This is the result first presented in (1), that the optimal ignition-to-burnout mass ratios are the same for all stages.

Inclusion of Gravity, Drag and Turning Corrections

To the extent that one is able to make rough estimates of velocity losses due to gravity, drag and turning, these losses can be included as modifications of the required V in Equation [5], and the corrections will be reflected in the determination of the μ_j. The resulting improvement is, of course, no better than the estimates of the losses; hence this method is not a substitute for accurate calculations with a high-speed digital computer. Instead, it has proved to be a valuable auxiliary, in providing very good slide-rule designs, thereby materially reducing the amount of expensive computer time needed to locate design optima.

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Optimum Rocket Trajectories With Aerodynamic Drag

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The problem considered is that of determining rocket thrust programs that use the least amount of fuel to propel a given mass in a vertical plane from one point to another without aerodynamic lift, where gravitational force and aerodynamic drag are known as functions of altitude and speed. It is solved by formulating it as a Mayer problem in the calculus of variations with consideration given to the fact that the rocket mass can never increase during the flight. A sample problem for a short-range ground-to-air rocket has been worked out using a digital computer to show some of the features of these least-fuel trajectories.

Introduction

GODDARD (1),[3] Hamel (2), Tsien and Evans (3), and Leitmann (4, 5) have considered the optimum thrust program for a vertical sounding rocket and have established that this program involves a rapid boost at the beginning of flight, usually followed by a period of continuous burning (sustain phase), and ending with a zero thrust period (coasting phase). Edwards (6) and Cicala and Miele (7) have formulated the present problem in considerable detail; the principal contribution of the present authors is to add to this formulation a way to handle the constraint of nonincreasing mass and some numerical calculations of a typical case. Actually, two constraints were considered: (a) that the rocket mass never becomes less than the final (empty) mass and (b) that the rate of change of rocket mass be zero or negative. It was found that specifying only (a) was sufficient for the present problem since the resulting thrust programs also satisfied (b). Earlier writers have considered calculus of variations problems with such constraints (8, 9).

Formulation of the Problem

We will consider the problem on a flat earth and neglect variations of the gravitational force with altitude, which is an adequate approximation for short-range rockets. (The more complete problem is a direct extension of the present one.) Adopting the coordinates shown in Fig. 1, the conservation of momentum parallel and perpendicular to the flight path requires that

$$m\dot{v} = -c\dot{m} - D(y, v) - mg \sin \gamma \qquad [1]$$

$$v\dot{\gamma} = -g \cos \gamma \qquad [2]$$

where m is the instantaneous mass of the rocket, D the aerodynamic drag and v the velocity. The rocket has been assumed to have high weathercock stability so that its axis is always parallel to the flight path. (The flight path is the path of the center of gravity of the rocket.)

We will also need the kinematic relations

$$\dot{y} = v \sin \gamma \qquad [3]$$

$$\dot{x} = v \cos \gamma \qquad [4]$$

The thrust of the rocket motor has been given as $-c\dot{m}$, where c is the effective exhaust velocity of the gases relative to the rocket and is assumed to be constant. Note that c/g is the specific impulse of the rocket propellant. Referring to the introduction we see that we need to require that

$$m \geq m_f \qquad [5]$$

and, further, we should check our solutions to be sure that $\dot{m} \leq 0$ throughout the flight since fuel cannot be "sucked back in" to the rocket.

Now, let us define an "admissible trajectory" as one that satisfies Equations [1 through 5] and in addition has certain

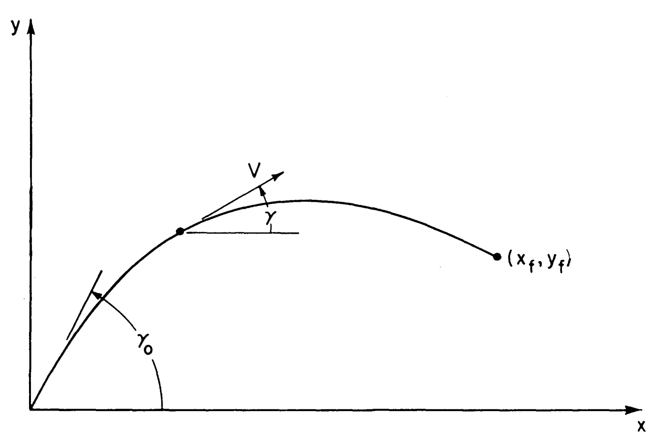

Fig. 1 Trajectory notation

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specified final values of m, x and y, and certain specified initial values of v, x and y.

