, we conclude that $\log(f(x))-\log(f(x-t))$ is nonincreasing in x, which implies that $f(x)/f(x-t)$ is nondecreasing in x. This concludes the proof. ■

Proof of Lemma 1: It is optimal for the seller to be always truthful if and only if any one step deviation from the truthful policy (i.e., the policy of always advertising items truthfully) does not yield a higher payoff. The seller may consider to exaggerate an item in his description when the value of the item is $v<v_H$. Let $\hat{V}(\vec{r})$ be the infinite horizon discounted expected value when the seller's current ratings are $\vec{r}$ and the seller is always truthful. Let $s_i(\vec{r})$ be the seller's score after i periods if he gets a rating of value 1 in all future periods, given that his current ratings are $\vec{r}$. Note that $s_0(\vec{r})=\sum_{i=1}^{T}r_i/T$ is the seller's current score. Then,

$$\hat{V}(1,\vec{r})-\hat{V}(\rho,\vec{r})=q\sum_{i=0}^{T-1}\delta^i(b_T(s_i(1,\vec{r}))-b_T(s_i(\rho,\vec{r})))$$

and $s_i(1,\vec{r})-s_i(\rho,\vec{r})=(1-\rho)/T$. By (4), the seller is not better off deviating from the truthful policy when his ratings are $\vec{r}$ if and only if

$$d\cdot b_T(s_0(\vec{r}))\leq\delta(\hat{V}(1,\vec{r})-\hat{V}(1-d,\vec{r}))$$

for $d\in[0,D]$, where $D=v_H-v_L$. It is optimal for the seller to be always truthful if and only if the previous condition holds for all $\vec{r}$. Substituting $\hat{V}(1,\vec{r})-\hat{V}(\rho,\vec{r})$ in the previous condition and setting $d=1-\rho$, we conclude that it is optimal for the seller to be truthful at all $\vec{r}$ if and only if

$$d\leq q\sum_{i=0}^{T-1}\delta^{i+1}\frac{b_T(s_i(1,\vec{r}))-b_T(s_i(1-d,\vec{r}))}{b_T(s_0(\vec{r}))}$$

for $d\in[0,D]$. This shows that $q\cdot F_p(T,\delta)\geq 1$ is a necessary condition for the seller to be always truthful; in particular, the condition is necessary and sufficient to ensure truthfulness when the current score of the seller is $s_0(\vec{r})=1$. To show that it is also sufficient we will show that

$$\frac{\sum_{i=0}^{T-1}\delta^i(b_T(1)-b_T(1-d/T))}{b_T(1)}\leq\frac{\sum_{i=0}^{T-1}\delta^i(b_T(s_i(1,\vec{r}))-b_T(s_i(1,\vec{r})-d/T))}{b_T(s_0(\vec{r}))}$$

for all $\vec{r}$ and $d > 0$. Since $s_i(1,\vec{r}) \geq s_0(\vec{r})$, it suffices that

$$(b_T(1) - b_T(1-t))b_T(k) \leq b_T(1)(b_T(k) - b_T(k-t))$$

for all $k < 1$ and $t \in (0,k]$, which holds (by Lemma EC.1). ■

Proof of Proposition 1: We will show that $\log(F_p(T,\delta))$ satisfies increasing differences. Let $T' \geq T$.

$$\log(F_p(T',\delta)) - \log(F_p(T,\delta)) = \log\left(\frac{1-\delta^{T'}}{1-\delta^T}\right) + t(T,T'),$$

where $t(T,T')$ does not depend on δ. Thus, to show that $\log(F_p(T,\delta))$ has increasing differences in (T,δ) it suffices to show that $(1-\delta^{T'})/(1-\delta^T)$ is increasing in δ. The first derivative with respect to δ is positive if and only if

$$\frac{T\cdot\delta^{T-1}}{1-\delta^T} \geq \frac{T'\cdot\delta^{T'-1}}{1-\delta^{T'}}.$$

