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# AA283
# Aircraft and Rocket Propulsion

## Chapter 8 - Multistage Rockets

1) With current technology and fuels and without greatly increasing $I_{sp}$ by airbreathing, a single stage rocket to orbit is still not possible.

2) The final velocity of an $n$ stage rocket is the sum of the velocity gains from each stage.

$$V_n = \Delta v_1 + \Delta v_2 + \Delta v_3 + \dots\dots\dots + \Delta v_n \qquad (8.1)$$

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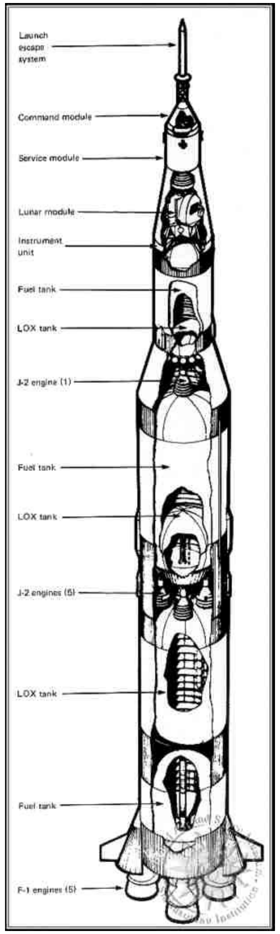
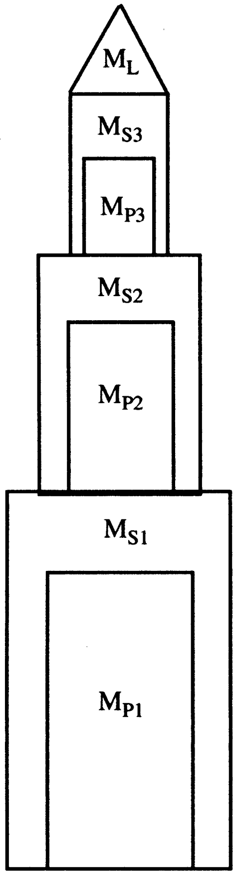
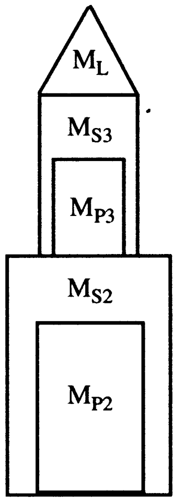
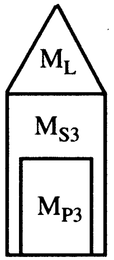
## 7.1 Notation




The index i refers to the ith stage

$M_{0i}$ - The total initial mass of the ith stage prior to firing including payload, ie, the mass of i, i+1, i+2, i+3, ...., n stages.

$M_{pi}$ - The mass of propellant in the ith stage.

$M_{si}$ - Structural mass of the ith stage alone including the mass of its engine, controllers and instrumentation as well as any residual propellant which is not expended by the end of the burn.

$M_L$ - The payload

## 7.2 Analysis

**Payload ratio**

$$\lambda_i = \frac{M_{0(i+1)}}{M_{0i} - M_{0(i+1)}}$$

$$\lambda_n = \frac{M_{0(n+1)}}{M_{0n} - M_{0(n+1)}} = \frac{M_L}{M_{0n} - M_L} \qquad (8.2)$$

**Structural coefficient**

$$\varepsilon_i = \frac{M_{Si}}{M_{0i} - M_{0(i+1)}} = \frac{M_{Si}}{M_{Si} + M_{Pi}} \qquad (8.3)$$

**Mass ratio**

$$R_i = \frac{M_{0i}}{M_{0i} - M_{Pi}} = \frac{1 + \lambda_i}{\varepsilon_i + \lambda_i} \qquad (8.4)$$

**Ideal velocity increment**

$$V_n = \sum_{i=1}^{n} C_i \ln R_i = \sum_{i=1}^{n} C_i \ln\left(\frac{1+\lambda_i}{\varepsilon_i + \lambda_i}\right) \quad (8.5)$$

**Payload fraction**

$$\Gamma = \frac{M_L}{M_{01}} = \left(\frac{M_{02}}{M_{01}}\right)\left(\frac{M_{03}}{M_{02}}\right)\left(\frac{M_{04}}{M_{03}}\right)\cdots\cdots\left(\frac{M_L}{M_{0n}}\right)$$
$$= \left(\frac{\lambda_1}{1+\lambda_1}\right)\left(\frac{\lambda_2}{1+\lambda_2}\right)\left(\frac{\lambda_3}{1+\lambda_3}\right)\cdots\cdots\left(\frac{\lambda_n}{1+\lambda_n}\right) \quad (8.6)$$

or

$$\ln\Gamma = \sum_{i=1}^{n} \ln\left(\frac{\lambda_i}{1+\lambda_i}\right) \quad (8.7)$$

## 8.3 The variational problem

Given $V_n$, maximize the payload fraction, $\Gamma$. It turns out that the opposite statement leads to equivalent results; namely given $\Gamma$ maximize the final velocity, $V_n$. In other words, maximize

$$\ln \Gamma = G(\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n) \tag{8.8}$$

for fixed

$$V_n = F(\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n) \tag{8.9}$$

or, maximize (8.9) for fixed (8.8). The structural coefficients, $\varepsilon_i$ and effective exhaust velocities, $C_i$, are known constants based on some prior choice of propellants and structural design for each stage.

The approach is to vary the payload ratios, $(\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n)$ so as to maximize $\Gamma$. Near a maximum, a small change in the $\lambda_i$ will not change $G$.

$$\delta G = \left(\frac{\partial G}{\partial \lambda_i}\right)\delta\lambda_i = 0 \tag{8.10}$$

The basic idea is shown schematically below.

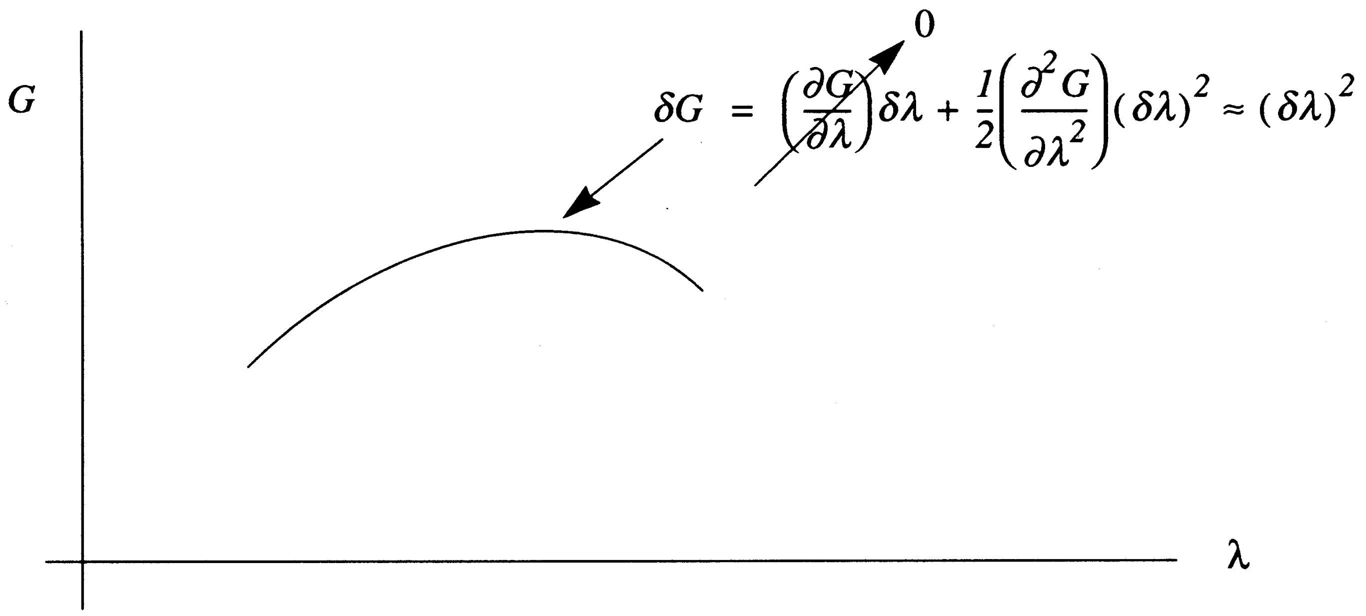



*Figure 8.1 Variation of G near a maximum*

The $\delta\lambda_i$ are not independent, they must be chosen so that $V_n$ is kept constant

$$\delta F = \left(\frac{\partial F}{\partial \lambda_i}\right)\delta\lambda_i = 0 \tag{8.11}$$

Thus only $n-1$ of the $\delta\lambda_i$ can be treated as independent. Without loss of generality let's choose $\lambda_n$ to be determined in terms of the other $\lambda s$. The sums (8.10) and (8.11) are,

$$\left.\begin{aligned} \sum_{i=1}^{n-1}\left(\frac{\partial G}{\partial \lambda_i}\right)\delta\lambda_i + \left(\frac{\partial G}{\partial \lambda_n}\right)\delta\lambda_n &= 0 \\ \sum_{i=1}^{n-1}\left(\frac{\partial F}{\partial \lambda_i}\right)\delta\lambda_i + \left(\frac{\partial F}{\partial \lambda_n}\right)\delta\lambda_n &= 0 \end{aligned}\right\} \tag{8.12}$$

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Use the second sum to replace $\lambda_n$ in the first.

$$\sum_{i=1}^{n-1}\left\{\left(\frac{\partial G}{\partial \lambda_i}\right)+\frac{1}{\alpha}\left(\frac{\partial F}{\partial \lambda_i}\right)\right\}\delta\lambda_i = 0 \tag{8.13}$$

where

$$\alpha = -\left(\frac{\partial F}{\partial \lambda_n}\right)\Big/\left(\frac{\partial G}{\partial \lambda_n}\right) \tag{8.14}$$

plays the role of a Lagrange multiplier. Since the equality (8.13) must hold for arbitrary $\delta\lambda_i$, the coefficients in brackets must be individually zero.

