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AA283
Aircraft and Rocket Propulsion

Chapter 07 - Rocket Performance

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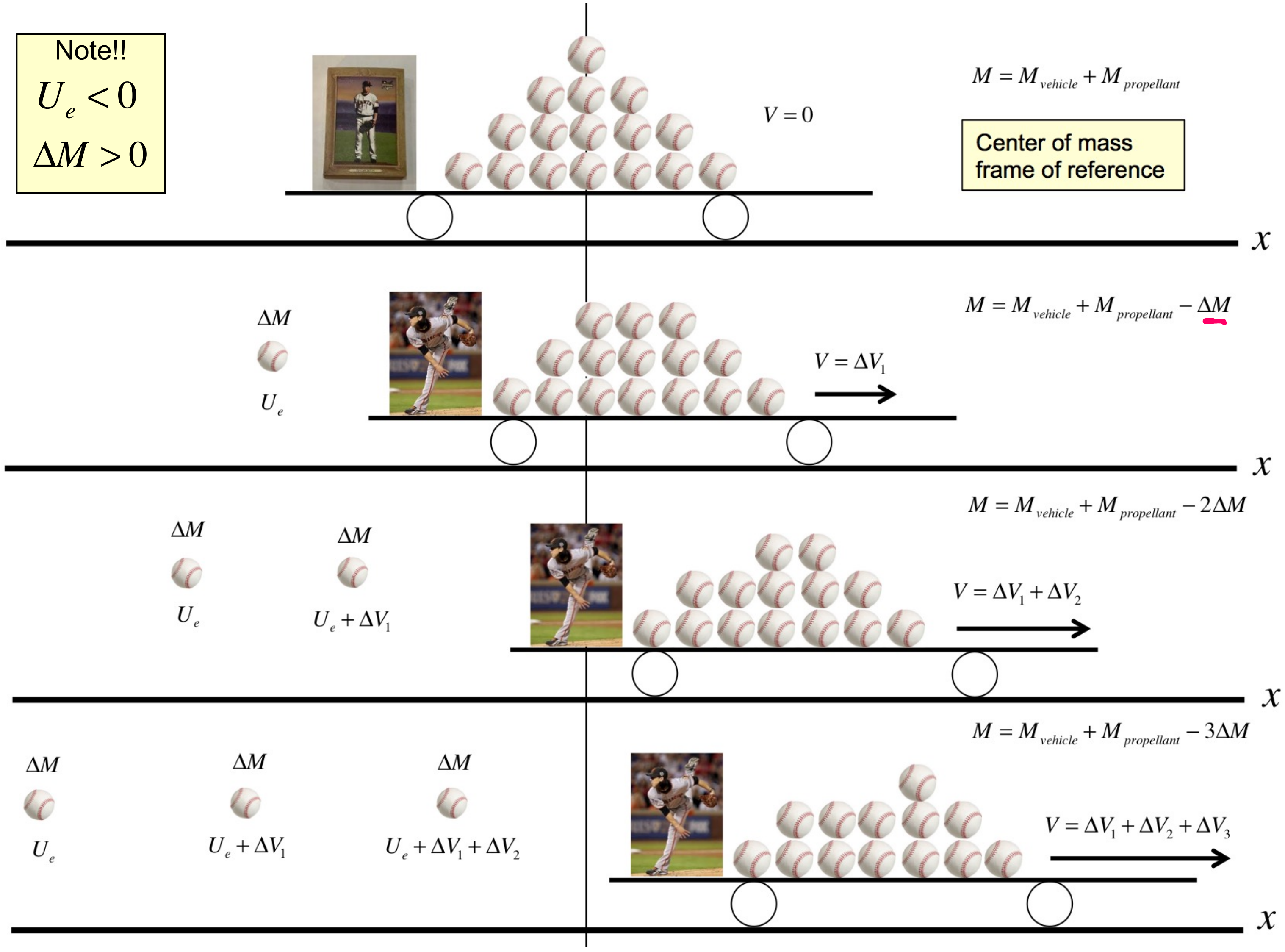
Thrust

Conservation of momentum – Tim Lincecum throwing baseballs from a frictionless cart



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Bookkeeping momentum in the center-of-mass frame of reference

	mass and velocity of vehicle	total momentum in the center-of-mass frame
momentum of exhaust mass	$M = M_{vehicle} + M_{propellant} \quad V = 0$	$0 = \left(M_{vehicle} + M_{propellant}\right) \times 0$
$\Delta M U_e$	$M = M_{vehicle} + M_{propellant} - \Delta M$ $V = \Delta V_1$	$0 = \left(M_{vehicle} + M_{propellant} - \Delta M\right)\Delta V_1 + \Delta M U_e$
$\Delta M U_e$ $\Delta M\left(U_e + \Delta V_1\right)$	$M = M_{vehicle} + M_{propellant} - 2\Delta M$ $V = \Delta V_1 + \Delta V_2$	$0 = \left(M_{vehicle} + M_{propellant} - 2\Delta M\right)\left(\Delta V_1 + \Delta V_2\right) + 2\Delta M U_e + \Delta M \Delta V_1$ Subtract $0 = \left(M_{vehicle} + M_{propellant} - \Delta M\right)\Delta V_1 + \Delta M U_e$ Equals $0 = \left(M_{vehicle} + M_{propellant} - 2\Delta M\right)\Delta V_2 + \Delta M U_e$
$\Delta M U_e$ $\Delta M\left(U_e + \Delta V_1\right)$ $\Delta M\left(U_e + \Delta V_1 + \Delta V_2\right)$	$M = M_{vehicle} + M_{propellant} - 3\Delta M$ $V = \Delta V_1 + \Delta V_2 + \Delta V_3$	$0 = \left(M_{vehicle} + M_{propellant} - 3\Delta M\right)\left(\Delta V_1 + \Delta V_2 + \Delta V_3\right) + 3\Delta M U_e + 2\Delta M \Delta V_1 + \Delta M \Delta V_2$ Subtract $0 = \left(M_{vehicle} + M_{propellant} - 2\Delta M\right)\left(\Delta V_1 + \Delta V_2\right) + 2\Delta M U_e + \Delta M \Delta V_1$ Equals $0 = \left(M_{vehicle} + M_{propellant} - 3\Delta M\right)\Delta V_3 + \Delta M U_e$

$$U_e < 0$$
$$\Delta M > 0$$

Conservation of momentum

$$M \Delta V = -\Delta M U_e$$

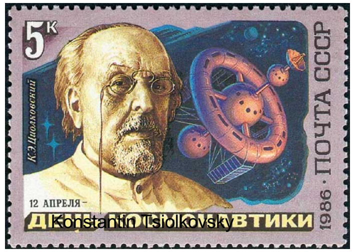
Thrust

$$T = M \frac{\Delta V}{\Delta t} = -U_e \frac{\Delta M}{\Delta t}$$

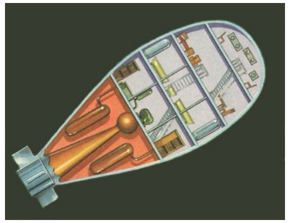
Back to rockets - What is the final speed of the cart?

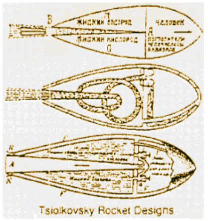
Basic equation of motion

$$M \Delta V = -\Delta M U_e$$



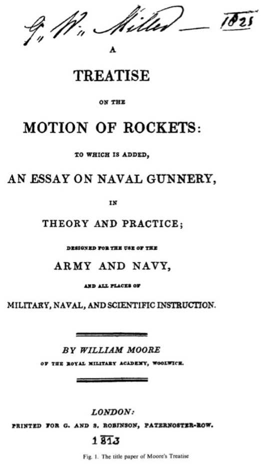
Convert to a differential. Take the limit $\Delta M \rightarrow 0$





Note the change in sign of dM. The differential change in mass is now the change in mass of the vehicle.

$$dV = U_e \frac{dM}{M}$$



A
TREATISE
ON THE
MOTION OF ROCKETS:
TO WHICH IS ADDED,
AN ESSAY ON NAVAL GUNNERY,
IN
THEORY AND PRACTICE;
DESIGNED FOR THE USE OF THE
ARMY AND NAVY,
AND ALL PLACES OF
MILITARY, NAVAL, AND SCIENTIFIC INSTRUCTION.