Since $T' \geq T$ it suffices to show that $r(x) \equiv (x\cdot\delta^{x-1})/(1-\delta^x)$ is decreasing. We proceed by differentiating r:

$$r'(x) = \frac{\delta^{x-1}}{(1-\delta^x)^2}(1-\delta^x - x\ln(1/\delta)).$$

To complete the proof, we show that $\delta^T + T\ln(1/\delta) > 1$ holds for $T \geq 1$, $\delta \in (0,1)$. First note that $\delta^T + T\ln(1/\delta)$ is increasing in T, since

$$\frac{\partial(\delta^T + T\ln(1/\delta))}{\partial T} = \ln(1/\delta)\cdot(1-\delta^T) > 0.$$

So it suffices to show that $\hat{g}(\delta) \equiv \delta + \ln(1/\delta) > 1$. g is strictly decreasing in $(0,1)$, because

$$\hat{g}'(\delta) = 1 + \frac{-1/\delta^2}{1/\delta} = \frac{\delta-1}{\delta} < 0,$$

and $\hat{g}(1) = 1$. So, $\hat{g}(\delta) > 1$ for $\delta \in (0,1)$.

This proves that $\log(F_p(T,\delta))$ has increasing differences in (T,δ); the result follows by applying Topkis' Theorem (Topkis 1998). ■

Proof of Lemma 2: It is optimal for the seller to be always truthful if and only if any one step deviation from the truthful policy (i.e., the policy of always advertising items truthfully) does not yield a higher payoff. The seller may consider to exaggerate an item in his description if in that period he has a low value item for sale. Let $\hat{V}(\vec{r})$ be the expected infinite horizon discounted payoff to the seller if he is always truthful and his current vector of ratings is $\vec{r}$. The seller does not deviate from the truthful policy at $\vec{r}$ if and only if

$$(v_H - v_L)\cdot b_T\left(\sum_{i=1}^{T} r_i/T\right) \leq \delta[(1-p_{tL})\hat{V}(1,\vec{r}) - (1-p_{uL})\hat{V}(0,\vec{r}) + (p_{tL}-p_{uL})\hat{V}(\vec{r})]. \quad \text{(EC.2)}$$

With probability $a(i,j) \equiv \binom{i}{j}(1-p)^j p^{i-j}$ exactly j new ratings arrive in i periods. Let $s_j(\vec{r})$ be the score of the seller after j good rating arrive with initial rating vector $\vec{r}$. Then $s_0(\vec{r}) = \sum_{i=1}^{T} r_i/T$, and

$$\begin{aligned}
&(1-p_{tL})\hat{V}(1,\vec{r}) - (1-p_{uL})\hat{V}(0,\vec{r}) + (p_{tL}-p_{uL})\hat{V}(\vec{r}) = \\
&q\cdot\sum_{i=0}^{\infty}\delta^i \sum_{j=0}^{\min(T-1,i)} a(i,j)[(1-p_{tL})b_T(s_j(1,\vec{r})) - (1-p_{uL})b_T(s_j(0,\vec{r})) + (p_{tL}-p_{uL})b_T(s_j(\vec{r}))] = \\
&q\cdot\sum_{i=0}^{\infty}\delta^i \sum_{j=0}^{\min(T-1,i)} a(i,j)[(1-p_{tL})(b_T(s_j(1,\vec{r})) - b_T(s_j(\vec{r}))) + (1-p_{uL})(b_T(s_j(\vec{r})) - b_T(s_j(0,\vec{r})))]
\end{aligned}$$

This shows that $q \cdot F_m(T,\delta) \geq 1$ is equivalent to (EC.2) for $\vec{r} = \vec{1}$, and thus it is a necessary condition for the seller to be always truthful. To show sufficiency, it suffices to show that for $j < T$,

$$\frac{(1-p_{tL})(b_T(s_j(1,\vec{r})) - b_T(s_j(\vec{r}))) + (1-p_{uL})(b_T(s_j(\vec{r})) - b_T(s_j(0,\vec{r})))}{b_T(s_0(\vec{r}))} \geq$$
$$\frac{(1-p_{uL})(b_T(1) - b_T(1-1/T))}{b_T(1)}$$

Note that $s_j(1,\vec{r}) \geq s_j(\vec{r}) \geq s_j(0,\vec{r})$ and $s_j(1,\vec{r}) = s_j(0,\vec{r}) + 1/T$. Therefore, either $s_j(1,\vec{r}) = s_j(\vec{r}) + 1/T$ or $s_j(\vec{r}) = s_j(0,\vec{r}) + 1/T$.