The problem, simply stated, is to find the admissible trajectory that minimizes the initial mass.

The Equations for a Stationary Trajectory

We first change from time to horizontal distance, x, as independent variable using Equation [4]

$$dt = \frac{dx}{v \cos \gamma}$$

Next we introduce dimensionless variables as

$$\xi = \frac{gx}{c^2};\ \mu = \frac{m}{m_f};\ \nu = \frac{v}{c};\ \eta = \frac{gy}{c^2};\ \tau = \frac{gt}{c};\ \Omega = \frac{De^{\nu}}{m_f g} \ldots [6]$$

Next we eliminate μ as a variable in favor of the quantity $\phi = \mu e^{\nu}$, which does not change during an impulsive burning of fuel (see Equation [1]). Log $\phi = \nu + \log \mu$ has the properties of a "potential velocity" since it is the velocity that could be attained at any point in the flight by burning all the rest of the fuel instantaneously, i.e., decreasing μ to 1. Such an instantaneous "boost," while not possible, is a convenient idealization to a very rapid burning of fuel. Rewriting Equations [1, 2, 3, 5] in the new variables

$$J_1 = \phi' + \frac{\sec \gamma}{\nu} (\phi \sin \gamma + \Omega) = 0 \qquad \text{where } \Omega = \Omega(\nu, \eta) \ldots [7]$$

$$J_2 = \gamma' + \frac{1}{\nu^2} = 0 \ldots [8]$$

$$J_3 = \eta' - \tan \gamma = 0 \ldots [9]$$

$$J_4 = \phi - e^{\nu} - \zeta^2 = 0 \ldots [10]$$

where $()' = d()/d\xi$ and we have introduced a new real variable ζ in Equation [10] to insure that $\mu \geq 1$. These are four equations in five variables, ϕ, γ, η, ν and ζ so one of them is arbitrary; we shall regard η as the arbitrary function. Note that the time can be computed from

$$\tau' = \frac{\sec \gamma}{\nu} \ldots [11]$$

Consider now an admissible trajectory, characterized by the functions ϕ, γ, η, ν and ζ, and another admissible trajectory characterized by $\phi + \delta\phi$, $\gamma + \delta\gamma$, $\eta + \delta\eta$, $\nu + \delta\nu$ and $\zeta + \delta\zeta$ where $\delta()$ signifies a small variation. Since both are admissible trajectories, it follows that

$$\delta J_i = J_i(\phi + \delta\phi, \gamma + \delta\gamma, \eta + \delta\eta, \nu + \delta\nu, \zeta + \delta\zeta) - J_i(\phi, \gamma, \eta, \nu, \zeta) = 0 \ldots [12]$$

Therefore

$$0 = \int_0^{\xi_f} \sum_{i=1}^{4} \lambda_i(\xi) \delta J_i(\xi) d\xi \ldots [13]$$

where $\lambda_i(\xi)$, $i = 1, 2, 3, 4$, are four more dependent variables known as Lagrange multiplier functions, as yet unspecified. Integrating Equation [13] by parts in the usual manner, we find

$$0 = [\lambda_1\delta\phi + \lambda_2\delta\gamma + \lambda_3\delta\eta]_0^{\xi_f} - \int_0^{\xi_f} \left\{ \left[\lambda_1' - \frac{\lambda_1 \tan \gamma}{\nu} - \lambda_4 \right] \delta\phi + \left[\lambda_2' - \frac{\lambda_1 \sec^2 \gamma}{\nu} (\phi + \Omega \sin \gamma) + \lambda_3 \sec^2 \gamma \right] \delta\gamma + \left[\lambda_3' - \lambda_1 \frac{\Omega_\eta \sec \gamma}{\nu} \right] \delta\eta + \left[\lambda_1 \frac{\sec \gamma}{\nu^2} (\phi \sin \gamma + \Omega - \nu\Omega_\nu) + \frac{2\lambda_2}{\nu^3} + \lambda_4 e^{\nu} \right] \delta\nu + 2\lambda_4\zeta\delta\zeta \right\} d\xi \ldots [14]$$