BY WILLIAM MOORE
OF THE ROYAL MILITARY ACADEMY, WOOLWICH.

LONDON:
PRINTED FOR G. AND S. ROBINSON, PATERNOSTER-ROW.
1813

Fig. 1. The title paper of Moore's Treatise

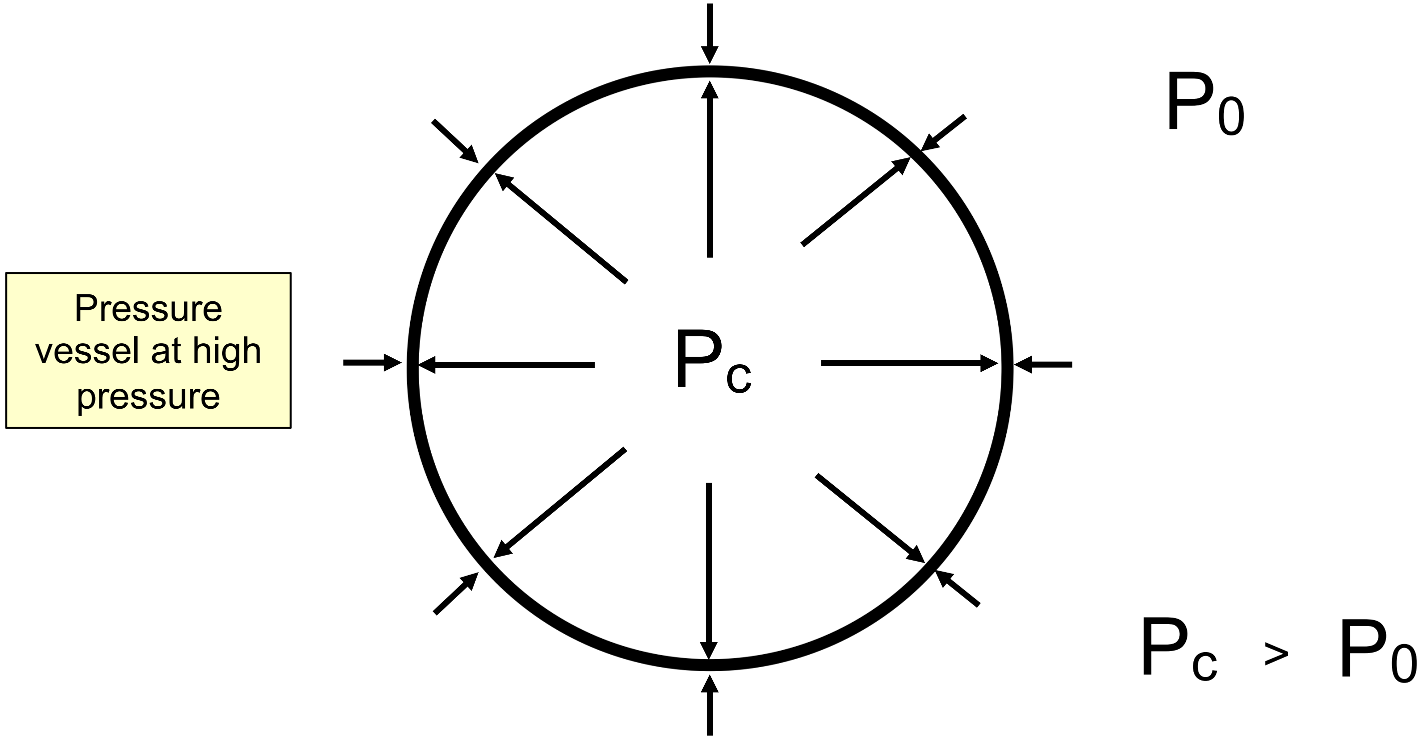
Integrate to produce the classical Moore-Tsiolkovsky rocket equation

$$V_{final} = U_e \ln\left(\frac{M_{final}}{M_{initial}}\right) = -U_e \ln\left(\frac{M_{delivered} + M_{propellant}}{M_{delivered}}\right)$$

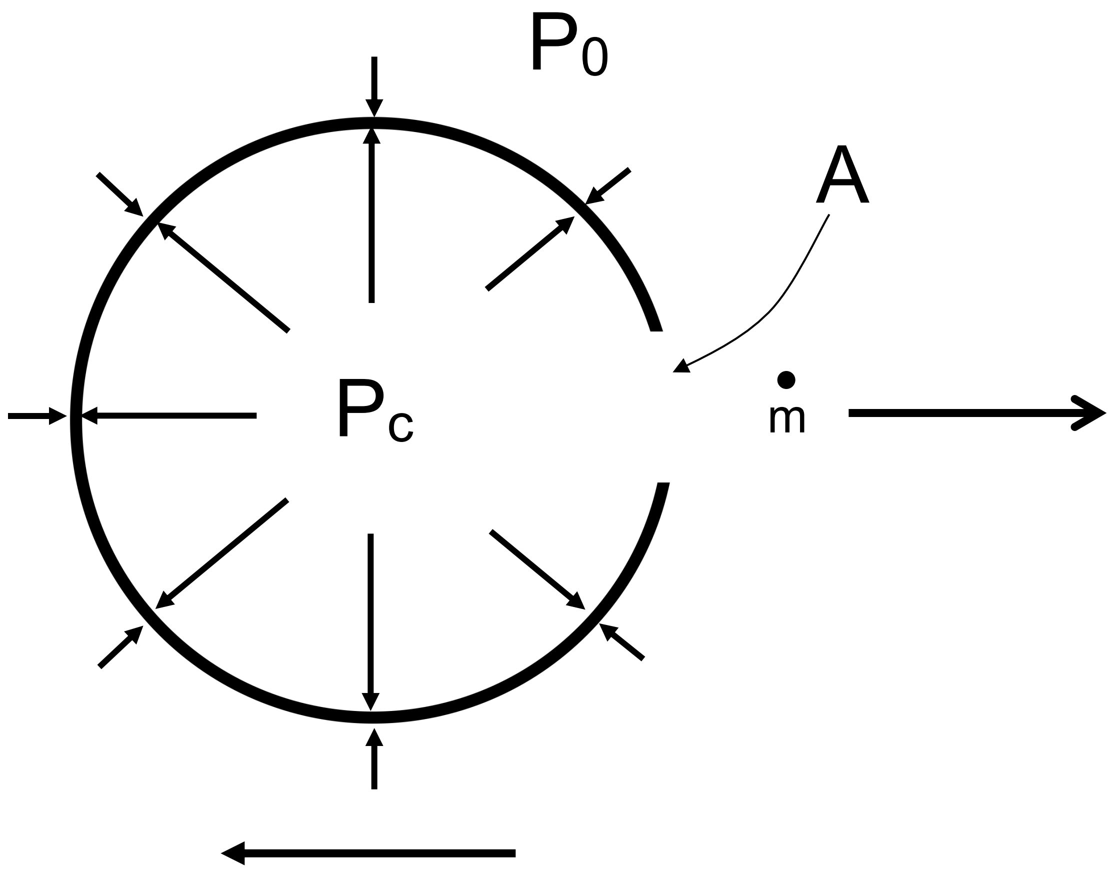
The motion of the cart is precisely analogous to the motion of a rocket in free space