$$\left(\frac{\partial G}{\partial \lambda_i}\right)+\frac{1}{\alpha}\left(\frac{\partial F}{\partial \lambda_i}\right) = 0 \quad ; \qquad i = 1, 2, \ldots, n-1 \tag{8.15}$$

From the definition of $\alpha$

$$\left(\frac{\partial G}{\partial \lambda_n}\right)+\frac{1}{\alpha}\left(\frac{\partial F}{\partial \lambda_n}\right) = 0 \tag{8.16}$$

We now have $n + 1$ equations in the $n + 1$ unknowns $(\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n, \alpha)$.

$$\left(\frac{\partial G}{\partial \lambda_i}\right) + \frac{1}{\alpha}\left(\frac{\partial F}{\partial \lambda_i}\right) = 0 \quad ; \qquad i = 1, 2, \ldots, n$$

$$V_n = \sum_{i=1}^{n} C_i \ln\left(\frac{1 + \lambda_i}{\varepsilon_i + \lambda_i}\right) \tag{8.17}$$

If we supply the expressions for $F$ and $G$ in (8.17), the result for the optimal set of payload ratios is

$$\lambda_i = \frac{\alpha \varepsilon_i}{\{C_i - C_i \varepsilon_i - \alpha\}} \tag{8.18}$$

The Lagrange multiplier is determined from the expression for $V_n$

$$V_n = \sum_{i=1}^{n} C_i \ln\left(\frac{C_i - \alpha}{\varepsilon_i C_i}\right) \tag{8.19}$$

Note that $\alpha$ has units of velocity. Finally, the optimum overall payload fraction is,

$$ln\Gamma = \sum_{i=1}^{n} ln\left(\frac{\alpha\varepsilon_i}{C_i - \varepsilon_i C_i - \alpha + \alpha\varepsilon_i}\right) \qquad (8.20)$$

## 8.4 Example - Exhaust velocity and structural coefficient the same for all stages.

Let $C = C_i$ and $\varepsilon = \varepsilon_i$ be the same for all stages. In this case,

$$\alpha = C\left(1 - \varepsilon e^{\left(\frac{V_n}{nC}\right)}\right), \tag{8.21}$$

$$\lambda = \frac{1 - \varepsilon e^{\left(\frac{V_n}{nC}\right)}}{e^{\left(\frac{V_n}{nC}\right)} - 1} \tag{8.22}$$

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The payload ratio is

$$\Gamma = \left( \frac{1 - \varepsilon e^{\left(\frac{V_n}{nC}\right)}}{(1 - \varepsilon) e^{\left(\frac{V_n}{nC}\right)}} \right)^n \tag{8.23}$$

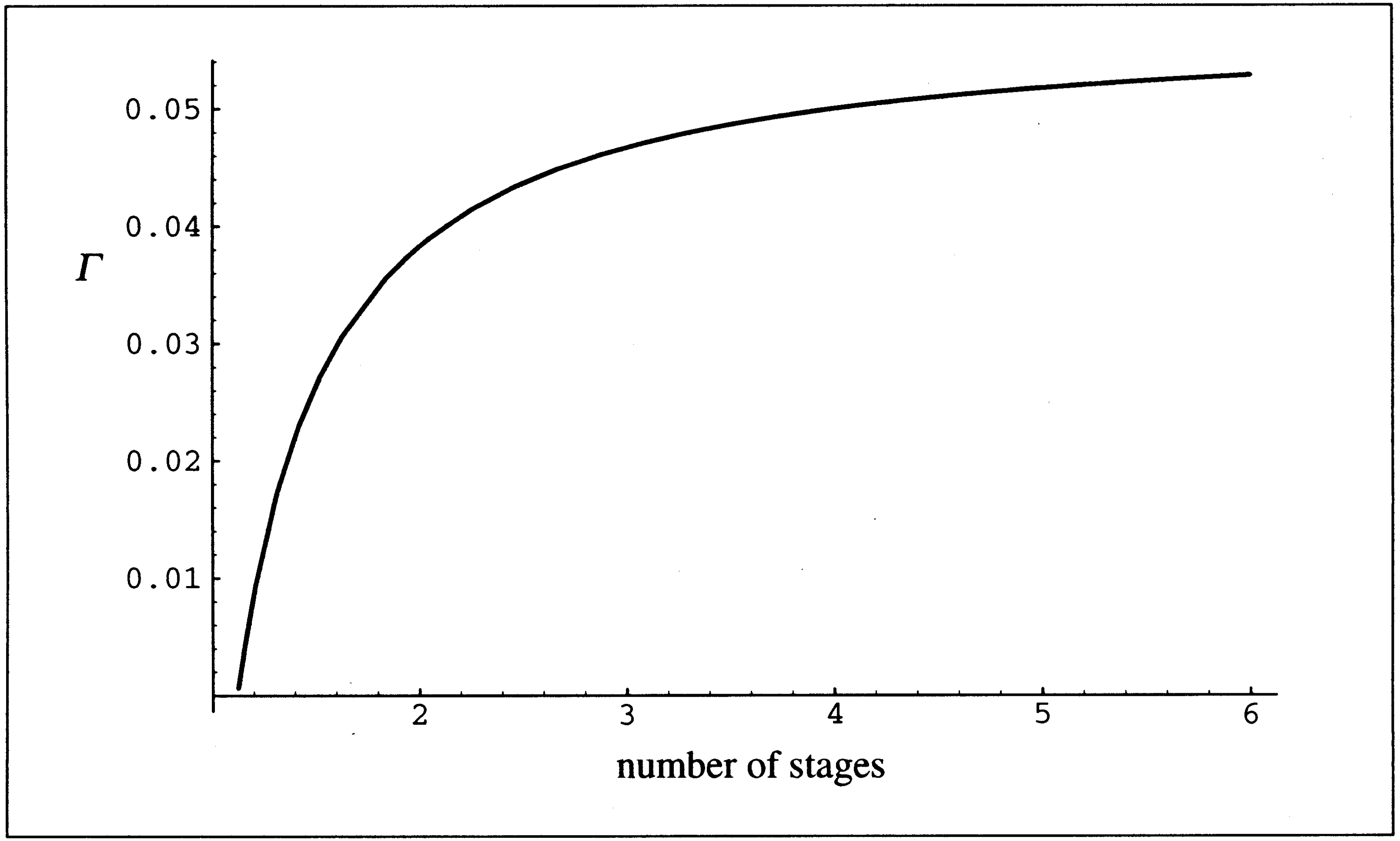
and the mass ratio is

$$R = e^{\left(\frac{V_n}{nC}\right)} \tag{8.24}$$

Consider a liquid oxygen, kerosene system. Take the specific impulse to be *360* sec implying $C = 3528$ M/sec. Let $V_n = 9077$ needed to reach orbital speed. The structural coefficient is $\varepsilon = 0.1$ and let the number of stages be $n = 3$. The stage design results are $\alpha = 2696$ M/sec, $\lambda = 0.563$, $R = 2.3575$ and the payload ratio is

$$\Gamma = 0.047 \tag{8.25}$$

Less than 5% of the overall mass of the vehicle is payload.




There is very little advantage to using more than about three stages.

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**TABLE 10.3** *Saturn V Apollo* flight configuration

| Mass and thrust features | Stage 1 | Stage 2 | Stage 3 |
|---|---|---|---|
| Engine | F-1 | J-2 | J-2 |
| Fuel | RP1 (hydrocarbon) | $LH_2$ | $LH_2$ |
| Oxidant | $LO_2$ | $LO_2$ | $LO_2$ |
| Number of engines | 5 | 5 | 1 |
| Total thrust | | | |
| $lb_f$ | 7,500,000 | 1,000,000 | 200,000 |
| kN | 33,400 | 4,450 | 890 |
| Total initial mass | | | |
| lb | 6,115,000 | 1,488,000 | 473,000 |
| kg | 2,780,000 | 677,000 | 215,000 |
| Mass of propellant | | | |
| lb | 4,393,000 | 943,000 | 239,000 |
| kg | 1,997,000 | 429,000 | 109,000 |
| Mass of structure and engines | | | |
| lb | 234,000 | 71,600 | 56,500 |
| kg | 106,000 | 32,600 | 25,700 |
| $\epsilon_i$ | 0.050 | 0.071 | 0.191 |
| Payload | | | |
| lb | | | 178,000 |
| kg | | | 81,100 |
| $\lambda_i$ | 0.321 | 0.466 | 0.603 |

$$C_1 = 2500 \qquad C_2 = 4250 \quad C_3 = 4250$$

$$V_3 = C_1 Ln\left(\frac{1+\lambda_1}{\varepsilon_1+\lambda_1}\right) + C_2 Ln\left(\frac{1+\lambda_2}{\varepsilon_2+\lambda_2}\right) + C_3 Ln\left(\frac{1+\lambda_3}{\varepsilon_3+\lambda_3}\right)$$

$$V_3 = 2500 Ln\left(\frac{1+0.321}{0.05+0.321}\right) + 4250 Ln\left(\frac{1+0.466}{0.071+0.466}\right) + 4250 Ln\left(\frac{1+0.603}{0.191+0.603}\right) = 10429 M/\sec$$

# Alternative approach.

The final velocity of a multistage system can be expressed as

$$V_n = \sum_{i=1}^{n} C_i \ln\left(\frac{M_{0i}/M_{Pi}}{M_{0i}/M_{Pi}-1}\right)$$

Consider a two stage design

$$V_2 = C_1 Ln\left(\frac{M_{01}/M_{P1}}{M_{01}/M_{P1}-1}\right) + C_2 Ln\left(\frac{M_{02}/M_{P2}}{M_{02}/M_{P2}-1}\right)$$

$$M_{01} = M_{S1} + M_{P1} + M_{S2} + M_{P2} + M_L$$

$$M_{02} = M_{S2} + M_{P2} + M_L$$

$$M_{s1} = \left(\frac{\varepsilon_1}{1-\varepsilon_1}\right) M_{P1} \qquad M_{s2} = \left(\frac{\varepsilon_2}{1-\varepsilon_2}\right) M_{P2} \qquad \Gamma = \frac{M_L}{M_{01}}$$