Now, we *choose* λ_1, λ_2, λ_3 and λ_4 so that the coefficients of $\delta\phi$, $\delta\gamma$, $\delta\nu$ and $\delta\zeta$ all vanish, i.e.

$$\lambda_1' - \frac{\lambda_1 \tan \gamma}{\nu} - \lambda_4 = 0 \ldots [15]$$

$$\lambda_2' - \frac{\lambda_1 \sec^2 \gamma}{\nu} (\phi + \Omega \sin \gamma) + \lambda_3 \sec^2 \gamma = 0 \ldots [16]$$

$$\lambda_1(\phi \sin \gamma + \Omega - \nu\Omega_\nu) + \frac{2\lambda_2 \cos \gamma}{\nu} + \nu^2 \cos \gamma \lambda_4 e^{\nu} = 0 \ldots [17]$$

$$\lambda_4 \zeta = 0 \ldots [18]$$

Referring to the terms before the integral in [14], we see that for an admissible trajectory $[\delta\eta]_{\xi=0} = [\delta\eta]_{\xi=\xi_f} = 0$. Now $\gamma(0)$ and $\gamma(\xi_f)$ are unspecified but we may *choose*

$$\lambda_2(0) = \lambda_2(\xi_f) = 0 \ldots [19a]$$

Similarly, since $\nu(\xi_f)$ is unspecified, it follows that $\phi(\xi_f)$ is unspecified, but we may choose

$$\lambda_1(\xi_f) = 0 \ldots [19b]$$

Equations [19a, 19b] are known as "natural boundary conditions" of the problem. Because of our choice of λ's through Equations [15–19], Equation [14] reduces to

$$[\lambda_1\delta\phi]_{\xi=0} = \int_0^{\xi_f} \left(\lambda_3' - \lambda_1 \frac{\Omega_\eta \sec \gamma}{\nu} \right) \delta\eta \ldots [20]$$

From [18] it follows that at any given time during the flight either $\lambda_4 = 0$ or $\zeta = 0$. From [10] it can be seen that $\zeta = 0$ corresponds to $m = m_f$, i.e., constant mass which means zero thrust. Now we have used the variable ϕ for the express purpose of allowing for an instantaneous boost at $\xi = 0$ since $\phi(0+) = \phi(0-)$, i.e., ϕ does not change during such a boost. Since we wish to find the minimum initial mass $\mu(0-)$ for a given initial velocity $\nu(0-)$ and $\mu(0-) = e^{-\nu(0-)}\phi(0)$, it will suffice to minimize $\phi(0)$. Referring to Equation [20] we see that if we put

$$\lambda_1(0) = \text{const} \neq 0 \ldots [21]$$

we can obtain a stationary value of $\phi(0)$ for arbitrary small $\delta\eta$ if

$$\lambda_3' - \lambda_1 \frac{\Omega_\eta \sec \gamma}{\nu} = 0 \ldots [22]$$

Since all of the equations involving the λ's are homogeneous in them, we may set

$$\lambda_1(0) = 1 \ldots [23]$$

An admissible trajectory that gives a stationary value of initial mass will thus be obtained by the simultaneous solution of the seven first-order ordinary differential equations [7, 8, 9, 10, 16, 17, 22] plus the two equations [15, 18] in the nine dependent variables ϕ, ν, η, γ, ζ, λ_1, λ_2, λ_3 and λ_4 with the seven boundary conditions

At $\xi = 0$	At $\xi = \xi_f$
$\lambda_2 = \eta = 0$	$\lambda_1 = \lambda_2 = 0$
$\lambda_1 = 1$	$\mu = 1, \eta = \eta_f$

. . . [24]