The Basics of Rocket Thrust

Thrust is produced, not by expelling baseballs, but by expelling a gas



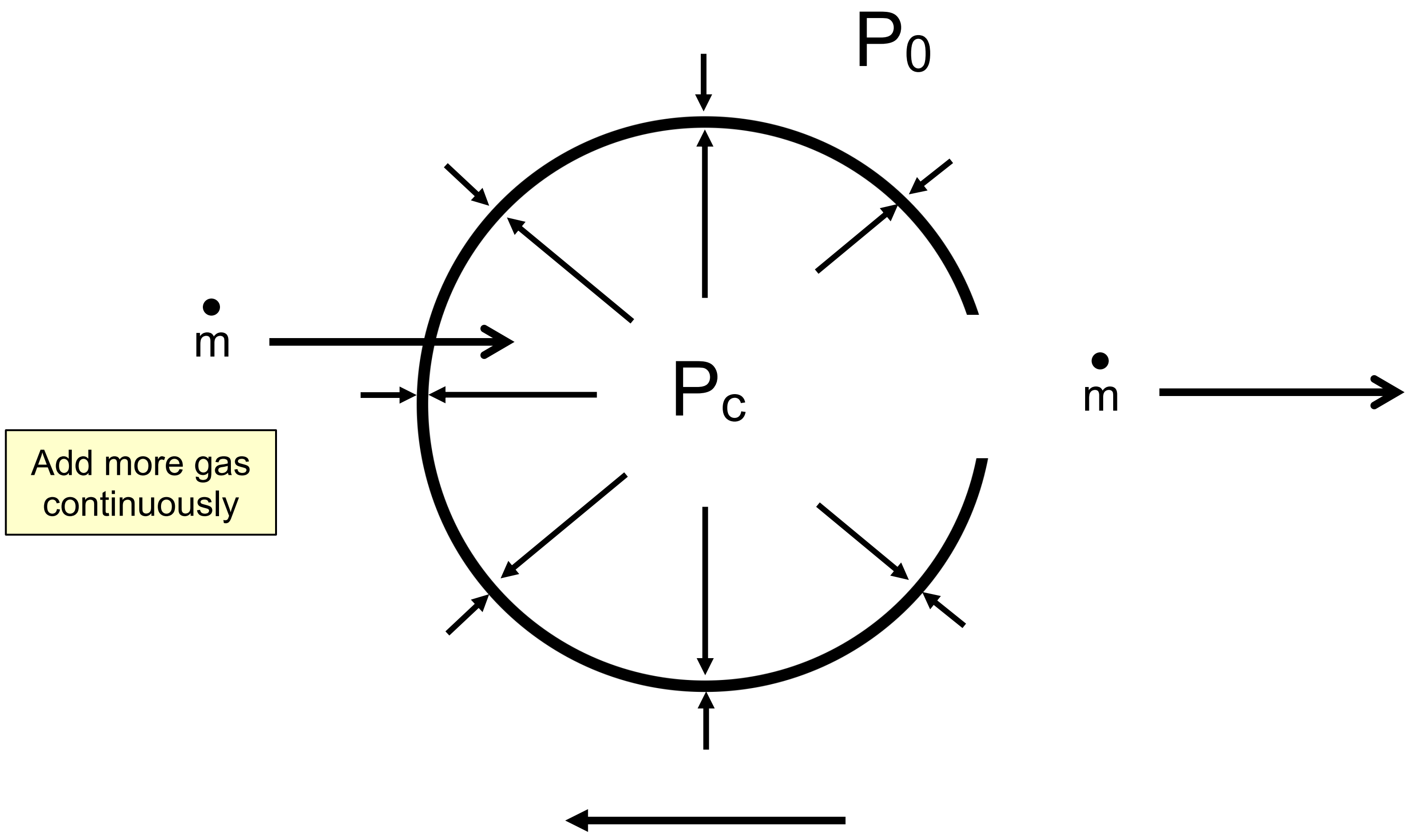
The Basics of Rocket Thrust



Open a hole

Thrust ~ $(P_c - P_0) \times A$

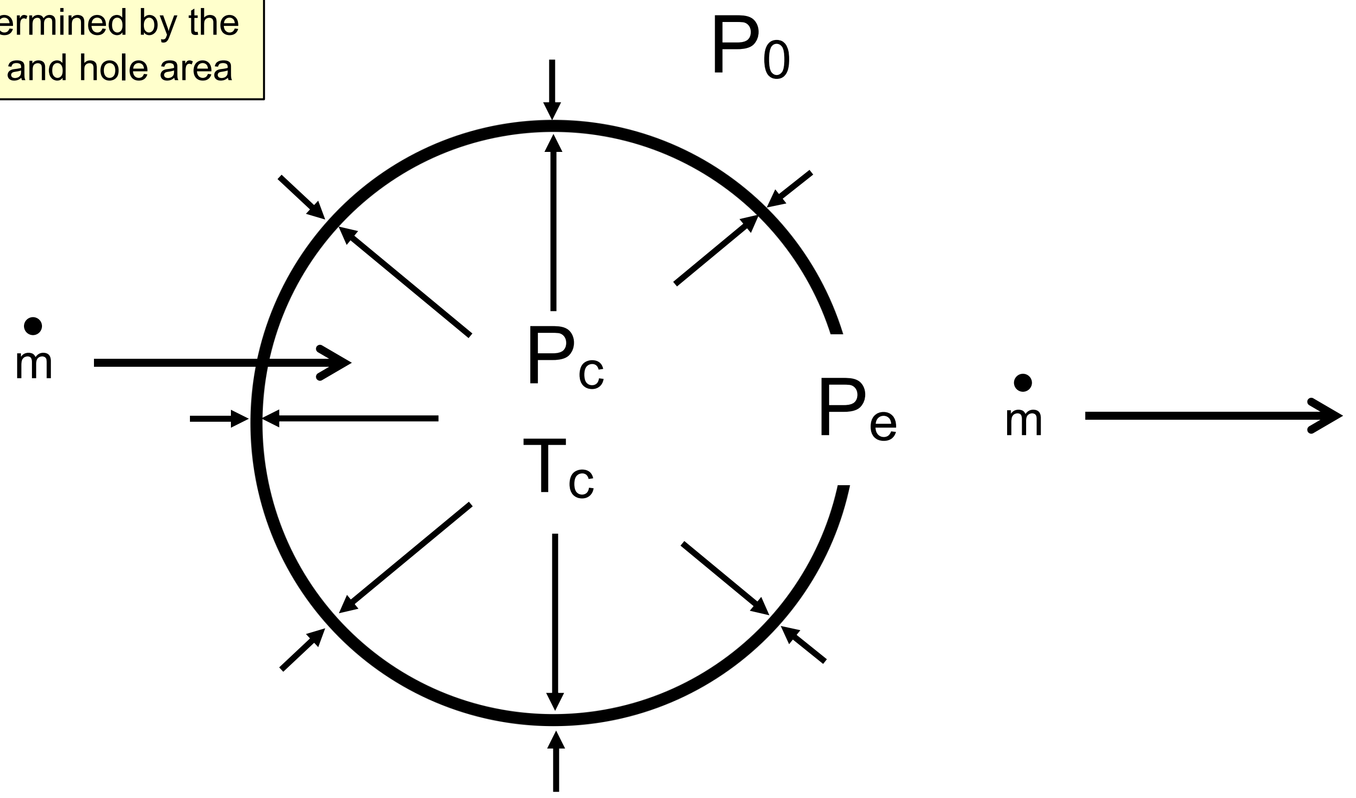
The Basics of Rocket Thrust



Thrust ~ $(P_c - P_0) \times A$

The Basics of Rocket Thrust

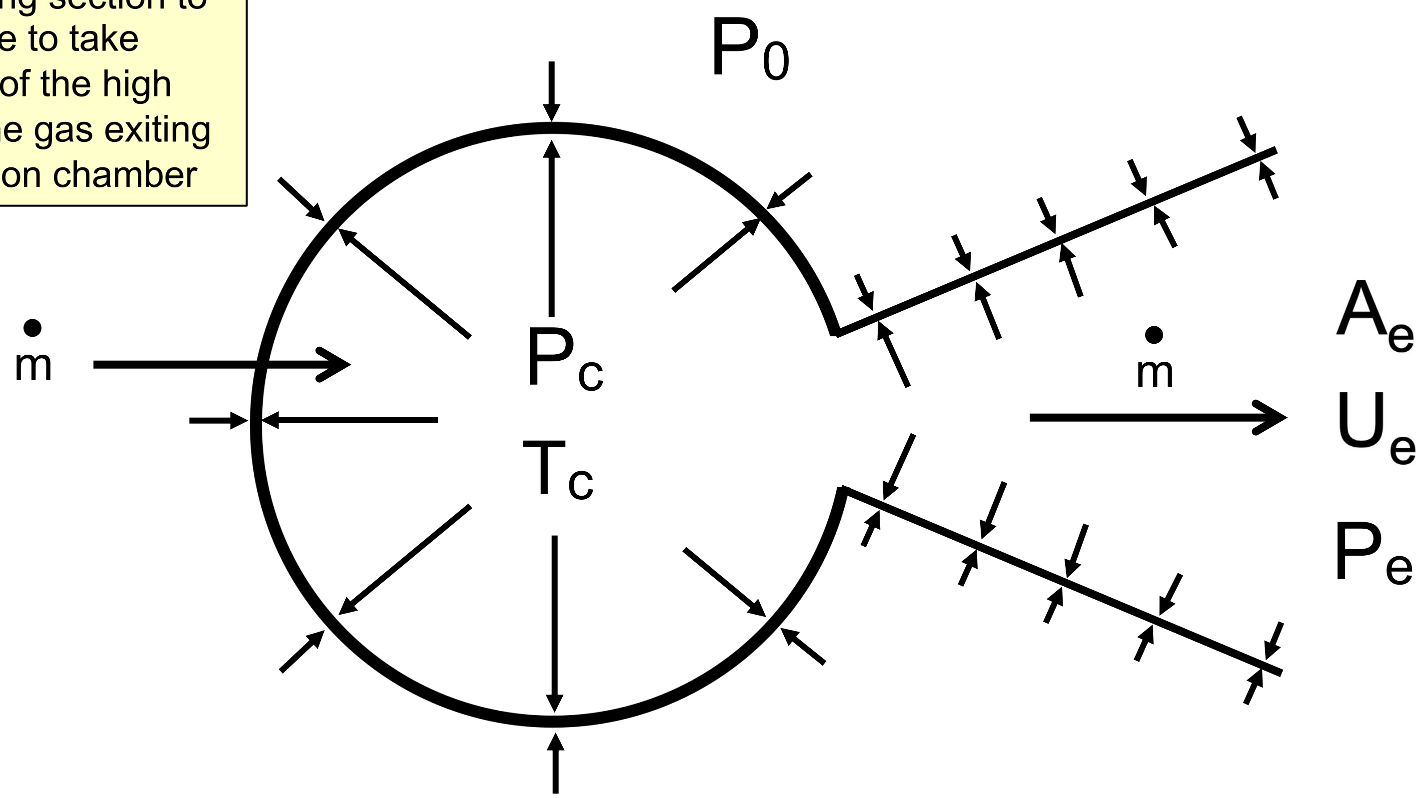
The combustion chamber pressure is determined by the mass flow rate and hole area



$$\text{Thrust} \sim P_c A + (P_e - P_0) A \sim \dot{m} \times \sqrt{\frac{R_u T_c}{M_w}} + (P_e - P_0) A$$

The Basics of Rocket Thrust

Add a diverging section to the nozzle to take advantage of the high pressure of the gas exiting the combustion chamber



$$\text{Thrust} = \dot{m} \times U_e + (P_e - P_0) \times A_e$$

7.1 Thrust – A more fluids-based approach

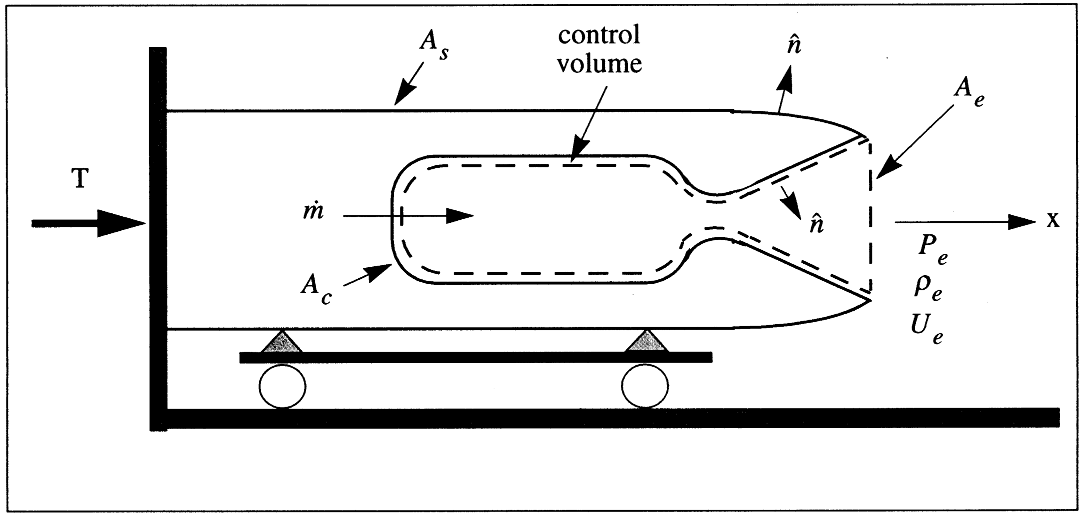


Figure 7.1 Rocket thrust schematic

$$
\begin{aligned}
A_s &= \text{outside surface of the vehicle exposed to } P_0 \\
A_c &= \text{inside surface of the combustion chamber} \\
A_e &= \text{nozzle exit area} \\
\hat{n} &= \text{outward unit normal} \\