We first assume that $s_j(1,\vec{r}) = s_j(\vec{r}) + 1/T$ and $s_j(\vec{r}) = s_j(0,\vec{r})$. Then

$$\frac{(1-p_{tL})(b_T(s_j(1,\vec{r})) - b_T(s_j(\vec{r}))) + (1-p_{uL})(b_T(s_j(\vec{r})) - b_T(s_j(0,\vec{r})))}{b_T(s_0(\vec{r}))} =$$
$$(1-p_{tL})\frac{b_T(s_j(1,\vec{r})) - b_T(s_j(\vec{r}))}{b_T(s_0(\vec{r}))} \geq$$
$$(1-p_{tL})\frac{b_T(1) - b_T(1-1/T)}{b_T(1)} \geq$$
$$(1-p_{uL})\frac{b_T(1) - b_T(1-1/T)}{b_T(1)}$$

This first inequality is a consequence of the fact that b_T is logarithmically concave, together with Lemma EC.1; and the second inequality holds because $p_{uL} \geq p_{tL}$.

We next assume that $s_j(1,\vec{r}) = s_j(\vec{r})$ and $s_j(\vec{r}) = s_j(0,\vec{r}) + 1/T$. Then

$$\frac{(1-p_{tL})(b_T(s_j(1,\vec{r})) - b_T(s_j(\vec{r}))) + (1-p_{uL})(b_T(s_j(\vec{r})) - b_T(s_j(0,\vec{r})))}{b_T(s_0(\vec{r}))} =$$
$$(1-p_{uL})\frac{b_T(s_j(\vec{r})) - b_T(s_j(0,\vec{r}))}{b_T(s_0(\vec{r}))} \geq$$
$$(1-p_{uL})\frac{b_T(1) - b_T(1-1/T)}{b_T(1)}$$

This again holds by applying (EC.1) of Lemma EC.1.

We conclude that if $p_{uL} \geq p_{tL}$, then the seller is always truthful if and only if $q \cdot F_m(T,\delta) \geq 1$. ■

Proof of Proposition 2: Let

$$g(T,\delta) \equiv \frac{b_T(1) - b_T(1-1/T)}{b_T(1)} \sum_{i=0}^{\infty} f(T,i)\delta^i,$$

$$f(T,i) = \sum_{j=0}^{\min(T-1,i)} \binom{i}{j} (1-p)^j p^{i-j}.$$

Also let $\alpha(i,j) \equiv \binom{i}{j}(1-p)^j p^{i-j}$. Clearly, $T_m^*(\delta) = \arg\max_T\{g(T,\delta)\}$.

We will show that g satisfies the single crossing property in (T,δ), i.e., that $g(T',\delta) > g(T,\delta)$ implies $g(T',\delta') > g(T,\delta')$ for $\delta' > \delta$ and $T' > T$. This will imply that $T^*(\delta)$ is increasing in δ (Milgrom and Shannon 1994). Let

$$c \equiv \frac{(b_{T'}(1) - b_{T'}(1-1/T'))/b_{T'}(1)}{(b_T(1) - b_T(1-1/T))/b_T(1)}.$$

Equivalently we will show that if

$$\sum_{i=0}^{\infty}(c \cdot f(T',i))\delta^i > \sum_{i=0}^{\infty} f(T,i)\delta^i,$$

then the inequality also holds for $\delta' > \delta$.

We first show that if $c \cdot f(T',i) > f(T,i)$, then $c \cdot f(T',i+1) > f(T,i+1)$. We consider the cases $i < T$ and $i \geq T$ separately.