A First Integral of the Problem

Equations [15–18] plus [22] are the Euler-Lagrange equations corresponding to the integrand of Equation [13]. Since the integrand of [13] is not an explicit function of ξ, a first integral of the five Euler-Lagrange equations exists, namely

$$\phi' \frac{\partial G}{\partial \phi'} + \nu' \frac{\partial G}{\partial \nu'} + \eta' \frac{\partial G}{\partial \eta'} + \gamma' \frac{\partial G}{\partial \gamma'} + \zeta' \frac{\partial G}{\partial \zeta'} - G = \text{const} \ldots [25]$$

where

$$G = \sum_{i=1}^{4} \lambda_i J_i$$

Substituting into [25] and multiplying through by ctn γ, the first integral becomes

$$\lambda_3 = -k_1 \operatorname{ctn} \gamma + \frac{\lambda_1}{\nu} (\phi + \Omega \csc \gamma) + \frac{\lambda_2 \operatorname{ctn} \gamma}{\nu^2} \quad [26]$$

where k_1 = const. Substituting [26] into [16] gives

$$\lambda_2' = \sec \gamma \csc \gamma \left(k_1 - \frac{\lambda_1 \Omega \cos \gamma}{\nu} - \frac{\lambda_2}{\nu^2} \right) \quad [27]$$

A Second Integral for the Sustain Phase

If we use an exponential law for the variation of drag with altitude, i.e.

$$\Omega(\nu, n) = \bar{\Omega}(\nu) e^{-\beta\eta} \quad [28]$$

then it follows that $\Omega_\eta = -\beta\Omega$. Using this in Equation [16] and multiplying [7] by λ_1 and [22] by ϕ and adding the resulting three equations together, we obtain an exact differential equation if $\lambda_4 = 0$ (as it is during sustain) which upon integration yields

$$\beta\lambda_1\phi - \lambda_3 = k_2 = \text{const} \quad [29]$$

Relations for the Sustain Phase

During the sustain phase $\lambda_4 = 0$, so we have three linear simultaneous algebraic equations for λ_1, λ_2, λ_3, namely [17, 26, 29]. Solving for λ_1, we obtain

$$\lambda_1 = \frac{2\nu(k_1 \cos \gamma - k_2 \sin \gamma)}{(1 - 2\beta\nu)\phi \sin \gamma + \Omega + \nu\Omega_\nu} \quad [30]$$

Substituting this into [15] with $\lambda_4 = 0$, the following first-order ordinary differential equation for the velocity ν is obtained

$$\nu' = \frac{k_1 \sin \gamma \{\Omega + \nu\Omega_\nu + \nu^2[(\beta\nu - 1)\Omega_\nu - \beta\Omega] - (2\beta\nu - 1)\phi \csc \gamma\} + k_2 \cos \gamma \{\Omega + \nu\Omega_\nu - \nu^2 \tan^2 \gamma \, [(\beta\nu - 1)\Omega_\nu - \beta\Omega]\}}{\nu[\nu^2\Omega_{\nu\nu} - (\phi \sin \gamma + \Omega - \nu\Omega_\nu)](k_1 \cos \gamma - k_2 \sin \gamma)} \quad [31]$$

This last equation, solved simultaneously with the equations of motion [7, 8], and the kinematic relation, [9], determines the sustain phase of the optimum trajectory. From [17, 24] the boundary condition $\lambda_2(0) = 0$ yields

$$[\phi \sin \gamma + \Omega - \nu\Omega_\nu]_{\xi=0} = 0 \quad [32]$$

Utilizing [32] and $\lambda_1(0) = 1$ in [30] we find that

$$[(\beta\nu - 1)\Omega_\nu - \beta\Omega + k_1 \cos \gamma - k_2 \sin \gamma]_{\xi=0} = 0 \quad [33]$$

Substituting [32, 33] into [31] we obtain at $\xi = 0$

$$\nu'(0) = \left[\frac{\nu^2 \tan \gamma [(\beta\nu - 1)\Omega_\nu - \beta\Omega] + 2\nu k_1 \csc \gamma - (\Omega + \nu\Omega_\nu) \operatorname{ctn} \gamma}{\nu^3 \Omega_{\nu\nu}} \right]_{\xi=0} \quad [34]$$

Equation [32] can be thought of as determining $\nu(0)$ in terms of initial ϕ, η and γ. Equations [33, 34] can be thought of as determining the constants k_1 and k_2 in terms of initial ϕ, η, γ and ν'.