P_e &= \text{area averaged exit gas pressure} \\
\rho_e &= \text{area averaged exit gas density} \\
U_e &= \text{area averaged x-component of velocity at the nozzle exit}
\end{aligned}
\tag{7.1}
$$

The total force on the rocket is zero

X-momentum of injected propellant

$$0 = T + \int_{A_s} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n}\, dA \Big|_x + \int_{A_c} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n}\, dA \Big|_x + \dot{m} U_{xm} \qquad (7.2)$$

If the rocket were inactive (turned off)

$$0 = \int_{A_s} P_0 \bar{\bar{I}} \bullet \hat{n}\, dA \Big|_x + \int_{A_c} P_0 \bar{\bar{I}} \bullet \hat{n}\, dA \Big|_x \qquad (7.3)$$

In this situation the control volume contains fluid all at rest.

$$0 = \int_{A_c} P_0 \bar{\bar{I}} \bullet \hat{n}\, dA \Big|_x + \int_{A_e} P_0 \bar{\bar{I}} \bullet \hat{n}\, dA \Big|_x \qquad (7.4)$$

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The last equation can be written as

$$0 = \left. \int_{A_c} P_0 \bar{\bar{I}} \bullet \hat{n} dA \right|_x + P_0 A_e . \tag{7.5}$$

Thus (7.3) can be written

$$0 = \left. \int_{A_s} P_0 \bar{\bar{I}} \bullet \hat{n} dA \right|_x - P_0 A_e \tag{7.6}$$