Express payload mass in terms of propellant masses and payload fraction

$$M_L = \left(\frac{\Gamma}{1-\Gamma}\right)\left(\frac{1}{1-\varepsilon_1}\right) M_{P1} + \left(\frac{\Gamma}{1-\Gamma}\right)\left(\frac{1}{1-\varepsilon_2}\right) M_{P2}$$

Express stage mass ratios in terms of propellant mass ratios

$$\frac{M_{01}}{M_{P1}} = \left(\frac{1}{1-\Gamma}\right)\left(\left(\frac{1}{1-\varepsilon_1}\right) + \left(\frac{1}{1-\varepsilon_2}\right)\frac{M_{P2}}{M_{P1}}\right)$$

$$\frac{M_{02}}{M_{P2}} = \left(\frac{1}{1-\Gamma}\right)\left(\left(\frac{\Gamma}{1-\varepsilon_1}\right)\left(\frac{1}{M_{P2}/M_{P1}}\right) + \left(\frac{1}{1-\varepsilon_2}\right)\right)$$

$$V_2 = C_1 Ln\left(\frac{M_{01}/M_{P1}}{M_{01}/M_{P1} - 1}\right) + C_2 Ln\left(\frac{M_{02}/M_{P2}}{M_{02}/M_{P2} - 1}\right)$$

$$V_2 = C_1 Ln\left( \frac{\left(\frac{1}{1-\Gamma}\right)\left(\left(\frac{1}{1-\varepsilon_1}\right)+\left(\frac{1}{1-\varepsilon_2}\right)\frac{M_{P2}}{M_{P1}}\right)}{\left(\frac{1}{1-\Gamma}\right)\left(\left(\frac{1}{1-\varepsilon_1}\right)+\left(\frac{1}{1-\varepsilon_2}\right)\frac{M_{P2}}{M_{P1}}\right)-1} \right) + C_2 Ln\left( \frac{\left(\frac{1}{1-\Gamma}\right)\left(\left(\frac{\Gamma}{1-\varepsilon_1}\right)\left(\frac{1}{M_{P2}/M_{P1}}\right)+\left(\frac{1}{1-\varepsilon_2}\right)\right)}{\left(\frac{1}{1-\Gamma}\right)\left(\left(\frac{\Gamma}{1-\varepsilon_1}\right)\left(\frac{1}{M_{P2}/M_{P1}}\right)+\left(\frac{1}{1-\varepsilon_2}\right)\right)-1} \right)$$

For given values of

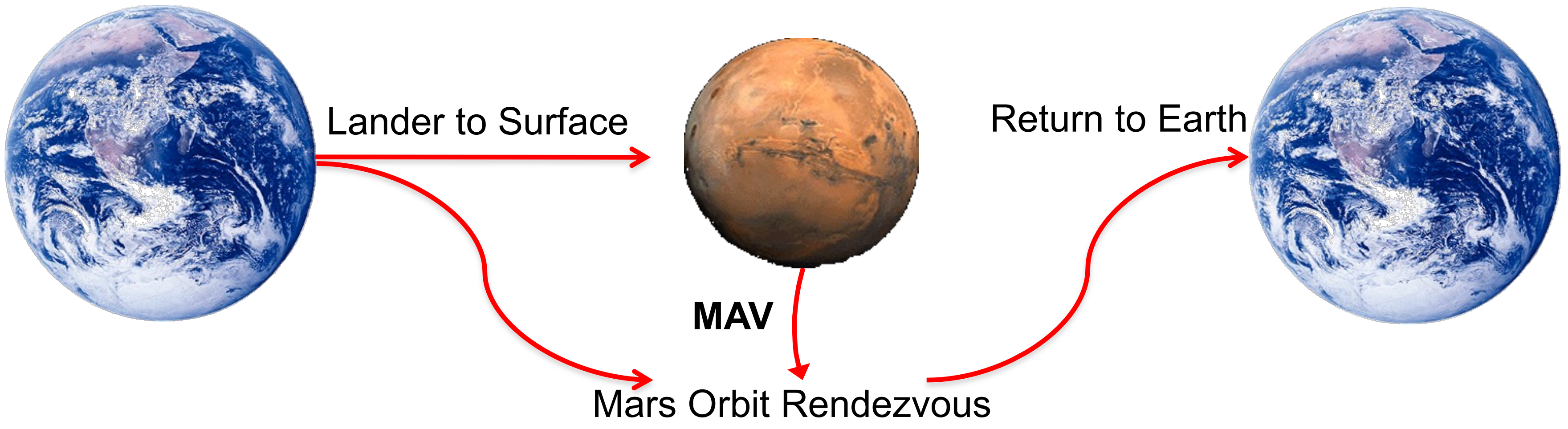
$$\Gamma, C_1, C_2, \varepsilon_1, \varepsilon_2$$

The final velocity is a function of the propellant ratio.

$$V_2 = F\left(\frac{M_{P2}}{M_{P1}}\right)$$

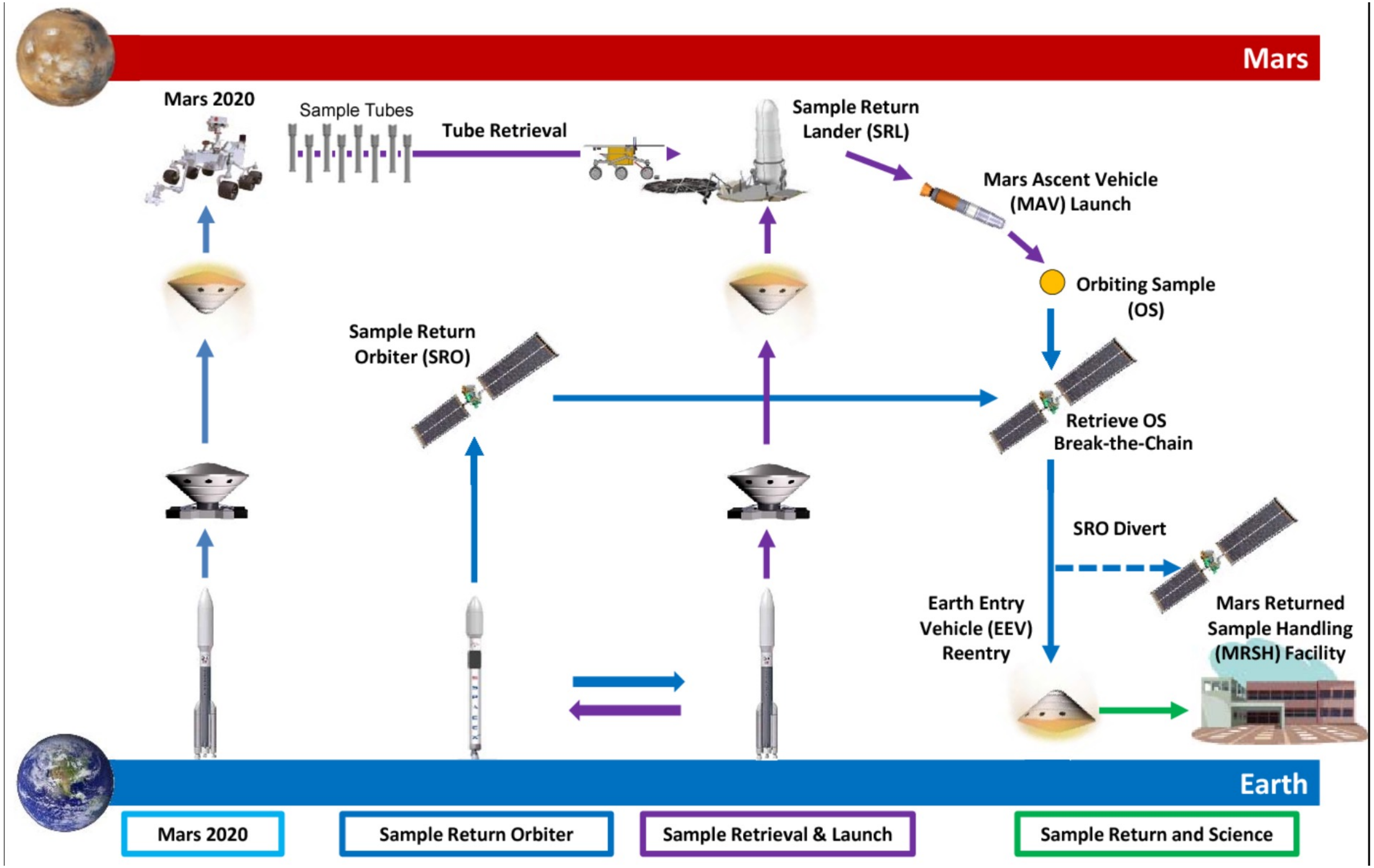
It is now just a matter of differentiating with respect to the propellant ratio to identify a maximum.

# Application - Mars Sample Return Campaign 2020-2030




- Next major step in Mars Science
- Requires international collaboration
- Multiple new developments
  - **Mars Ascent Vehicle (MAV)**
  - Sample acquisition and handling
  - Precision entry descent and landing

# Mars Sample Return Mission Architecture



# Mars 2020 landed in Jezero Crater Feb 18, 2021

The rover Perseverance has drilled rock samples and placed them in individual containers.

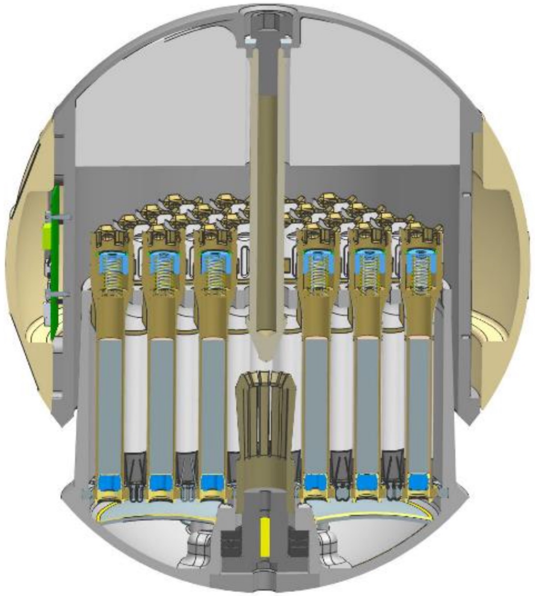
The samples have been left on the surface of Mars for later pick up by a second rover.