Boost Velocity

Equation [17] with $\lambda_4 = 0$ provides us with a way to calculate the amount of the initial boost since $\mu^0 = \mu_0 e^{\nu_0}$, if $\nu^0 = 0$, where the superscript refers to conditions before the boost and the subscript to conditions after an instantaneous boost. Substituting this into Equation [17] we have

$$\Omega(0, \nu_0) - \nu_0 \Omega_\nu(0, \nu_0) + \mu^0 \sin \gamma_0 = 0 \quad [35]$$

where $\Omega = \Omega(\eta, \nu)$. Thus if μ^0, γ_0, and the drag function Ω are given, Equation [35] is a transcendental equation for ν_0. Fig. 2 shows a plot of this relation for the particular case of a quadratic drag law

$$D = D_0 \nu^2 e^{-\beta\eta} \quad [36]$$

where D_0 = drag at sea level at speed $v = c$.

Note: with $\gamma = 90°$, Equation [17] is the optimizing condition for the entire sustain phase of a vertical sounding rocket, a simple relation apparently missed by Hamel (2) and Tsien-Evans (3), but noticed by Leitmann (4).

Conditions at the Juncture of the Sustain Phase and the Coasting Phase

By considering the conditions at the juncture of a continuous burning arc and a coasting arc (both extremals of the problem) it is easily determined that all the λ's as well as ϕ, ν, γ, η and ζ are continuous across this juncture (see Bolza (10)). The first derivatives of these quantities may be discontinuous across the juncture.

Relations During the Coasting Phase

During the coasting phase, $\mu = 1$, and $\phi = e^\nu$, so that Equation [7] becomes simply

$$\nu' = \frac{-\sec \gamma}{\nu} (\Omega e^{-\nu} + \sin \gamma) \quad [37]$$

Also, eliminating λ_4 between [15, 22] yields

$$\lambda_1' = \frac{\lambda_1 \tan \gamma}{\nu^2} [\nu - 1 - e^{-\nu} \csc \gamma (\Omega - \nu\Omega_\nu)] - \frac{2\lambda_2 e^{-\nu}}{\nu^3} \quad [38]$$

During the coasting phase then, Equations [37, 38] must be solved simultaneously with [8, 9, 27] with initial values of ν, η, γ, λ_1 and λ_2 that are the final values of the sustain phase.

Method of Solution

It can be shown that the present problem of finding the thrust program that minimizes the initial mass of a ballistic rocket in traveling between two given points is identical to the problem of maximizing the range of a rocket with a given amount of propellant when the initial point and the final altitude are given (or maximizing final altitude with final range given). This latter viewpoint was taken in our sample problem. Thus we start with a given initial mass, take an arbitrary $\gamma(0)$, then use [35] to determine $\nu_0 = \nu(0+)$ and $m_0 = m^0 e^{-\nu_0}$ (the velocity and mass after boost), then proceed with the sustain phase by guessing an initial acceleration $\nu'(0)$. The problem is then integrated numerically by finite differences throughout the sustain phase until the fuel is exhausted; at that point, coasting flight is begun (zero thrust) and it is continued until $\lambda_2 = 0$. If λ_1 is not also zero at that point, the sustain phase is started over again with another value of initial acceleration $\nu'(0)$. It was found that the