The original force balance becomes

$$0 = T + P_0 A_e + \left. \int_{A_c} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n} dA \right|_x + \dot{m} U_{xm} \tag{7.7}$$

The assumption in this last relation is

$$\left(\int_{A_s} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n}\, dA \Bigg|_x \right)_{after\ engine\ turn\ on} = \left(\int_{A_s} P_0 \bar{\bar{I}} \bullet \hat{n}\, dA \Bigg|_x \right)_{before\ engine\ turn\ on} \qquad (7.8)$$

Momentum balance

$$\frac{d}{dt}\int_V \rho \bar{U}\, dV = \int_V \frac{\partial \rho \bar{U}}{\partial t} dV = -\int_V \nabla \bullet (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}})\, dV \qquad (7.9)$$

Convert the right hand side to a surface integral

$$0 = \int_V \nabla \bullet (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}})\, dV =$$

$$\int_{A_c} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n}\, dA + \int_{A_e} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n}\, dA \qquad (7.10)$$

On the combustion chamber surface the velocity is zero by the no-slip condition. Use area averaging over the nozzle exit.

$$\left.\int_{A_c} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n}\, dA \right|_x + \dot{m} U_{xm} + \rho_e U_e^2 A_e + P_e A_e = 0 \qquad (7.11)$$

Now the force balance on the rocket is

$$0 = T + P_0 A_e - (\rho_e U_e^2 A_e + P_e A_e). \qquad (7.12)$$

Finally our rocket thrust formula is

$$T = \rho_e U_e^2 A_e + (P_e - P_0) A_e \qquad (7.13)$$

The propellant mass flow is

$$\dot{m} = \rho_e U_e A_e \tag{7.14}$$

and so the thrust formula is often written as

$$\boxed{T = \dot{m} U_e + (P_e - P_0) A_e} \tag{7.15}$$

Same as the thrust definition for air-breathing engines without the incoming air momentum term.

7.2 Momentum conservation in the center-of-mass system

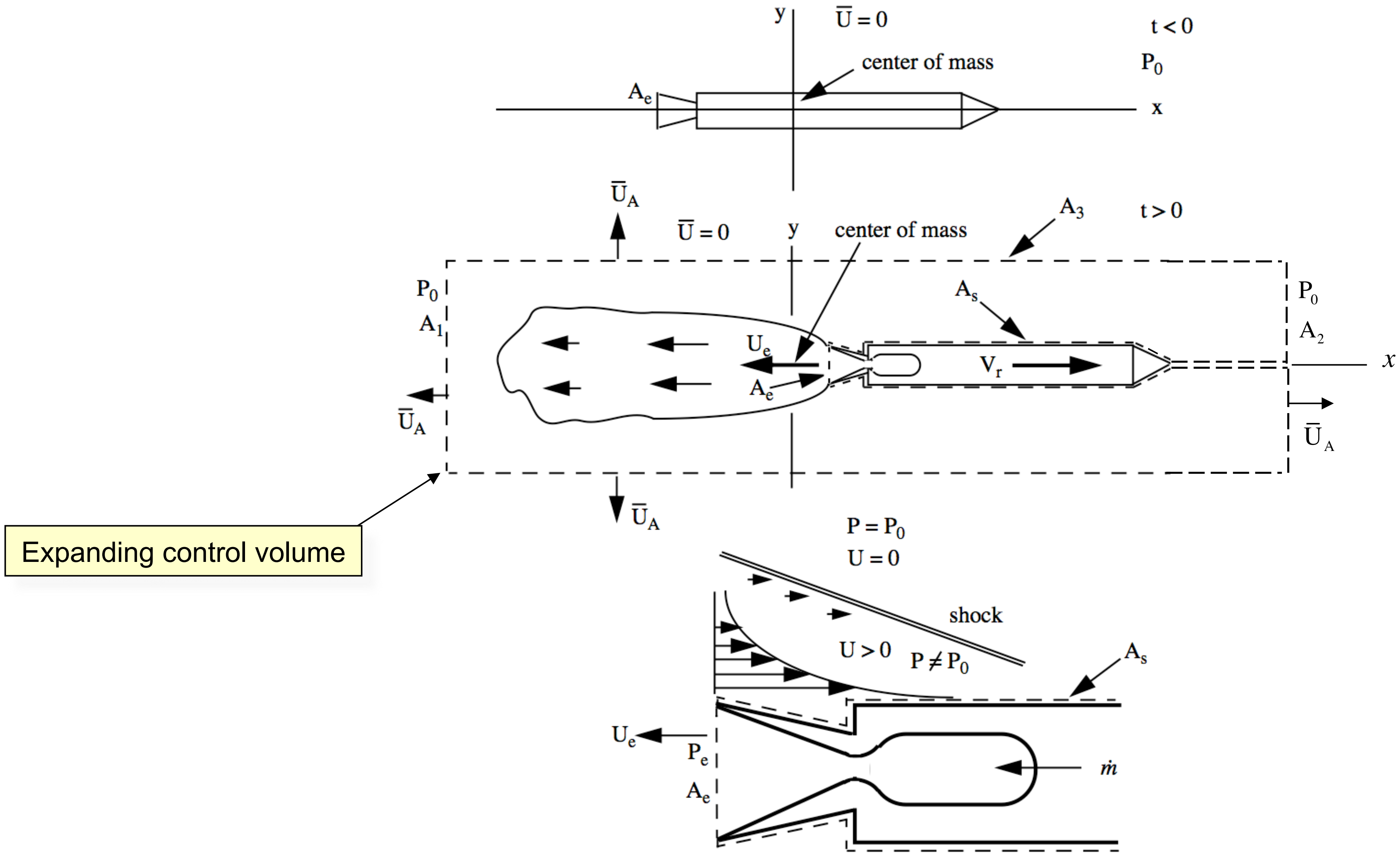

Figure 7.2 Center-of-mass description of the rocket and expelled propellant.

Conservation of momentum for the aggregate system comprising

1. Rocket vehicle plus onboard propellant
2. Expelled combustion gases
3. Air set into motion by the drag of the vehicle and through entrainment by the rocket plume.

$$\frac{D}{Dt}\left\{ \int_{V(t)} \rho \bar{U}\, dV \Big|_x + M_r(t) V_r(t) \right\} = 0 \qquad (7.16)$$

Mr(t) is the vehicle mass and Vr(t) is the vehicle velocity.

In the absence of gravity no external forces act on the system.

Momentum integral

$$\frac{D}{Dt}\int_{V(t)} \rho \bar{U} dV\Big|_x = -\int_{A(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA\Big|_x \qquad (7.17)$$

Over most of the control volume surface no additional momentum is being enclosed. The control volume is sufficiently large so that the fluid velocity on A1, A2 and A3 is zero and the pressure is P0. The pressure forces on A1 and A2 cancel and the pressure force on A3 has no component in the *x* direction. Thus the momentum balance becomes

$$\frac{D}{Dt}\int_{V(t)} \rho \bar{U} dV\Big|_x = -\int_{A_s(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA\Big|_x - \int_{A_e(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA\Big|_x \qquad (7.18)$$

Recall that the integral of the ambient pressure over the whole surface of the rocket is zero.

$$0 = \int_{A_c} P_0 \bar{\bar{I}} \bullet \hat{n} dA \Big|_x + \int_{A_e} P_0 \bar{\bar{I}} \bullet \hat{n} dA \Big|_x \qquad (7.4)$$

Add the above equation to the previous result

$$\frac{D}{Dt}\int_{V(t)} \rho \bar{U} dV \big|_x = -\int_{A_s(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + (P - P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \big|_x - \int_{A_e(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + (P - P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \big|_x \qquad (7.20)$$

On the no-slip surface of the rocket the fluid velocity is $\bar{U} = (V_r, 0, 0)$