The second rover will place the samples in the payload bay of the MAV which will then launch to Mars orbit, rendevous with an ESA orbiter and pass the OS on to the orbiter in a pitch and catch maneuver between the spacecraft.

Current OS
Reference Design

Mars 2020 Sample Tube
Assembly

Engineering OS ready for
impact testing

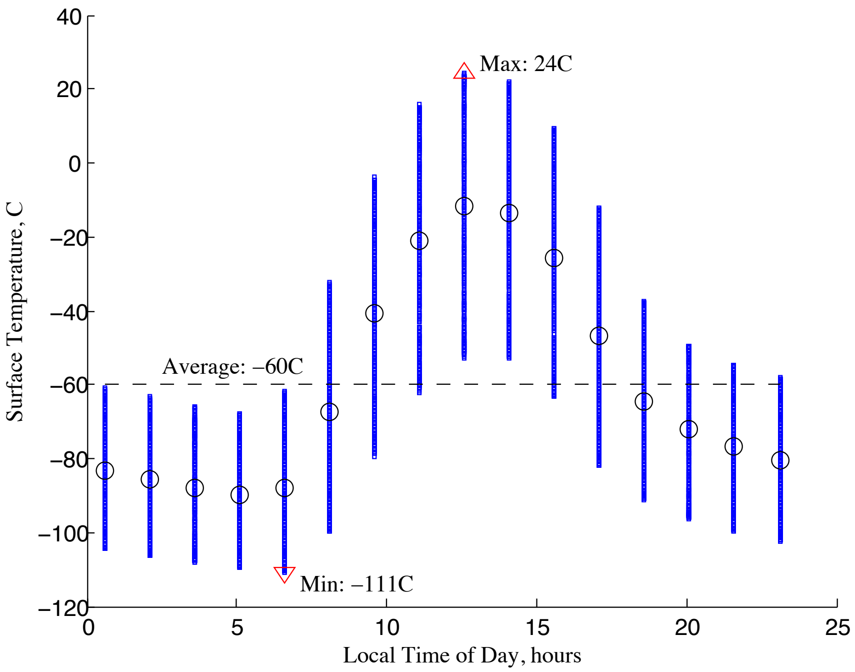


# The Mars Ascent Vehicle

The MAV takes a container with Mars rock samples into orbit around Mars. There the container is transferred to another spacecraft for the return journey to Earth.

# Critical Challenge: Mars Environmental Conditions

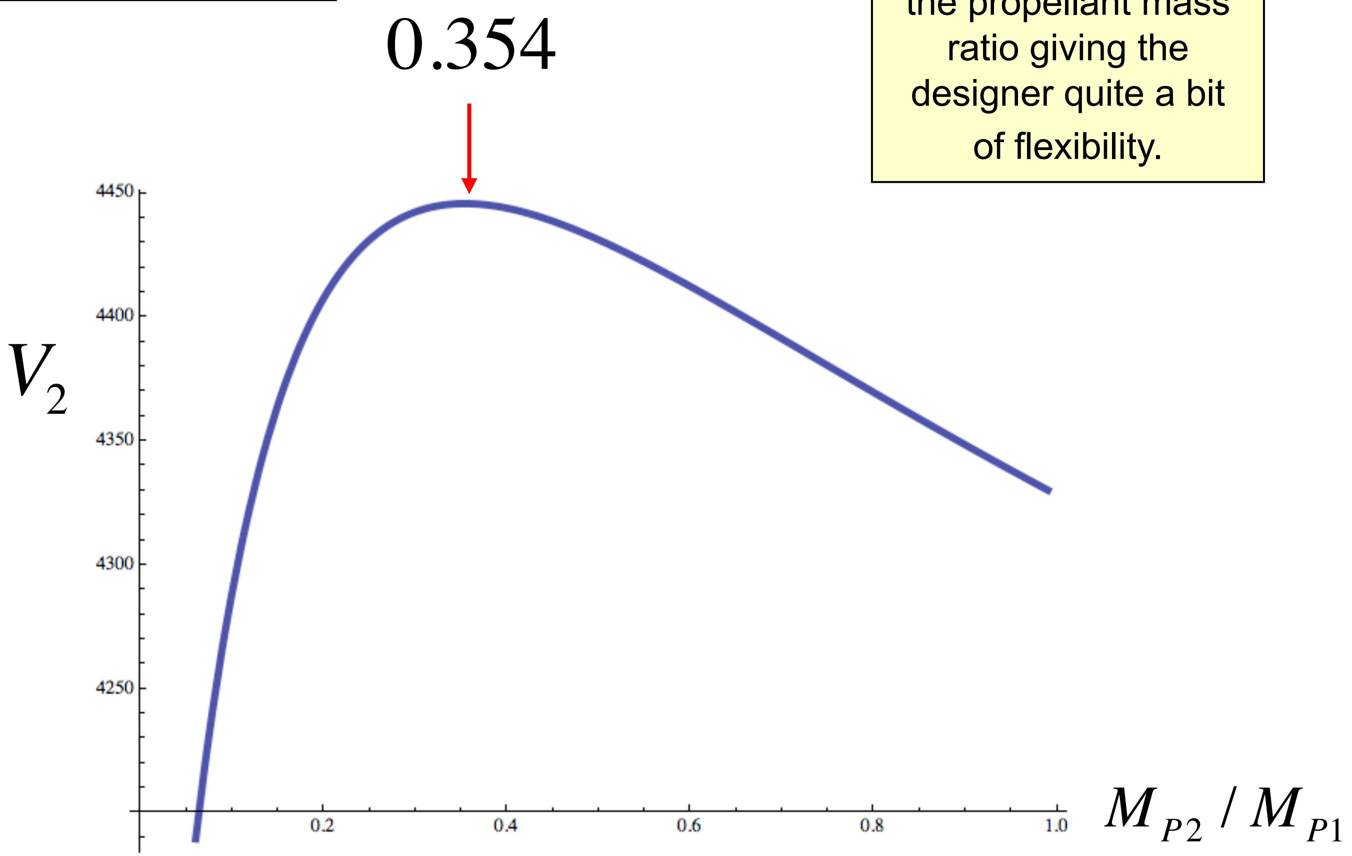
- Diurnal/seasonal minima and maxima (-111C to 24C)




Data from the NASA Ames Research Center Mars Global Climate Model for Holden Crater.

# Use a two stage design for the Mars ascent vehicle

One part of Ashley Karp's PhD project




Note that the final velocity is actually quite insensitive to the propellant mass ratio giving the designer quite a bit of flexibility.

Notional values.

$$\varepsilon_1 = 0.13$$

$$\varepsilon_2 = 0.155$$

$$C_1 = 2883$$

$$C_2 = 3026$$

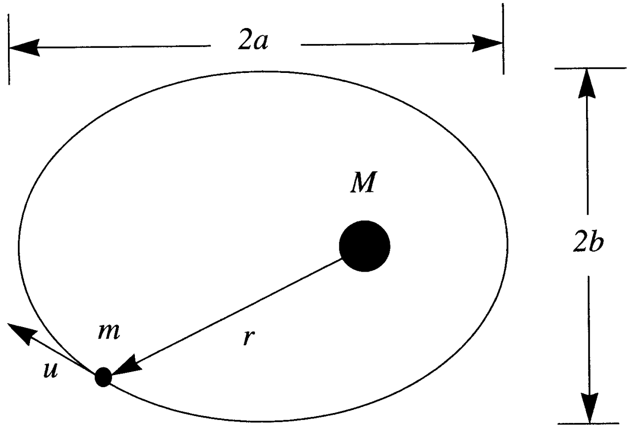
$$\Gamma = 0.147$$

In order to confirm the design it is necessary to fly it to orbit.

# Kepler's Equations

Kepler's equations govern the motion of objects near gravitating bodies. This is called the two body problem.

$$\bar{F} = -G\frac{Mm}{r^2}\left(\frac{\bar{r}}{r}\right)$$




Universal gravitational constant

$$G = 6.67 \times 10^{-11} \text{ m}^3/\text{kg-sec}^2$$

$$\ddot{x}(t) + M_{Planet} G \frac{x(t)}{r(t)^3} = 0 \qquad \ddot{y}(t) + M_{Planet} G \frac{y(t)}{r(t)^3} = 0 \qquad \ddot{z}(t) + M_{Planet} G \frac{z(t)}{r(t)^3} = 0$$

$$r(t) = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$$

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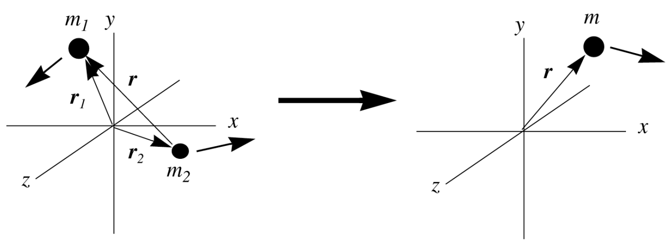
# Constants of the motion – two body problem




Reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

Energy

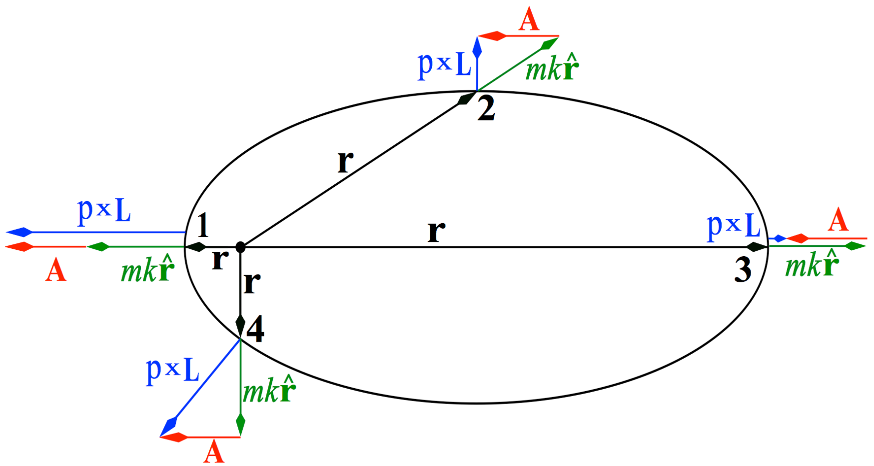
$$E = \frac{1}{2}mv^2 - \frac{k}{r}$$

Angular momentum

$$\bar{L} = \bar{r} \times \bar{p}$$

$$k = Gm_1 m_2$$

The Laplace vector




5 constants
6 degrees of freedom

$$\bar{A} = \bar{p} \times \bar{L} - mk\left(\frac{\bar{r}}{r}\right)$$

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# Orbital Period

$$\frac{GMT^2}{(r_{\text{mean}})^3} = F\left(\frac{m}{M}, e\right)$$

$$r_{\text{mean}} = \sqrt{ab}$$

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

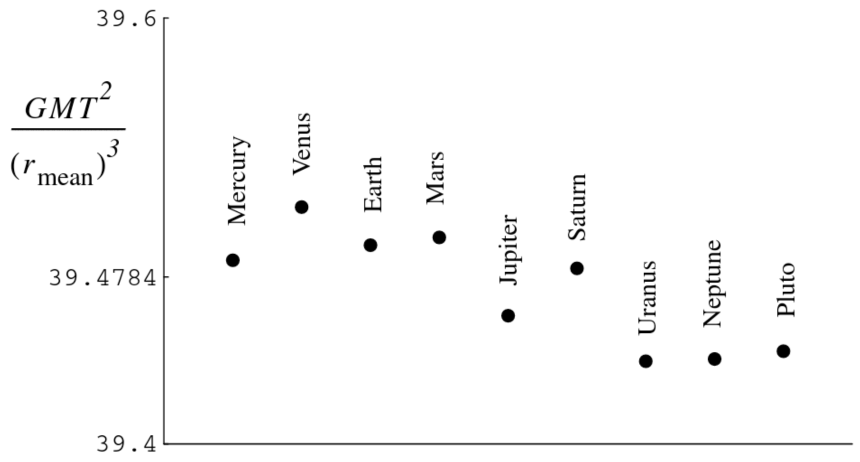
Kepler's theory gives

$$F\left(\frac{m}{M}, e\right) = 4\pi^2\left(\frac{1}{(1 + m/M)(1 - e^2)^{3/4}}\right)$$

# Orbital Periods of the Planets about the Sun

Table 2.1. *The planets and their orbits.*

| Heavenly body | Mass (Earth masses) | Diameter (Earth diameters) | Mean orbit Radius ($10^6$ km) | Eccentricity | Orbital period (years) |
|---|---|---|---|---|---|
| Sun | 332,488.0 | 109.15 | — | — | — |
| Mercury | 0.0543 | 0.38 | 57.9 | 0.2056 | 0.241 |
| Venus | 0.8136 | 0.967 | 108.1 | 0.0068 | 0.615 |
| Earth | 1.0000 | 1.000 | 149.5 | 0.0167 | 1.000 |
| Mars | 0.1069 | 0.523 | 227.8 | 0.0934 | 1.881 |
| Jupiter | 318.35 | 10.97 | 777.8 | 0.0484 | 11.862 |
| Saturn | 95.3 | 9.03 | 1426.1 | 0.0557 | 29.458 |
| Uranus | 14.58 | 3.72 | 2869.1 | 0.0472 | 84.015 |
| Neptune | 17.26 | 3.38 | 4495.6 | 0.0086 | 164.788 |
| Pluto | <0.1 | 0.45 | 5898.9 | 0.2485 | 247.697 |



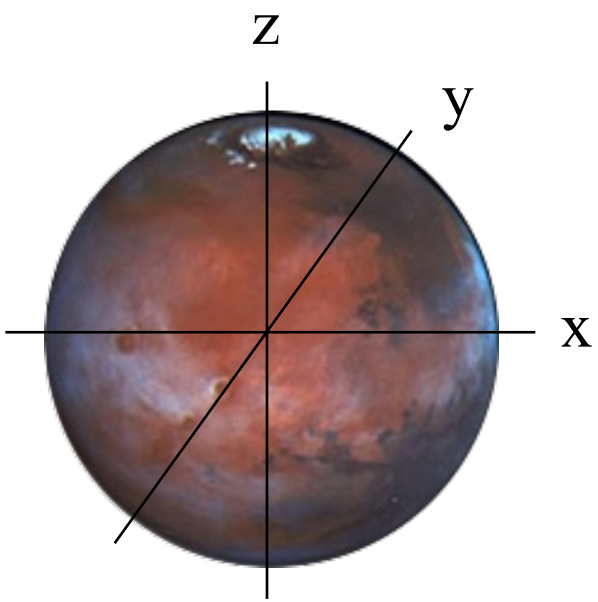
# Mars Ascent Vehicle - launch to orbit

Equations of motion

$$\ddot{x}(t) + m_{MARS}G\frac{x(t)}{r(t)^3} + \frac{F_{x_{DRAG}}(t)}{m(t)} - \frac{F_{x_{THRUST}}(t)}{m(t)} = 0$$

$$\ddot{y}(t) + m_{MARS}G\frac{y(t)}{r(t)^3} + \frac{F_{y_{DRAG}}(t)}{m(t)} - \frac{F_{y_{THRUST}}(t)}{m(t)} = 0$$

$$\ddot{z}(t) + m_{MARS}G\frac{z(t)}{r(t)^3} + \frac{F_{z_{DRAG}}(t)}{m(t)} - \frac{F_{z_{THRUST}}(t)}{m(t)} = 0$$




Mars radius=3.376 x $10^6$ m

Mars mass=6.418 x $10^{23}$ kg

$$g_{MARS} = m_{MARS}G / r^2 = 3.756 \ \text{m/sec}^2$$

Mars time scale: $\tau_{MARS} = \sqrt{\frac{r^3}{mG}} = 948.03$ sec

Mars velocity scale: $U_{MARS} = \sqrt{\frac{mG}{r}} = 3561$ m/sec

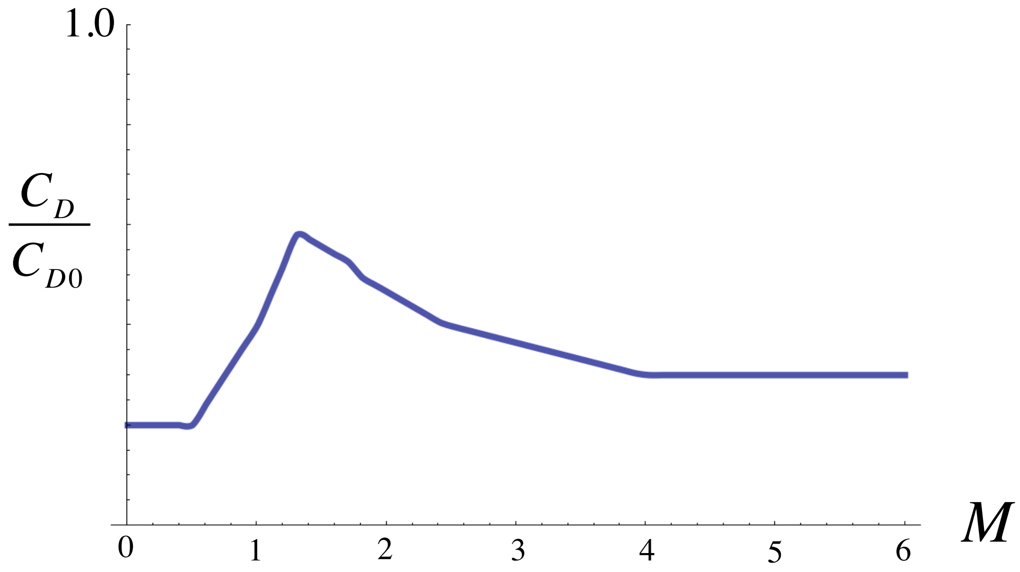
Universal gravitational constant

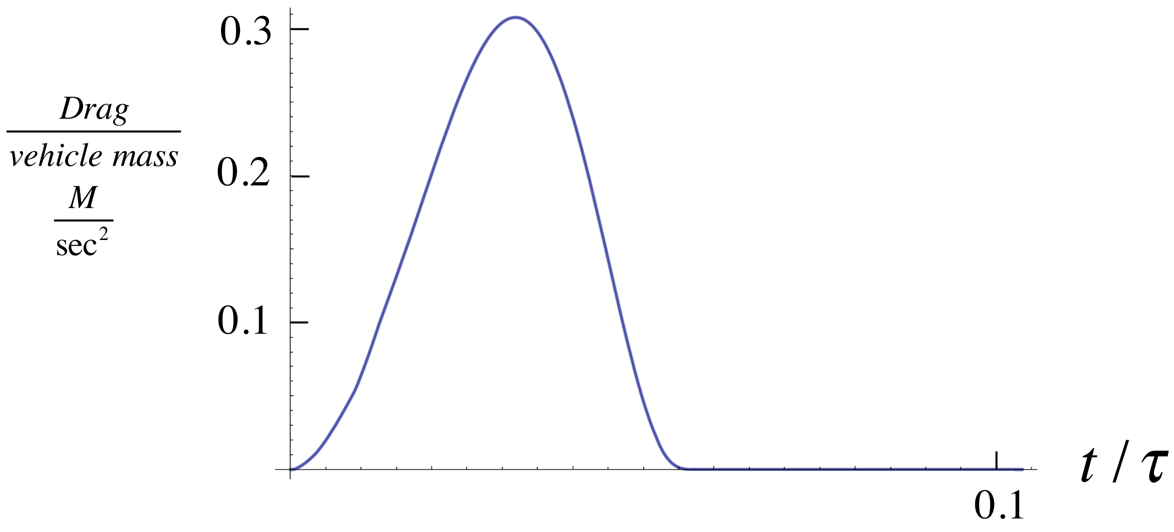
$$G = 6.67 \times 10^{-11} \ \text{m}^3\text{/kg-sec}^2$$

Vehicle mass

$$m(t)$$

# Aerodynamic drag



# Gravity turn equations

$$\ddot{x}(t)+m_{MARS}G\frac{x(t)}{r(t)^3}+\frac{F_{x_{DRAG}}(t)}{m(t)}-\frac{F_{x_{THRUST}}(t)}{m(t)}=0$$

$$\ddot{y}(t)+m_{MARS}G\frac{y(t)}{r(t)^3}+\frac{F_{y_{DRAG}}(t)}{m(t)}-\frac{F_{y_{THRUST}}(t)}{m(t)}=0$$

$$\ddot{z}(t)+m_{MARS}G\frac{z(t)}{r(t)^3}+\frac{F_{z_{DRAG}}(t)}{m(t)}-\frac{F_{z_{THRUST}}(t)}{m(t)}=0$$

Gravity turn - no drag

$$\bar{F}_{THRUST}\times\bar{V}=0$$

$$\bar{V}=(\dot{x},\dot{y},\dot{z})$$

See the paper *Universal Gravity Turn Trajectories* on my website.