Suppose $i < T$. Then $cf(T',i) > f(T,i)$ implies that

$$c\sum_{j=0}^{i} a(i,j) > \sum_{j=0}^{i} a(i,j),$$

which can only happen if $c > 1$. Also $\min(i+1, T'-1) = i+1$, while $\min(i+1, T-1)$ is $i+1$ if $i < T-1$; and i if $i = T-1$. In either case, $c \cdot f(T',i+1) > f(T,i+1)$.

Now suppose that $i \geq T$, and let $k \equiv \min(i, T'-1)$. Assume $c \cdot f(T',i) - f(T,i) > 0$. Then

$$\sum_{j=T}^{k} a(i,j) > \frac{1-c}{c}\sum_{j=0}^{T-1} a(i,j).$$

We observe that

$$a(i+1,j) = (1-p)\frac{i+1}{i+1-j}a(i,j) = (1-p)\left(1 + \frac{j}{i+1-j}\right)a(i,j).$$

Then,

$$\begin{aligned}
&\sum_{j=T}^{k} a(i+1,j) - \frac{1-c}{c}\sum_{j=0}^{T-1} a(i+1,j) \\
&= (1-p)\sum_{j=T}^{k}\left(1 + \frac{j}{i+1-j}\right)a(i,j) - (1-p)\frac{1-c}{c}\sum_{j=0}^{T-1}\left(1 + \frac{j}{i+1-j}\right)a(i,j)
\end{aligned}$$

Moreover,

$$\sum_{j=T}^{k}\frac{i+1}{i+1-j}a(i,j) \geq \frac{i+1}{i+1-T}\sum_{j=T}^{k} a(i,j) > \frac{1-c}{c}\frac{i+1}{i+1-T}\sum_{j=0}^{T-1} a(i,j) \geq \frac{1-c}{c}\sum_{j=0}^{T-1}\frac{i+1}{i+1-j}a(i,j).$$

Since

$$\sum_{j=T}^{k} a(i,j) - \frac{1-c}{c}\sum_{j=0}^{T-1} a(i,j) > 0,$$

we have that

$$\sum_{j=T}^{k}\frac{i+1}{i+1-j}a(i,j) \geq \frac{i+1}{i+1-T}\sum_{j=T}^{k} a(i,j) > \frac{1-c}{c}\frac{i+1}{i+1-T}\sum_{j=0}^{T-1} a(i,j) \geq \frac{1-c}{c}\sum_{j=0}^{T-1}\frac{i+1}{i+1-j}a(i,j)$$

We conclude that if $c \cdot f(T',i) - f(T,i) > 0$, then

$$\sum_{j=T}^{k} a(i+1,j) > \frac{1-c}{c}\sum_{j=0}^{T-1} a(i+1,j).$$

Since $a(i+1,j)\geq 0$, $\min(i+1,T'-1)\geq k$ and $\min(i+1,T-1)=T-1$, this implies that

$$c\cdot f(T',i+1)-f(T,i+1)>0.$$

The final step of the proof is to show that if $\sum_{i=0}^{\infty}(c\cdot f(T',i))\delta^i>\sum_{i=0}^{\infty}f(T,i)\delta^i$, then the inequality also holds for $\delta'>\delta$. Let $T'>T$ and $e_i=c\cdot f(T',i)-f(T,i)$. We have shown that if $e_i>0$ then $e_{i+1}>0$. If $e_i>0$ for all i, then trivially $\sum_{i=0}^{\infty}e_i\delta^i>0$ for all δ.

Now suppose $e_0<0$ and let $k=\max\{i:e_i<0\}$. If $\sum_{i=0}^{\infty}e_i\delta^i>0$, then

$$\sum_{i=k+1}^{\infty}|e_i|\delta^i>\sum_{i=1}^{k}|e_i|\delta^i.$$

Moreover,

$$\sum_{i=k+1}^{\infty}i|e_i|\delta^{i-1}>\sum_{i=1}^{k}i|e_i|\delta^{i-1}.$$

The last inequality is equivalent to

$$\frac{\partial}{\partial\delta}\left(\sum_{i=0}^{\infty}e_i\delta^i\right)=\sum_{i=1}^{\infty}ie_i\delta^{i-1}>0,$$

which concludes the proof. ■

Proof of Lemma 3: It is optimal for the seller to describe a low value item truthfully at $\vec{r}$ if

$$(v_H-v_L)\cdot b_T\left(\sum_{i=1}^{T}r_i/T\right)\leq\delta(1-p_{uL}-p_{tL})(V(1,\vec{r})-V(0,\vec{r})).$$