On the rocket the control volume surface velocity is $\bar{U}_A = (V_r, 0, 0)$

$$\int_{A_s(t)} \rho \bar{U}(\bar{U} - \bar{U}_A) \bullet \bar{n} dV \Big|_x = 0 \qquad (7.21)$$

Now

$$\frac{D}{Dt} \int_{V(t)} \rho \bar{U} dV \Big|_x = - \int_{A_s(t)} ((P - P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_x - \qquad (7.22)$$

$$\int_{A_e(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + (P - P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_x$$

The combination of viscous skin-friction drag, pressure drag and wave drag are all accounted for by the first integral.

$$D = - \int_{A_s(t)} ((P - P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_x \qquad (7.23)$$

Now we can evaluate the rate-of-change of the integrated gas momentum over the control volume.

$$\frac{D}{Dt}\int_{V(t)} \rho\bar{U}dV\Big|_x = D - \int_{A_e(t)} (\rho\bar{U}(\bar{U}-\bar{U}_A) + (P-P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n}dA\Big|_x \qquad (7.24)$$

Fluid velocity at the nozzle exit in the center of mass frame

$$U = V_r + U_e \qquad (7.25)$$

Area average over the nozzle exit

$$\int_{A_e(t)} (\rho\bar{U}(\bar{U}-\bar{U}_A) + (P-P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n}dA\Big|_x$$

$$= \rho_e A_e(U_e + V_r)(U_e + V_r - V_r) + (P_e - P_0)A_e \qquad (7.26)$$

$$\frac{D}{Dt}\int_{V(t)} \rho\bar{U}dV\Big|_x = D - (\rho_e A_e(U_e + V_r)(U_e + V_r - V_r) + (P_e - P_0)A_e) \qquad (7.27)$$

Momentum conservation of the aggregate system is

$$\frac{D}{Dt}(M_r(t)V_r(t)) + D - (\rho_e(U_e + V_r)U_e + (P_e - P_0))A_e = 0 \qquad (7.28)$$

Carry out the differentiation

$$M_r\frac{dV_r}{dt} + V_r\frac{dM_r}{dt} + D - \rho_e U_e A_e V_r - (\rho_e U_e^2 + (P_e - P_0))A_e = 0 \qquad (7.29)$$

Note

$$\frac{dM_r}{dt} = \rho_e U_e A_e \qquad (7.30)$$

Finally

$$M_r\frac{dV_r}{dt} = (\rho_e U_e^2 A_e + (P_e - P_0)A_e) - D \qquad (7.31)$$

Rocket mass × acceleration = Thrust - Drag

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7.3 Effective exhaust velocity

Total impulse

$$I = \int_0^t T\,dt \qquad (7.33)$$

Total propellant expended

$$M_p = \int_0^t \dot{m}\,dt \qquad (7.34)$$

Effective exhaust velocity

$$C = \frac{dI}{dM_p} = \frac{T}{\dot{m}} = U_e + \frac{A_e}{\dot{m}}(P_e - P_0) \qquad (7.35)$$

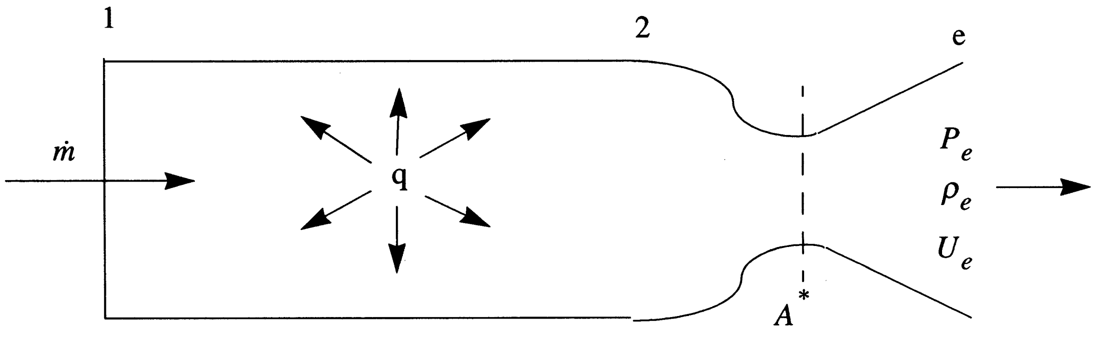
In terms of the exit Mach number

$$C = U_e\left(1 + \frac{P_e A_e}{\rho_e U_e^2 A_e}\left(1 - \frac{P_0}{P_e}\right)\right) \qquad (7.36)$$

$$C = U_e\left(1 + \frac{1}{\gamma M_e^2}\left(1 - \frac{P_0}{P_e}\right)\right) \qquad (7.37)$$

Note that for a large area ratio exhaust the pressure contribution is a diminishing proportion of the thrust.

Theoretical maximum exhaust velocity



$$h_{t2} = h_{t1} + q = h_e + \frac{1}{2}U_e^2 \quad (7.38)$$

$$C_{max} \cong \sqrt{2q} \quad (7.39)$$

$$C_{max} \cong \sqrt{2C_pT_{t2}} = \sqrt{\frac{2\gamma}{\gamma - 1}RT_{t2}} . \quad (7.40)$$

Note that

$$R = \frac{R_u}{M_w} \qquad (7.41)$$

The theoretical maximum exhaust velocity depends on the combustion chamber temperature and the molecular weight of the exhaust gases.

$$C_{max} \cong \sqrt{\frac{2\gamma}{\gamma - 1}\left(\frac{R_u}{M_w}\right)T_{t2}} \qquad (7.42)$$

7.4 Specific impulse

The specific impulse is defined as

$$I_{sp} = \frac{T}{\dot{m}g_0} = \frac{C}{g_0} \tag{7.46}$$

Note that for rockets the specific impulse is defined in terms of the total propellant mass flow whereas for air-breathing engines only the fuel mass flow is used. Note also that the specific impulse increases as the nozzle expansion ratio increases and as the ambient pressure decreases.

Typical solid rockets have ideal (fully expanded) specific impulses in the range of 230 to 290 sec.

Liquid rockets using hydrocarbon fuel burning with Oxygen have ideal specific impulses in the range of 360 to 370 sec. Hydrogen - Oxygen systems reach 450 sec.

A meaningful statement about the specific impulse of a rocket motor should always specify the chamber pressure and area ratio of the motor and whether the information refers to ideal or delivered impulse.

7.5 Chamber pressure

Recall the all-important mass flow relation

$$\dot{m} = \rho U A = \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \frac{\gamma P_t A}{\sqrt{\gamma R T_t}} f(M) \qquad (7.47)$$

where

$$f(M) = \frac{A^*}{A} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{M}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \qquad (7.48)$$

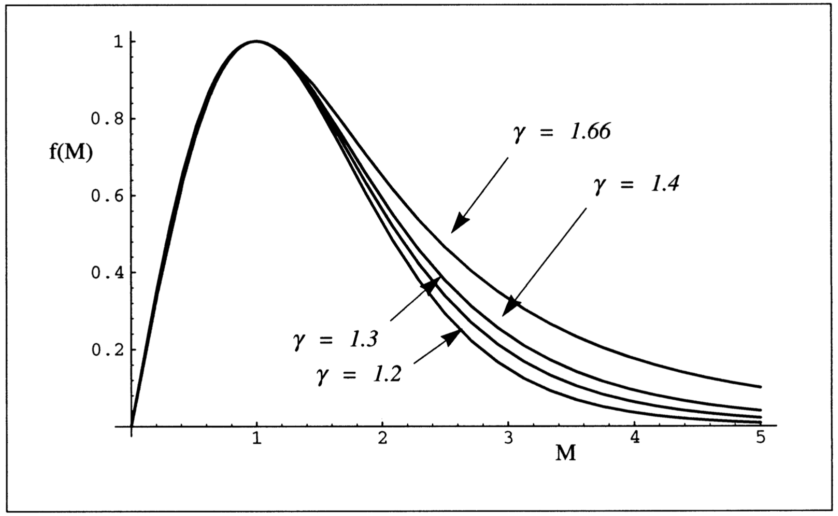


Figure 7.3 Area-Mach number relation

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The stagnation temperature at station 2 is the sum of the incoming stagnation temperature and temperature increase due to heat release.

$$T_{t2} = T_{t1} + \frac{q}{C_p} \qquad (7.49)$$

To a good approximation the stagnation temperature in the combustion chamber is independent of chamber stagnation pressure.