Assume there is no lift on the rocket. Combine thrust and drag.

$$\left(\bar{F}_{THRUST}-\bar{F}_{DRAG}\right)\times\bar{V}=0$$

$$\bar{V}=(\dot{x},\dot{y},\dot{z})$$

$\longrightarrow$

$$\dot{y}(t)F_z(t)-\dot{z}(t)F_y(t)=0$$

$$\dot{z}(t)F_x(t)-\dot{x}(t)F_z(t)=0$$

$$\dot{x}(t)F_y(t)-\dot{y}(t)F_x(t)=0$$

$\longrightarrow$

$$F_y(t)=\frac{\dot{y}(t)}{\dot{x}(t)}F_x(t)$$

$$F_z(t)=\frac{\dot{z}(t)}{\dot{x}(t)}F_x(t)$$

Vehicle acceleration

$$a(t)=\frac{\left(F_x(t)^2+F_y(t)^2+F_z(t)^2\right)^{1/2}}{m(t)}=\frac{F_x(t)}{\dot{x}(t)m(t)}\left(\dot{x}(t)^2+\dot{y}(t)^2+\dot{z}(t)\right)^{1/2}$$

$$\frac{F_x(t)}{m(t)}=\frac{\dot{x}(t)}{\dot{r}(t)}a(t)$$

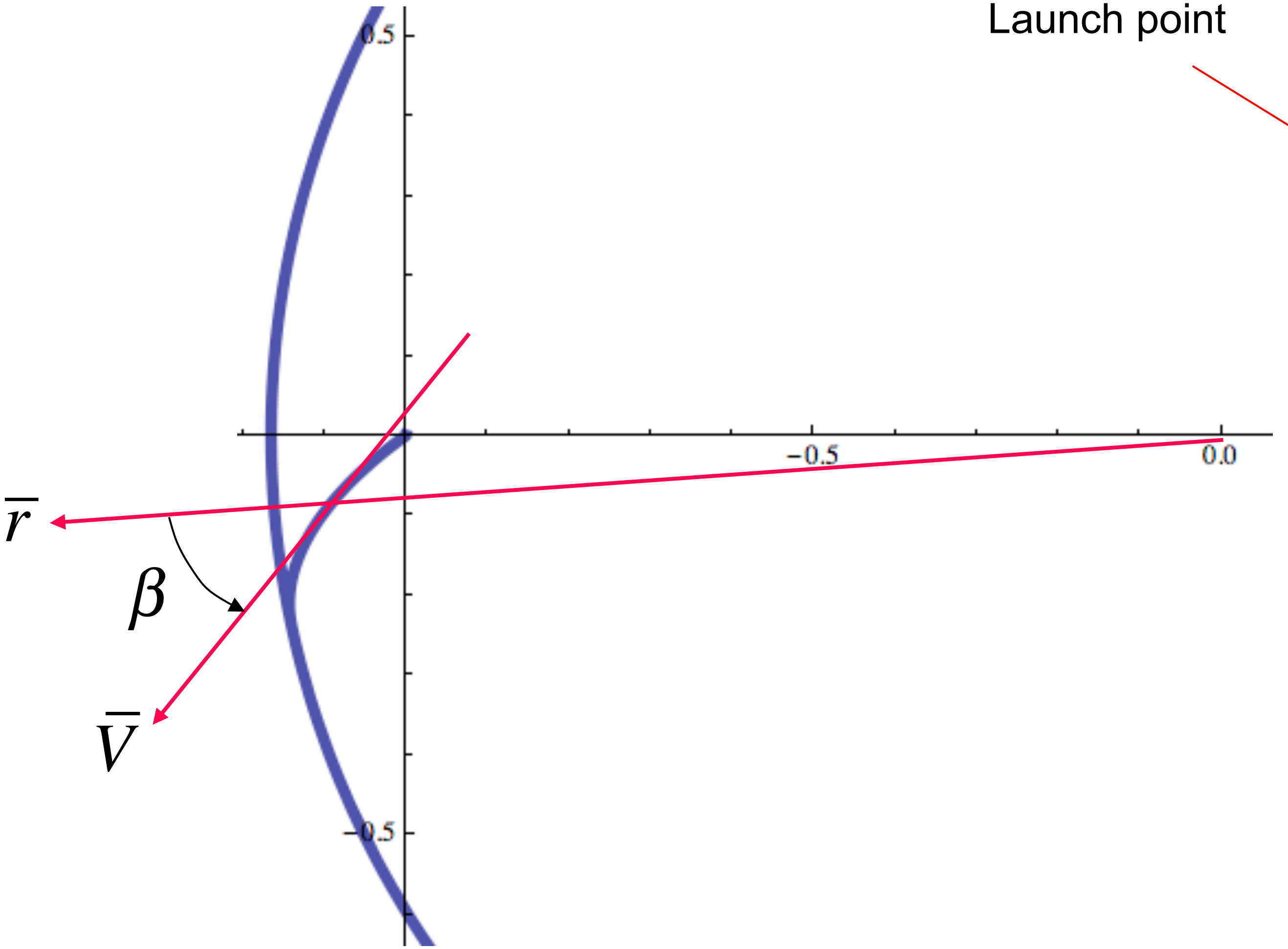
$$\ddot{x}(t)+m_{MARS}G\frac{x(t)}{r(t)^3}-a(t)\frac{\dot{x}(t)}{\dot{r}(t)}=0$$

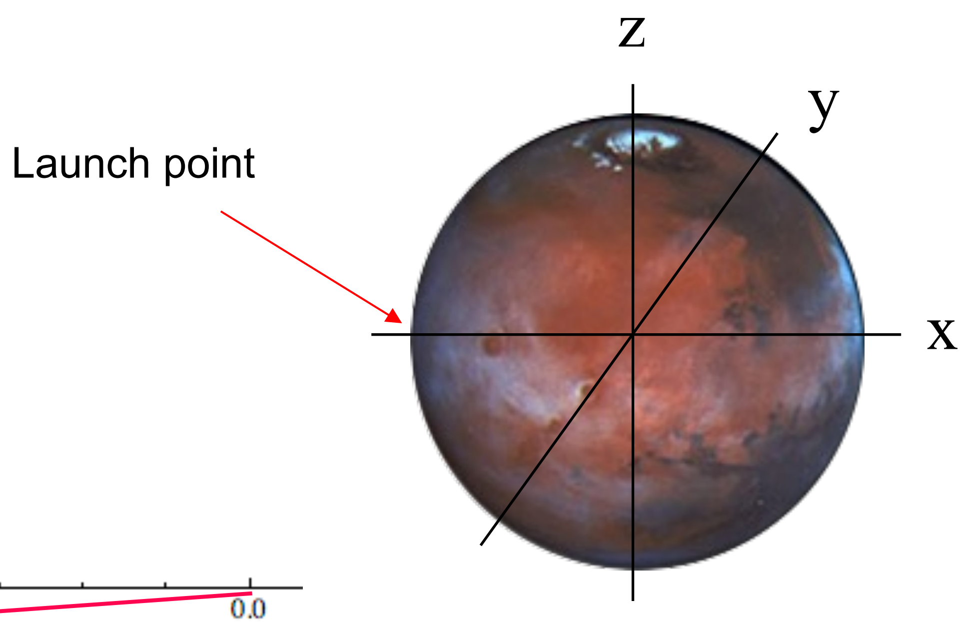
$$\ddot{y}(t)+m_{MARS}G\frac{y(t)}{r(t)^3}-a(t)\frac{\dot{y}(t)}{\dot{r}(t)}=0$$