It is optimal for the seller to be always truthful if and only if any one step deviation from the truthful policy (i.e., the policy of always advertising items truthfully) does not yield a higher payoff. The seller may consider to exaggerate an item in his description if in that period he has a low value item for sale. Let $\hat{V}(\vec{r})$ be the infinite horizon discounted payoff to the seller if he is always truthful. The seller will not deviate from being truthful when his ratings are $\vec{r}$ if

$$(v_H-v_L)\cdot b_T\left(\sum_{j=1}^{T}r_j/T\right)\leq\delta(1-p_{uL}-p_{tL})(\hat{V}(1,\vec{r})-\hat{V}(0,\vec{r})).$$

We observe that the payments to the seller from $\hat{V}(1,\vec{r})$ and $\hat{V}(0,\vec{r})$ may differ in the next T periods, but not after that. Let $s_i(\vec{r})$ be the seller's score in i periods if his current rating vector is $\vec{r}$ and he gets good ratings in all future periods. Then,

$$\hat{V}(1,\vec{r})-\hat{V}(0,\vec{r})=q\sum_{k=0}^{i}\binom{i}{k}p^k(1-p)^{i-k}\left(b_T(s_i(1,\vec{r})-k/T)-b_T(s_i(1,\vec{r})-(k+1)/T)\right)$$

In particular, after i periods the seller gets k bad ratings (which are inaccurate) with probability $a(i,k)\equiv\binom{i}{k}p^k(1-p)^{i-k}$. We conclude that the seller will not deviate from being truthful when his ratings are $\vec{r}$ if

$$(v_H-v_L)\cdot b_T(s_0(\vec{r}))\leq\delta(1-p_{uL}-p_{tL})q\sum_{k=0}^{i}a(i,k)\left(b_T(s_i(1,\vec{r})-k/T)-b_T(s_i(1,\vec{r})-(k+1)/T)\right).$$

To prove the Lemma it suffices to show that

$$\sum_{k=0}^{i} a(i,k)\left(\frac{b_T(s_i(1,\vec{r})-k/T)-b_T(s_i(1,\vec{r})-(k+1)/T)}{b_T(s_0(\vec{r}))}-\frac{b_T(1-k/T)-b_T(1-(k+1)/T)}{b_T(1)}\right)\geq 0. \tag{EC.3}$$

This implies that if the seller does not deviate from being truthful at $\vec{1}$, then he does not deviate at any $\vec{r}$. The remainder of the proof shows that if (i), (ii) or (iii) is satisfied, then (EC.3) holds.

We first consider condition (i) and assume that b_T is concave. Then,

$$b_T(s_i(1,\vec{r})-k/T)-b_T(s_i(1,\vec{r})-(k+1)/T)\geq b_T(1-k/T)-b_T(1-(k+1)/T)$$

by the concavity of b_T, and

$$b_T(1)\geq b_T\left(\sum_{j=1}^{T} r_j/T\right)$$

since b_T is increasing. We conclude that in this case

$$\left(\frac{b_T(s_i(1,\vec{r})-k/T)-b_T(s_i(1,\vec{r})-(k+1)/T)}{b_T\left(\sum_{i=1}^{T} r_i/T\right)}-\frac{b_T(1-k/T)-b_T(1-(k+1)/T)}{b_T(1)}\right)\geq 0,$$

and thus (EC.3) holds.