The chamber pressure is determined by the mass flow and chamber temperature. Virtually all rocket chambers run at a high enough pressure to insure that the nozzle is choked.

$$P_{t2} = \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \frac{\sqrt{\gamma R T_{t2}}}{\gamma A^*} \dot{m} \qquad (7.50)$$

7.6 Combustion chamber stagnation pressure drop

Recall the Rayleigh line relation

$$\frac{P_{t2}}{P_{t1}} = \left\{\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}\right\}\left\{\frac{1 + \frac{\gamma - 1}{2} M_2^2}{1 + \frac{\gamma - 1}{2} M_1^2}\right\}^{\frac{\gamma}{\gamma - 1}} \qquad (7.51)$$

The static pressure ratio is

$$\frac{P_2}{P_1} = \left\{\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}\right\} \qquad (7.52)$$

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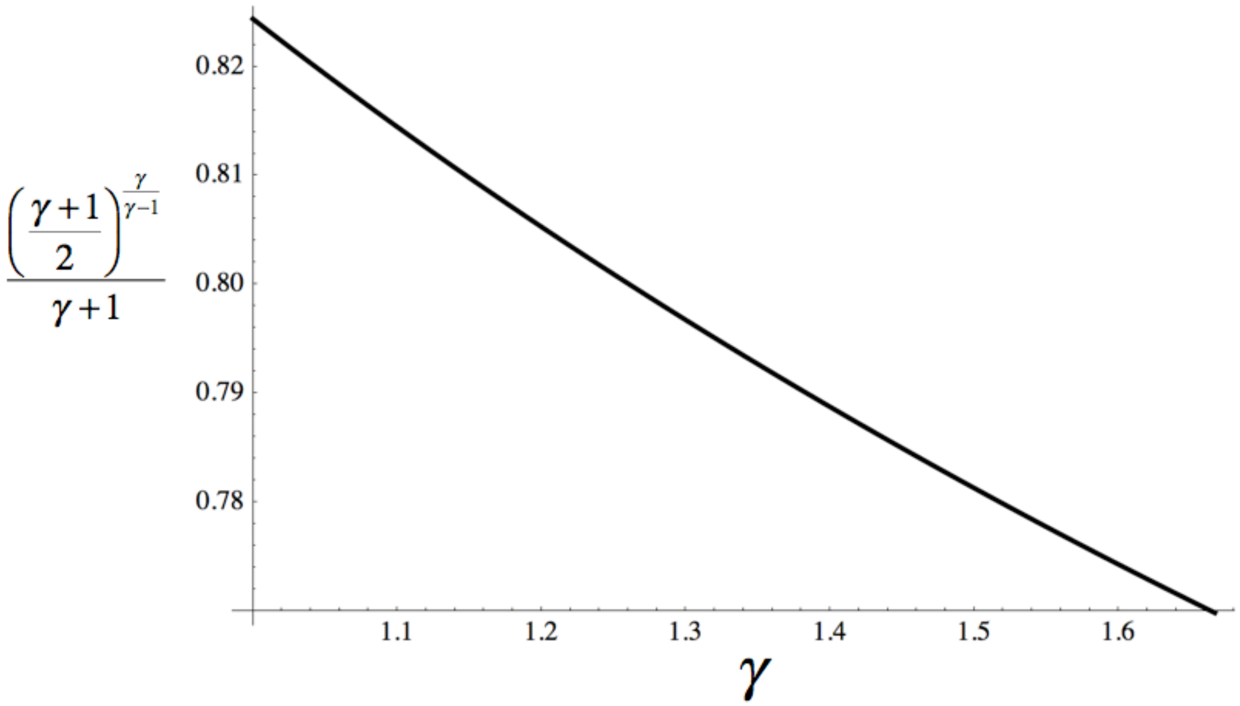
The maximum stagnation pressure loss due to heat addition

$$\left(\frac{P_{t2}}{P_{t1}}\right)_{\substack{M_1=0\\M_2=1}} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}} = \frac{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}}{(1+\gamma)}$$



At station 1 the Mach number is small and so we can approximate conditions at station 2 just in terms of the Mach number at station 2.

$$\frac{P_{t2}}{P_{t1}} = \left\{\frac{1}{1+\gamma M_2^2}\right\}\left\{1+\frac{\gamma-1}{2}M_2^2\right\}^{\frac{\gamma}{\gamma-1}}$$

$$\frac{P_2}{P_1} = \left\{\frac{1}{1+\gamma M_2^2}\right\} \qquad (7.53)$$

The Mach number at station 2 is determined by the nozzle area ratio.

$$\frac{A^*}{A_2} = \frac{M_2}{\left\{\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_2^2\right)\right\}^{\frac{\gamma+1}{2(\gamma-1)}}} \qquad (7.54)$$

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The internal area ratio of the nozzle determines the maximum stagnation pressure loss due to heat addition