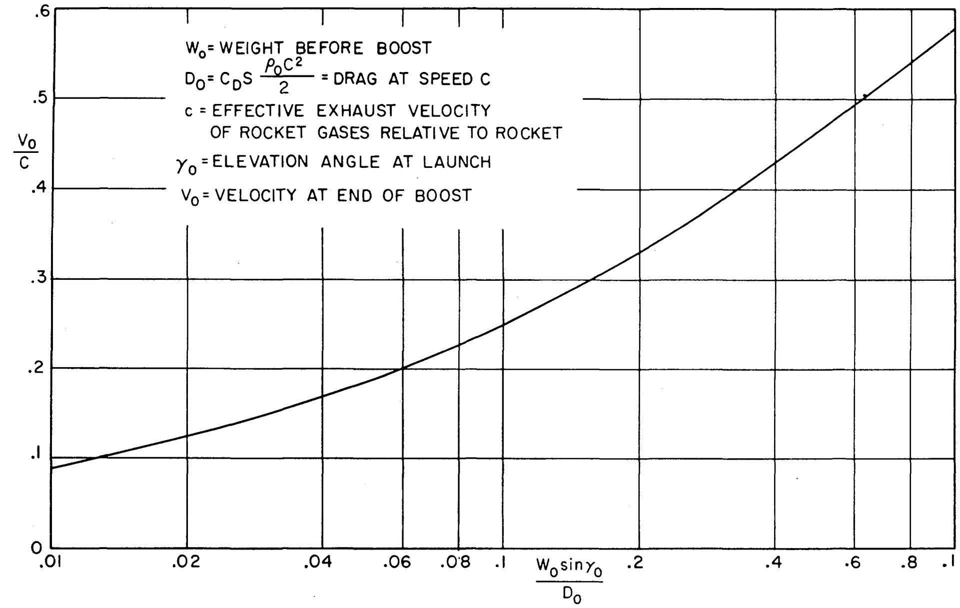

Fig. 2 Boost velocity for maximum range ballistic trajectory-quadratic drag law

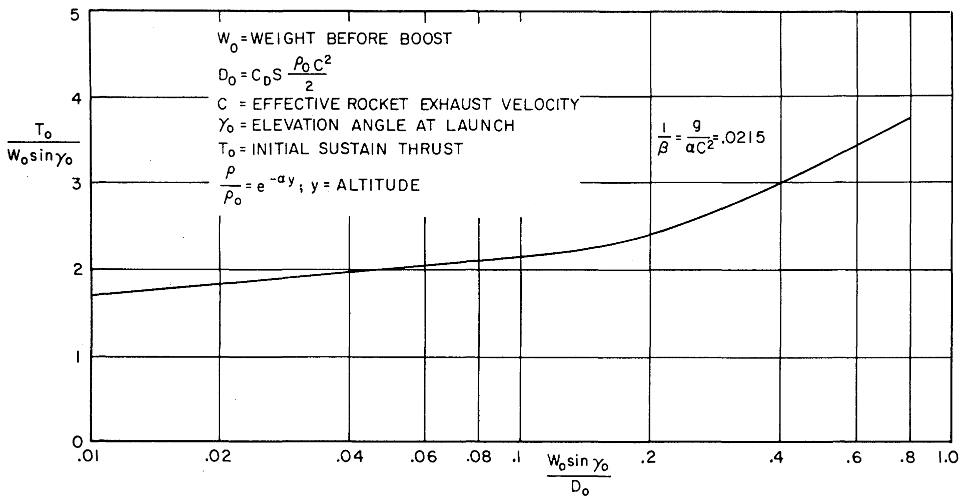

Fig. 3 Approximate initial sustain thrust for maximum range ballistic trajectory (quadratic drag law and exponential atmosphere)

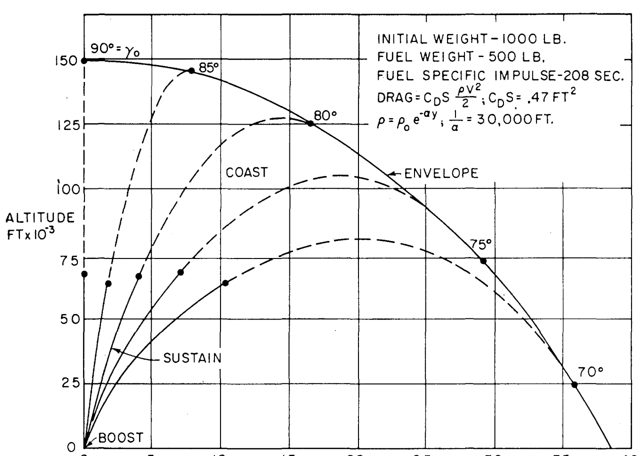

Fig. 4 Altitude-range envelope for optimum thrust programming of ground-to-air rocket for sample problem

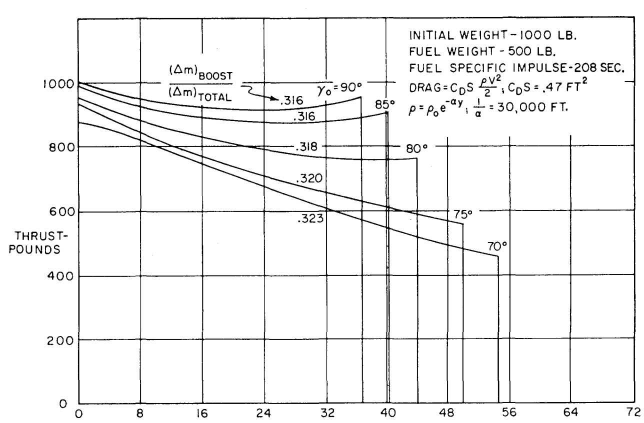

Fig. 5 Thrust vs. time curves for optimum thrust programming of ground-to-air rocket for sample problem

correct value of $\nu'(0)$ could be estimated rather closely by differentiating the coefficient of λ_1 in Equation [17] (see Fig. 3). If the final altitude came out below sea level, a higher value of $\gamma(0)$ was used on the next try. In this way, by taking the final value of η that came out of the problem, we say that *if* we had specified this value as η_f then the thrust program of this problem meets the necessary conditions for a stationary value of final range, ξ_f, for that altitude.