Now assume that $b_T(s)=e^{\alpha\cdot s+\beta}$ and $\alpha>0$ (condition (ii)). Then

$$\frac{b_T(s_i(1,\vec{r})-k/T)-b_T(s_i(1,\vec{r})-(k+1)/T)}{b_T\left(\sum_{i=1}^{T} r_i/T\right)}-\frac{b_T(1-k/T)-b_T(1-(k+1)/T)}{b_T(1)}=$$

$$\frac{e^{\alpha(s_i(1,\vec{r})-k/T)+\beta}-e^{\alpha(s_i(1,\vec{r})-(k+1)/T)+\beta}}{e^{\alpha\left(\sum_{j=1}^{T} r_j/T\right)+\beta}}-\frac{e^{\alpha(1-k/T)+\beta}-e^{\alpha(1-(k+1)/T)+\beta}}{e^{\alpha+\beta}}=$$

$$\frac{e^{\alpha(s_i(1,\vec{r})-k/T)}-e^{\alpha(s_i(1,\vec{r})-(k+1)/T)}}{e^{\alpha\left(\sum_{j=1}^{T} r_j/T\right)}}-\frac{e^{\alpha(1-k/T)}-e^{\alpha(1-(k+1)/T)}}{e^{\alpha}}=$$

$$\left(\frac{e^{\alpha(s_i(1,\vec{r}))}}{e^{\alpha\left(\sum_{j=1}^{T} r_j/T\right)}}-1\right)(e^{-\alpha k/T}-e^{-\alpha(k+1)/T})\geq 0$$

because $s_i(1,\vec{r})\geq\sum_{j=1}^{T} r_j/T$ and $\alpha>0$.

We now show (iii). We will show that if b_T is strictly logarithmically concave, then (EC.3) holds for sufficiently small p. If b_T is strictly log-concave, then $\log(b_T(x))-\log(b_T(x-w))$ is strictly decreasing in x. Thus $b_T(x-w)/b_T(x)$ is strictly increasing in x and $b_T(1-1/T)/b_T(1)>b_T(x-1/T)/b_T(x)$. Let

$$c(T)\equiv\frac{b_T(1-1/T)}{b_T(1)}-\frac{b_T(1-2/T)}{b_T(1-1/T)}.$$

Then

$$\frac{b_T(1-1/T)}{b_T(1)}-\frac{b_T(x-1/T)}{b_T(x)}\geq c(T)$$

for $x\in\{1/T,2/T,...,(T-1)/T\}$, and $c(T)>0$. Moreover, there exists λ such that

$$\frac{b_T(x-k/T)-b_T(x-(k+1)/T)}{b_T(x)}-\frac{b_T(1-k/T)-b_T(1-(k+1)/T)}{b_T(1)}\geq-\lambda$$

for $x \in \{1/T, 2/T, ..., (T-1)/T\}$ and $k \in \{1, 2, ..., Tx-1\}$. For instance, the aforementioned inequality holds for any premium function b_T if $\lambda = 2$, since

$$\frac{b_T(x-(k+1)/T)}{b_T(x)} + \frac{b_T(1-k/T)}{b_T(1)} \leq \frac{b_T(x)}{b_T(x)} + \frac{b_T(1)}{b_T(1)} = 2.$$

We conclude that if

$$p \leq 1 - \left(\frac{\lambda}{c(T)+\lambda}\right)^{1/T}$$

then condition (EC.3) holds. ∎

Proof of Proposition 3: Let

$$c(i) = \sum_{k=0}^{i} \binom{i}{k} p^k (1-p)^{i-k} f(k),$$

$$f(k) = \frac{b_T(1-k/T) - b_T(1-1/T-k/T)}{b_T(1)} - \frac{b_{T'}(1-k/T') - b_{T'}(1-1/T'-k/T')}{b_{T'}(1)},$$

$$g(T,\delta) = \sum_{i=0}^{T-1} \delta^i \sum_{k=0}^{i} \binom{i}{k} p^k (1-p)^{i-k} \frac{b_T(1-k/T) - b_T(1-(k+1)/T)}{b_T(1)}.$$

Clearly, $T_w^*(\delta) = \arg\max_T\{g(T,\delta)\}$.

The proof consists of three steps. First, we show that if $h_{T,T'}$ satisfies the assumption of the proposition, then $f(k) < 0$ implies $f(k+1) < 0$. The second step is to show that $c(i) < 0$ implies that $c(i+1) < 0$. Then we show that g satisfies the single crossing property in (T,δ) and conclude that $T_w^*(\delta)$ is increasing.