$$\ddot{z}(t)+m_{MARS}G\frac{z(t)}{r(t)^3}-a(t)\frac{\dot{z}(t)}{\dot{r}(t)}=0$$

$$g_{MARS}=m_{MARS}G/r^2=3.756\quad \text{m/sec}^2$$

# Angle between velocity vector and planet radius



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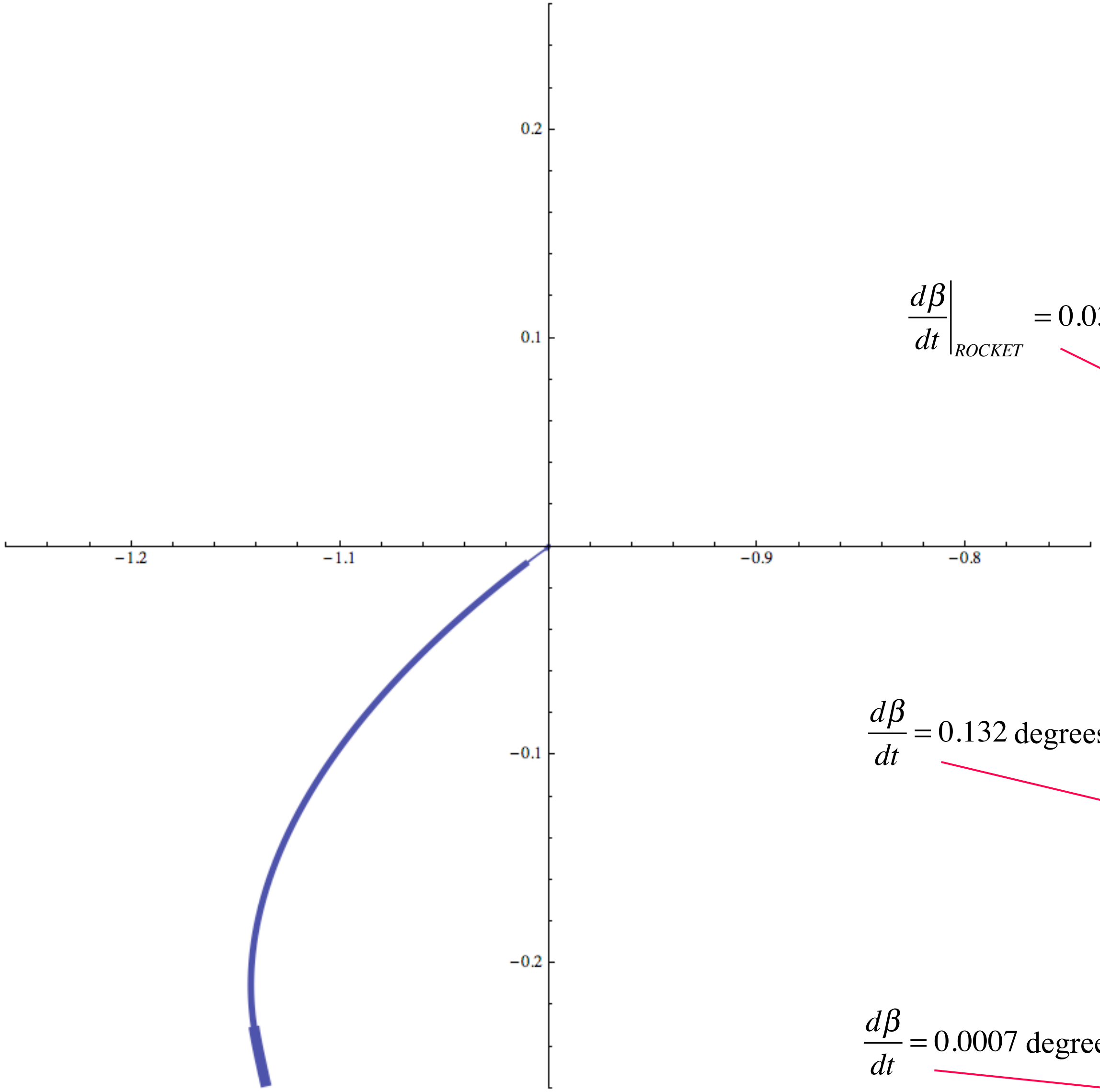
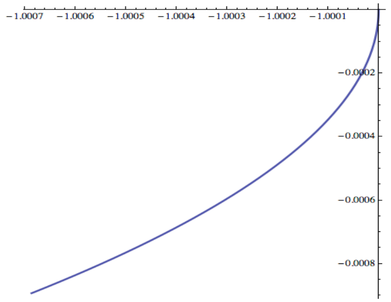
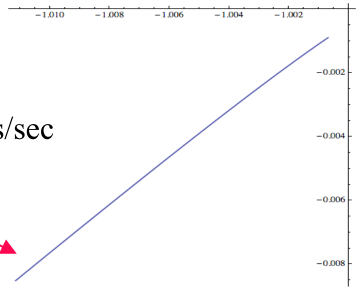
# Launch trajectory



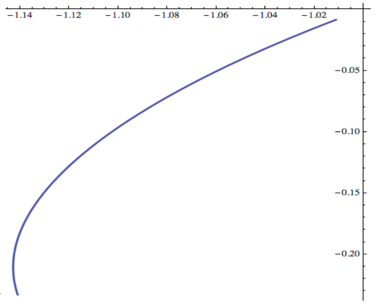
Surface speed at the equator=241.17 m/sec

Vertical launch
12.5 sec



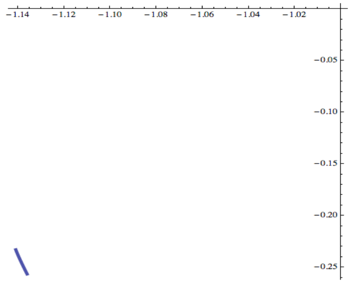
$$\left.\frac{d\beta}{dt}\right|_{ROCKET} = 0.038 \text{ degrees/sec}$$

First stage
gravity turn
36.6 sec



$$\frac{d\beta}{dt} = 0.132 \text{ degrees/sec}$$

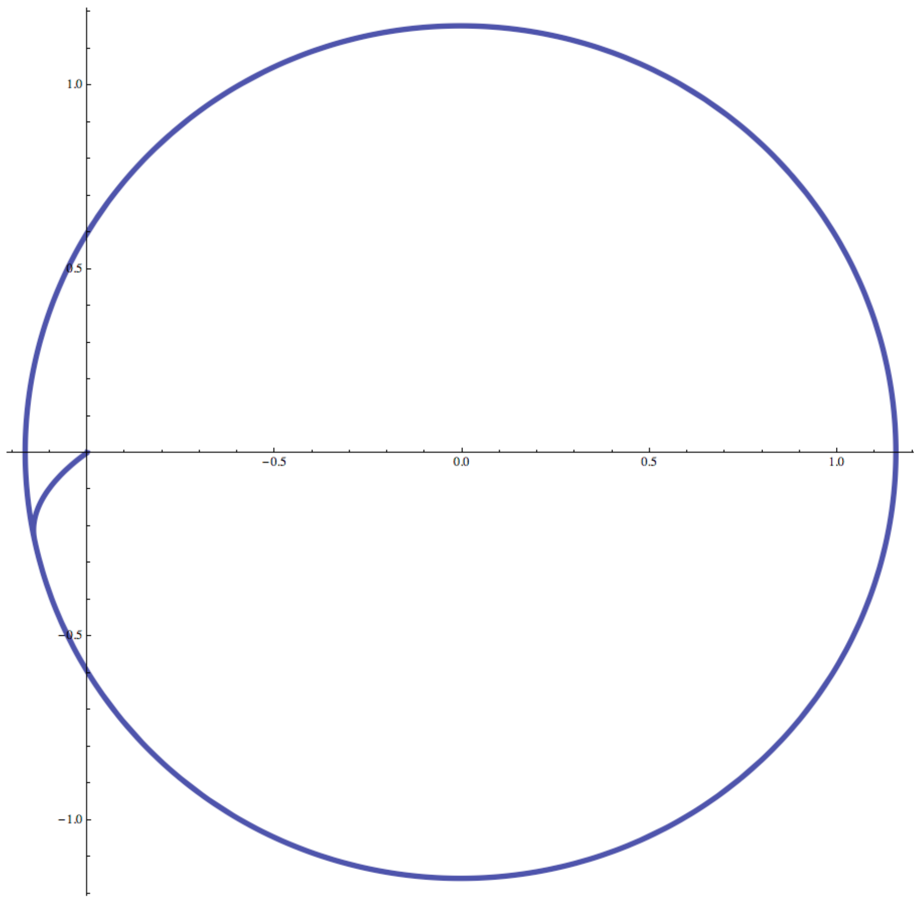
Coast
620 sec



$$\frac{d\beta}{dt} = 0.0007 \text{ degrees/sec}$$

Second stage
gravity turn to orbit
39.4 sec

# Launch and orbit trajectory




Maximum altitude

557.9 km

Minimum altitude

527.5 km

# Three stage design

$$V_3 = C_1 \ln\left(\frac{M_{01}/M_{P1}}{M_{01}/M_{P1}-1}\right) + C_2 \ln\left(\frac{M_{02}/M_{P2}}{M_{02}/M_{P2}-1}\right) + C_3 \ln\left(\frac{M_{03}/M_{P3}}{M_{03}/M_{P3}-1}\right)$$

The mass ratios can be written in terms of the payload fraction as follows.

$$M_{01}/M_{P1} = \left(\frac{1}{1-\Gamma}\right)\left(\frac{1}{1-\varepsilon_1} + \frac{1}{1-\varepsilon_2}\left(M_{P2}/M_{P1}\right) + \frac{1}{1-\varepsilon_3}\left(M_{P3}/M_{P1}\right)\right)$$

$$M_{02}/M_{P2} = \left(\frac{1}{1-\Gamma}\right)\left(\frac{1}{1-\varepsilon_2} + \frac{\Gamma}{1-\varepsilon_1}\left(\frac{1}{M_{P2}/M_{P1}}\right) + \frac{1}{1-\varepsilon_3}\left(\frac{M_{P3}/M_{P1}}{M_{P2}/M_{P1}}\right)\right)$$

$$M_{03}/M_{P3} = \left(\frac{1}{1-\Gamma}\right)\left(\frac{1}{1-\varepsilon_3} + \frac{\Gamma}{1-\varepsilon_1}\left(\frac{1}{M_{P3}/M_{P1}}\right) + \frac{\Gamma}{1-\varepsilon_2}\left(\frac{M_{P2}/M_{P1}}{M_{P3}/M_{P1}}\right)\right)$$