Sample Problem

A sample problem was calculated using the Harvard Univac. The parameters for the problem were

$$m^0g = 1000 \text{ lb}; \quad m_fg = 500 \text{ lb}; \quad \rho = \rho_0 e^{-\alpha y}; \quad C_DS = 0.47 \text{ ft}^2$$

and the drag law was chosen (for simplicity) to be

$$D = C_DS\,\frac{\rho V^2}{2}$$

For altitudes below about 50,000 ft, $1/\alpha = 30{,}000$ ft approximates the density variation with altitude fairly closely.

The problems calculated were "inverse problems," i.e., the maximum range at various altitudes for a rocket with a mass ratio of 2 and certain drag characteristics was calculated.

Fig. 4 shows five optimum trajectories. The envelope of these trajectories represents the maximum distance this rocket can reach with the best possible thrust programming. Note all the trajectories consist of an initial instantaneous boost, a sustain phase, and a coasting phase. The initial flight path angle for maximum range ground-to-ground appears to be about 68 deg, well above the value of 45 deg which it would be in the absence of drag. Fig. 5 shows thrust vs. time during the sustain phase; note about 32 per cent of the rocket fuel is used in the instantaneous initial boost and the remaining 68 per cent of it is used during the sustain phase in a slightly "regressive burning" motor; i.e., the thrust drops off with time. The burning time is longer for the lower altitude firings.

Fig. 6 shows velocity vs. time for each flight. For $\gamma = 90$ deg the analysis of Tsien and Evans[4] was used and it was gratifying to see that their results faired smoothly into the results of the numerical calculations.

Note that for $\gamma_0 = 70$ deg, the velocity began to decrease near the end of the flight whereas it did not for the other cases. This is undoubtedly due to the entry into high density air near sea level which the other trajectories did not experience.

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[4] As previously noted by others, there are some typographical errors in their Equations [31 and 32], p. 101; the equations (in their notation) should read

$$\alpha s = v - v_0 + \frac{\gamma}{2}\ln\frac{2v + (1-\beta) - \gamma}{2v + (1-\beta) + \gamma}\,\frac{2v_0 + (1-\beta) + \gamma}{2v_0 + (1-\beta) + \gamma}$$

$$+ \frac{\beta + 3}{2}\ln\frac{v^2 + (1-\beta)v - 2\beta}{v_0^2 + (1-\beta)v_0 - 2\beta} \quad [31]$$

$$\gamma = \sqrt{(1+\beta)^2 + 4\beta} \quad [32]$$

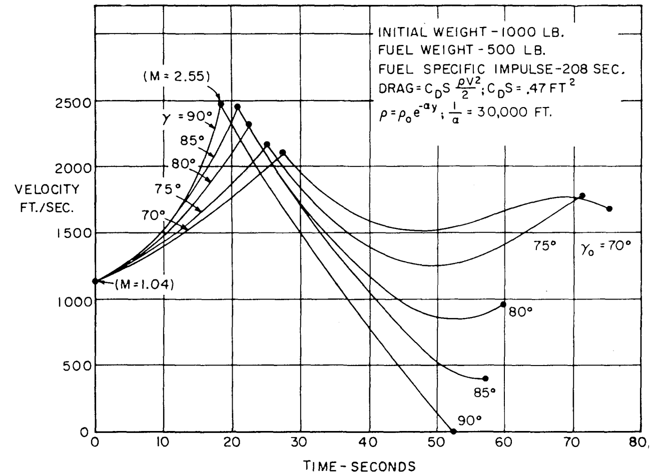

Fig. 6 Velocity vs. time curves for optimum thrust programming of ground-to-air rocket for sample problem

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