Step 1: Let

$$h(k) = \frac{b_{T'}(1-k/T')}{b_{T'}(1)} - \frac{b_T(1-k/T)}{b_T(1)}$$

We observe that

$$f(k) = \frac{b_{T'}(1-(k+1)/T')}{b_{T'}(1)} - \frac{b_T(1-(k+1)/T)}{b_T(1)} - \frac{b_{T'}(1-k/T')}{b_{T'}(1)} + \frac{b_T(1-k/T)}{b_T(1)} = h(k+1) - h(k).$$

Since there exists a $k_0 \in \{0, ..., T\}$ such that h_k is increasing in k for $k < k_0$ and decreasing in k for $k \geq k_0$, we conclude that $f_k < 0$ if and only if $k \geq k_0$. Thus, if $f(k) < 0$ then $f(k+1) < 0$.

Step 2: Let

$$a(i,k) = \binom{i}{k} p^k (1-p)^{i-k}.$$

The key property we exploit is that:

$$\frac{a(i+1,k)}{a(i,k)} = (1-p)\frac{i+1}{i+1-k}$$

is strictly increasing in k. We have shown that $f(k) \geq 0$ for $k < k_0$ and $f(k) < 0$ for $k \geq k_0$. Suppose $c(i) < 0$. Of course, in this case we must have $i \geq k_0$. Then:

$$\sum_{k=0}^{k_0-1} a(i,k) f(k) < -\sum_{k=k_0}^{i} a(i,k) f(k).$$

But now note that for all $k < k_0$, $i+1-k > i+1-k_0$; and for all k such that $k_0 \le k < i$, $i+1-k \le i+1-k_0$. So we get:

$$\begin{aligned}\sum_{k=0}^{k_0-1} a(i+1,k)f(k) &= (i+1)(1-p)\sum_{k=0}^{k_0-1} a(i,k)f(k)/(i+1-k)\\ &< (i+1)(1-p)\sum_{k=0}^{k_0-1} a(i,k)f(k)/(i+1-k_0)\\ &< -(i+1)(1-p)\sum_{k=k_0}^{i} a(i,k)f(k)/(i+1-k_0)\\ &< -(i+1)(1-p)\sum_{k=k_0}^{i} a(i,k)f(k)/(i+1-k)\\ &< -\sum_{k=k_0}^{i+1} a(i+1,k)f(k)\end{aligned}$$

where the last inequality follows since $f(i+1) < 0$. We conclude that $c(i+1) < 0$, as required.
Step 3: Let $T' > T$ and $\delta' > \delta$. The function g satisfies the single crossing property in (T,δ) if $g(T',\delta) > g(T,\delta)$ implies that $g(T',\delta') > g(T,\delta')$. We observe that

$$\begin{aligned}&g(T',x) - g(T,x) =\\ &\sum_{i=T}^{T'-1}\delta^i \sum_{k=0}^{i}\binom{i}{k}p^k(1-p)^{i-k}\frac{b_T(1-k/T') - b_T(1-(k+1)/T')}{b_{T'}(1)} - \sum_{i=0}^{T-1}\delta^i\sum_{k=0}^{i}\binom{i}{k}p^k(1-p)^{i-k}f(k) =\\ &\sum_{i=T}^{T'-1}\delta^i \sum_{k=0}^{i}\binom{i}{k}p^k(1-p)^{i-k}\frac{b_T(1-k/T) - b_T(1-(k+1)/T)}{b_T(1)} - \sum_{i=0}^{T-1}\delta^i c(i)\end{aligned}$$