For given values of

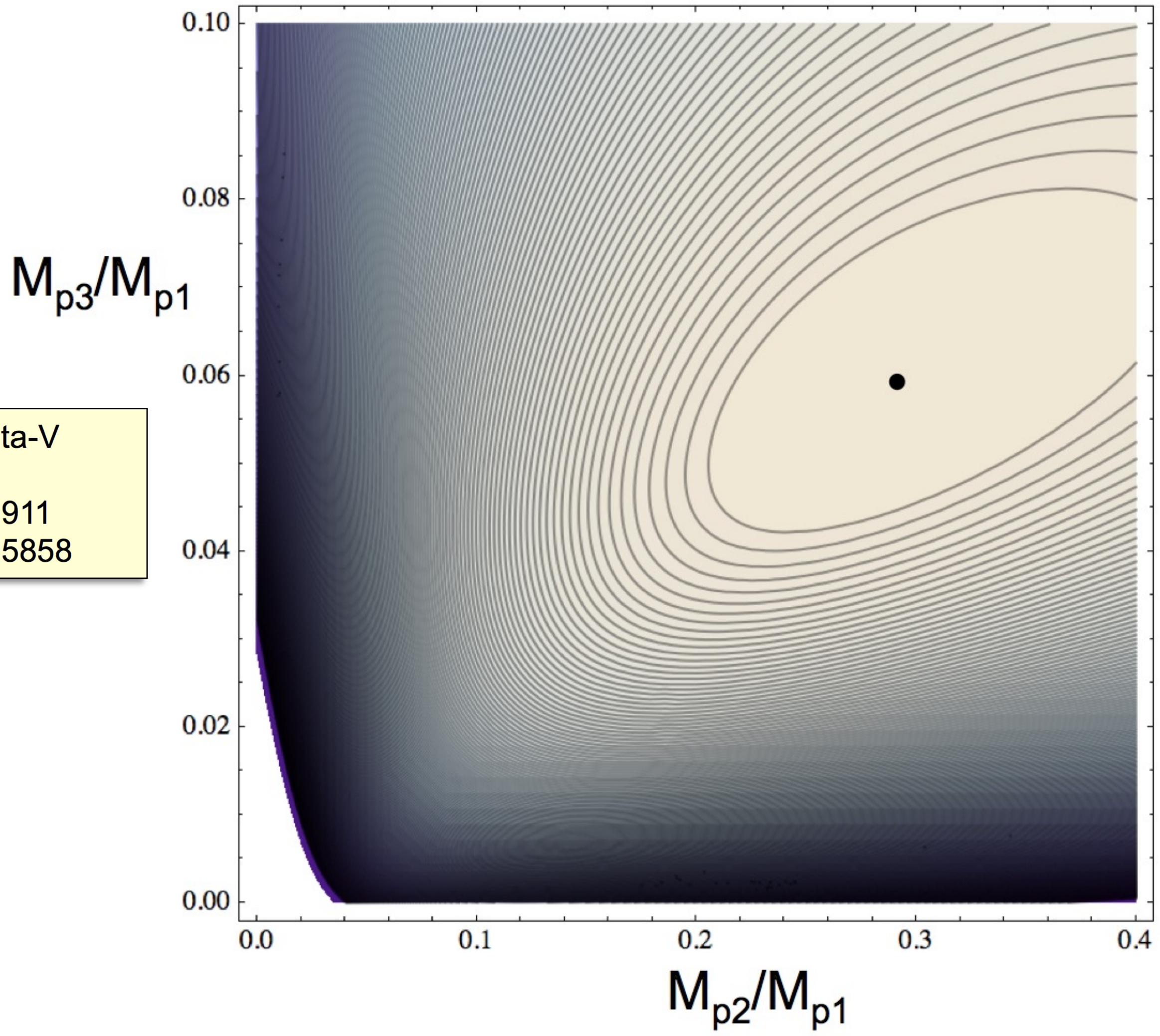
$$\Gamma, C_1, C_2, C_3, \varepsilon_1, \varepsilon_2, \varepsilon_3$$

The final velocity is a function of of the propellant ratios.

$$V_3 = F\left(\frac{M_{P2}}{M_{P1}}, \frac{M_{P3}}{M_{P1}}\right)$$

Differentiate with respect to the two variables to identify a maximum.

# Three stage launch vehicle for small satellites




Maximum delta-V occurs at
$M_{p2}/M_{p1} = 0.2911$
$M_{p3}/M_{p1} = 0.05858$

$\varepsilon_1 = 0.116$

$\varepsilon_2 = 0.117$

$\varepsilon_3 = 0.148$

$C_1 = 3040\ m/\sec$

$C_2 = 3210\ m/\sec$

$C_3 = 3350\ m/\sec$

$\Gamma = 0.01125$

$\Delta V = 10{,}800\ m/\sec$

$Payload\ mass\ =\ 85\ kg$

$M_{01} = 7557\ kg$

## 8.5 Problems

**Problem 1** - A two stage rocket is to be used to put a payload of 1000kg into low earth orbit. The vehicle will be launched from Kennedy Space Center where the speed of rotation of the Earth is 427 M/sec. Assume gravitational velocity losses of about 1200M/sec and aerodynamic velocity losses of 500M/sec. The first stage burns Kerosene and Oxygen producing a mean specific impulse of 320 seconds averaged over the flight, while the upper stage burns Hydrogen and Oxygen with an average specific impulse of 450 seconds. The structural coefficient of the first stage is 0.05 and that of the second is 0.07. Determine the payload ratios and the total mass of the vehicle. Suppose the same vehicle is to be used to launch a satellite into a north-south orbit from Kodiak island in Alaska. How does the mass of the payload change?

**Problem 2** - A group of universities join together to launch a four stage rocket with a small payload to the Moon. The fourth stage needs to reach the earth escape velocity of 11,176 M/sec. The vehicle will be launched from Kennedy Space Center where the speed of rotation of the Earth is 427 M/sec. Assume gravitational velocity losses of about 1500M/sec and aerodynamic velocity losses of 600M/sec. To keep cost down four stages with the same effective exhaust velocity $C$ and structural coefficient $\varepsilon$ are used. Each stage burns Kerosene and Oxygen producing a mean specific impulse of 330 seconds averaged over each segment of the flight. The structural coefficient of each stage is $\varepsilon = 0.1$ . Determine the payload ratio (if any).

**Problem 3** - A low-cost four stage rocket is to be used to launch small payload to orbit. The concept proposed for the system utilizes propellants that are safe and cheap but provide a specific impulse of only 200 seconds. All four stages are identical. What structural efficiency is required to reach orbit with a finite payload?

**Problem 4** - One of the most unusual launch systems in use by the U.S. is called Pegasus. It was developed to provide rapid response, all-weather, satellite launch capability from essentially anywhere in the world to essentially any orbit inclination. The three-stage launch vehicle with a mass of $22,004\,kg$ is carried aloft by a modified Lockheed L-1011, one of the original wide-body commercial aircraft developed in the 1960's. A typical mission profile is shown in Figure 8.7. The Pegasus is released at an altitude of $12,400\,m$ and speed of $240\,m/sec$. Each stage is a solid propellant rocket motor with the following properties.

$$Stage\,1 - M_{s1} = 1369\,kg,\ M_{p1} = 15014\,kg,\ C_1 = 2891\,m/sec$$

$$Stage\,2 - M_{s2} = 391\,kg,\ M_{p2} = 3915\,kg,\ C_2 = 2832\,m/sec \tag{8.41}$$

$$Stage\,3 - M_{s3} = 102.1\,kg,\ M_{p3} = 770.2\,kg,\ C_3 = 2813\,m/sec$$

In Figure 8.7 the stage 3 burnout is at an orbit altitude of $512.4\,km$. The payload carried to this orbit is $M_L = 443\,kg$.

(i) Assuming the orbit to be circular, what orbital speed is this? Determine the ideal $\Delta V$ of the Pegasus system. What is the velocity loss due to gravity plus drag in reaching this orbit?

(ii) How close to optimal is the Pegasus launch vehicle? Keeping the same structural efficiency, $\epsilon$, and effective exhaust velocity, $C$, for each stage, determine the distribution of propellant masses that would make the system optimal and compare to the current values. For the same payload mass and total propellant mass what would be the ideal optimal $\Delta V$?

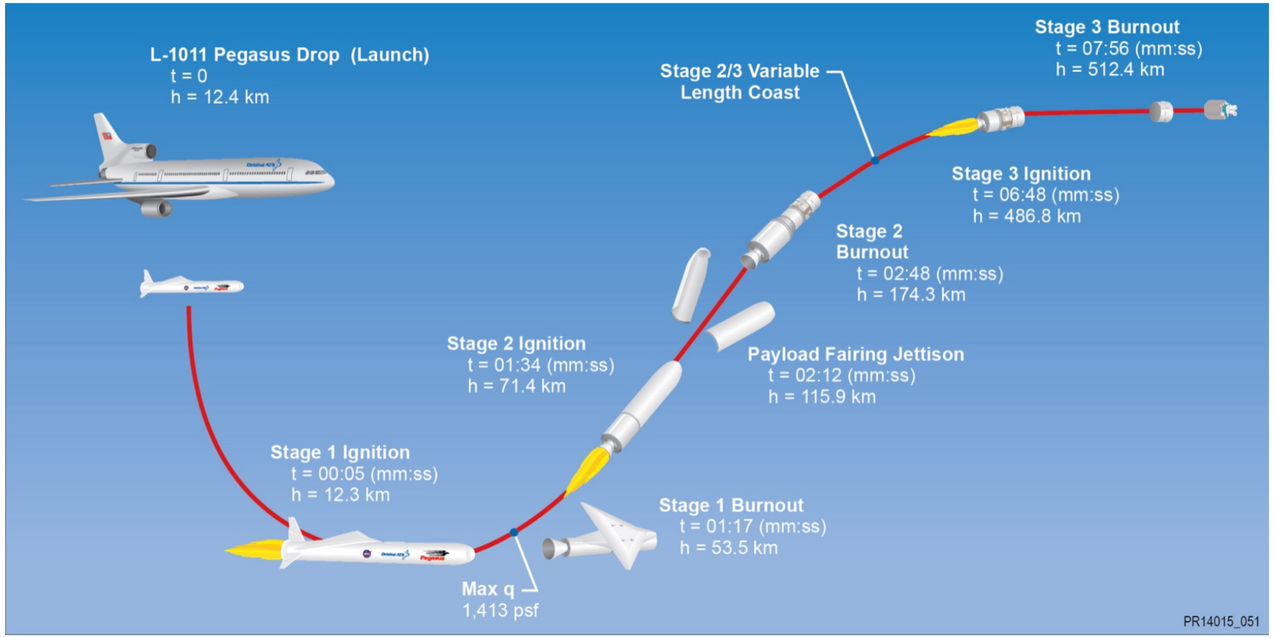



Figure 8.7: *Pegasus launch profile.*