According to Step 2, there exists some i_0 such that we can rewrite the previous difference as

$$g(T',x) - g(T,x) = -\sum_{i=0}^{i_0-1} x^i d_i + \sum_{i=i_0}^{T'-1} x^i d_i,$$

where $d_i \ge 0$ for all i. Assume that $g(T',\delta) - g(T,\delta) > 0$. Then,

$$\sum_{i=0}^{i_0-1} i\delta^{i-1}d_i = \sum_{i=0}^{i_0-1}\frac{i}{\delta}\delta^i d_i \le \frac{i_0-1}{\delta}\sum_{i=0}^{i_0-1}\delta^i d_i \le \frac{i_0-1}{\delta}\sum_{i=i_0}^{T'-1}\delta^i d_i = \sum_{i=T}^{T'-1}(i_0-1)\delta^{i-1}d_i \le \sum_{i=i_0}^{T'-1} i\delta^{i-1}d_i.$$

This implies that if $g(T',\delta) - g(T,\delta) > 0$, then $g'(T',\delta) - g'(T,\delta) \ge 0$. We conclude that if $g(T',\delta) - g(T,\delta) > 0$, then $g(T',\delta') - g(T,\delta') > 0$ for $\delta' > \delta$. This shows that the objective satisfies the single crossing property. Thus, we apply Theorem 4 from Milgrom and Shannon (1994) to conclude the proof. ■

Proof of Corollary 1: We will first show a stronger result: if $b'(1-y) - yb''(1-y) < 0$ for some $y \in [0,1]$ implies that $b'(1-z) - zb''(1-z) < 0$ for $z > y$, then $T_w^*(\delta)$ is increasing in δ. Then, we will show that this is the case if $\log(b'(s))$ is concave.

Let

$$g(x) \equiv b(1)\cdot h'_{T',T}(x) = \frac{1}{T}b'(1-x/T) - \frac{1}{T'}b'(1-x/T').$$

Clearly, $g(0) > 0$. It suffices to show that if $g(x) < 0$ then $g(x') < 0$ for $x' > x$ in order to apply Proposition 3.

$$\begin{aligned} g(x) &= \frac{1}{T}b'(1 - x/T) - \frac{1}{T'}b'(1 - x/T') \\ &= \int_{1/T'}^{1/T} \frac{\partial}{\partial y}[yb'(1 - yx)]dy \\ &= \int_{1/T'}^{1/T} [b'(1 - yx) - yxb''(1 - yx)]dy \\ &= \frac{1}{x}\int_{x/T'}^{x/T} [b'(1 - z) - zb''(1 - z)]dz \end{aligned}$$

If $b'(1 - y) - yb''(1 - y) > 0$ for all $y \in [0, 1]$ then $g(x) > 0$ for all $x \in [0, T]$. Otherwise there exists $z_0 \in (0, 1]$ such that $b'(1 - z_0) - z_0 b''(1 - z_0) = 0$ and $b'(1 - z) - zb''(1 - z) > 0$ for $z < z_0$; $b'(1 - z) - zb''(1 - z) < 0$ for $z > z_0$.

If $g(x) < 0$, then

$$\int_{x/T'}^{z_0} [b'(1 - z) - zb''(1 - z)]dz < \int_{z_0}^{x/T} |b'(1 - z) - zb''(1 - z)|dz.$$

Let $x' > x$. Then $x'/T' > x/T'$ and $x'/T > x/T$. If $x'/T' > z_0$, then $g(x') < 0$. If $x'/T' < z_0$, then

$$\int_{x/T'}^{z_0} [b'(1 - z) - zb''(1 - z)]dz > \int_{x'/T'}^{z_0} [b'(1 - z) - zb''(1 - z)]dz$$

and

$$\int_{z_0}^{x/T} |b'(1 - z) - zb''(1 - z)|dz < \int_{z_0}^{x'/T} |b'(1 - z) - zb''(1 - z)|dz.$$

We conclude that

$$\int_{x'/T'}^{z_0} [b'(1 - z) - zb''(1 - z)]dz < \int_{z_0}^{x'/T} |b'(1 - z) - zb''(1 - z)|dz$$

which implies that $g(x') < 0$.

We have shown the result for the case that $b'(1 - y) - yb''(1 - y) < 0$ implies that $b'(1 - z) - zb''(1 - z) < 0$ for $z > y$. A sufficient condition for this is that $(1 - x)b''(x)/b'(x)$ is decreasing. The latter holds if b' is logarithmically concave. ■