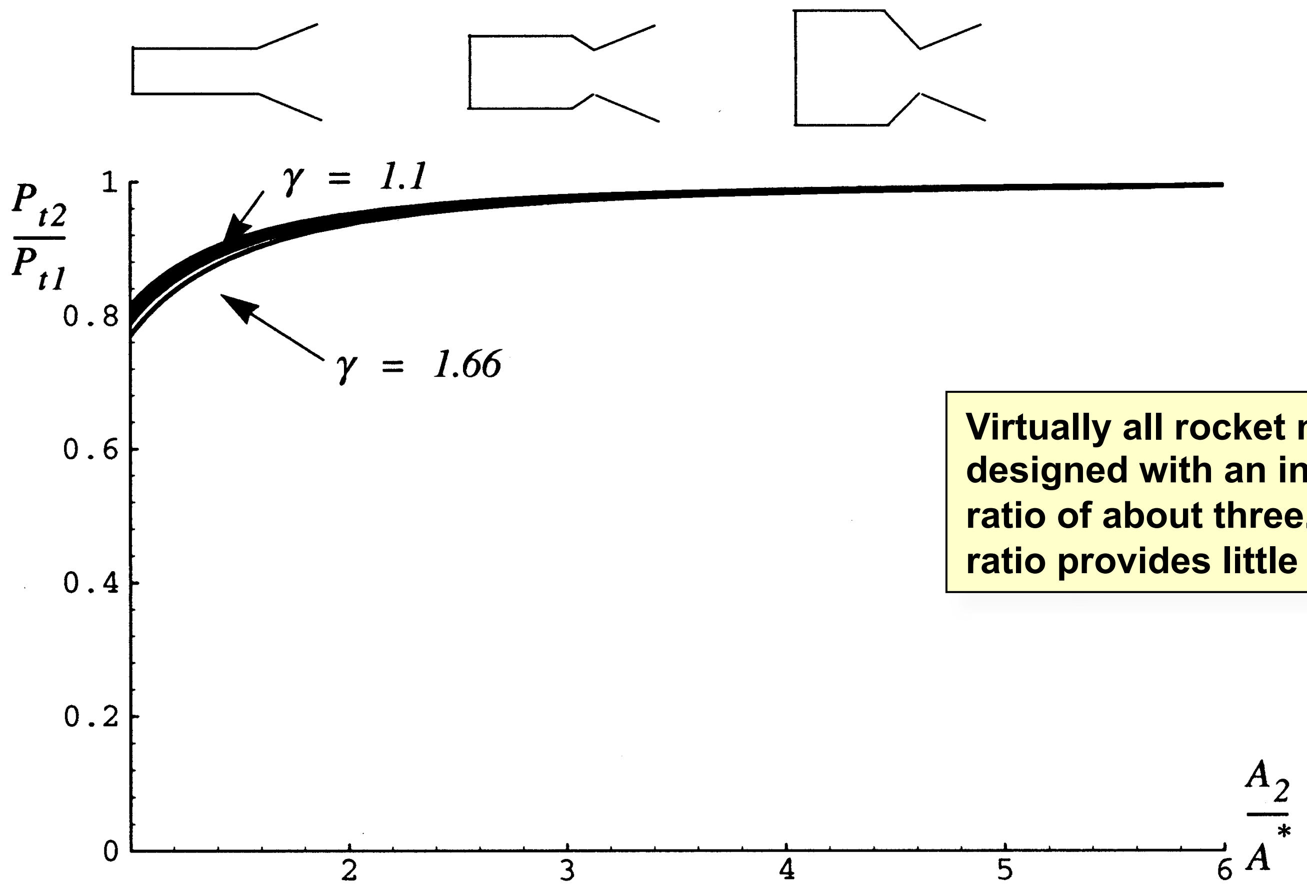

Figure 7.4 Combustion chamber stagnation pressure loss

Virtually all rocket nozzles are designed with an internal area ratio of about three. A larger area ratio provides little improvement.

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Description

The Walter 109-509C two-chambered rocket engine was intended for use on Germany's Meserschmitt Me 263 and Junkers Ju 248 second generation rocket-powered interceptor aircraft, but hostilities in Europe ended before it was made operational. It burned a fuel mixture of methyl alcohol and hydrazine in hydrogen peroxide. The photograph shows a sectioned version of the smaller chamber. The fuel was pumped between the two layers of its double skin to keep it cool.

7.7 C* efficiency

Define C*

$$\dot{m} = \frac{P_{t2}A^*}{C^*} \qquad (7.43)$$

Recall our mass flow relation – at the choked throat

$$\dot{m} = P_{t2}A^* \left(\frac{\gamma}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \sqrt{\gamma \frac{R_u}{M_w} T_{t2}}} \right)$$

Under a constant heat capacity assumption C* would be

$$C^* = \frac{P_{t2}A^*}{\dot{m}} = \frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \sqrt{\gamma \frac{R_u}{M_w} T_{t2}}$$

C* is directly proportional to the speed of sound in the rocket chamber and provides a very useful measure of completeness of combustion. The definition (7.43) enables C* to be determined without making any assumptions about the gas passing through the nozzle. For a given mass flow and throat area, higher C* implies higher chamber pressure and higher thrust.

Rocket test schematic

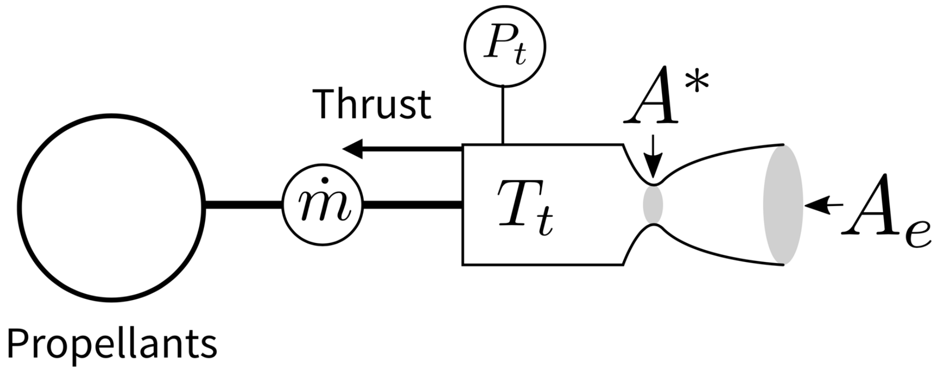


Figure – J Dyer

In a typical rocket test the mass flow, chamber pressure and throat area are relatively easy to measure with commonly available instrumentation allowing C* to be determined indirectly. The chamber temperature is very hard to measure. The C* efficiency is defined as

$$\eta_{C^*} \equiv \frac{\left(\dfrac{P_{t2}A^*}{\dot{m}}\right)_{measured}}{\left(\dfrac{P_{t2}A^*}{\dot{m}}\right)_{ideal}} \tag{7.44}$$

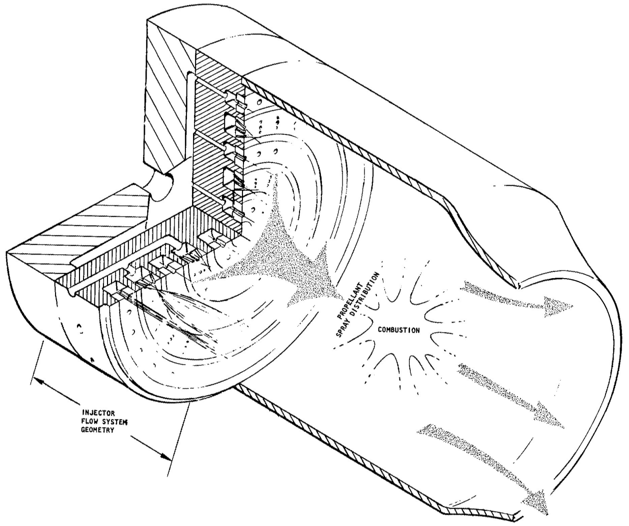
For the same mass flow and area the C* efficiency is determined by the chamber pressure.

$$\eta_{C^*} \equiv \frac{P_{t2_{measured}}}{P_{t2_{ideal}}} \tag{7.45}$$

The ideal chamber pressure is usually determined from an equilibrium combustion calculation.

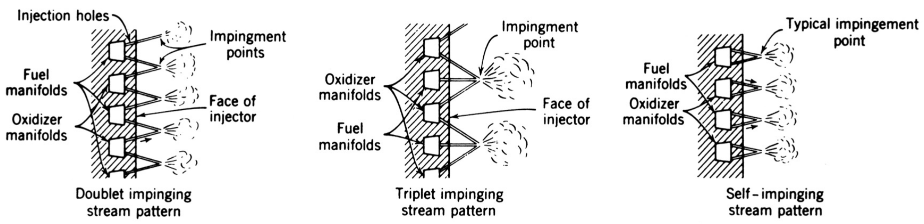
L*

The length of the combustion chamber is determined by the distance needed for the propellants to atomize, mix and fully react in order to release all their chemical energy before exiting the combustion chamber.



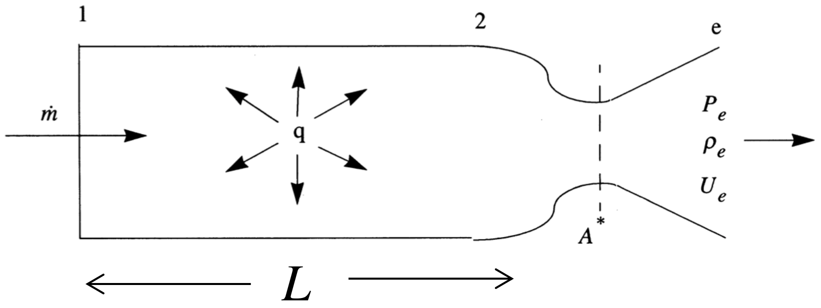
injector flow system

From Sutton: Rocket Propulsion Elements



Residence time t_r

Whether or not complete atomization, mixing and combustion is achieved is determined by the time that the propellants spend in the combustion chamber; the residence time.



$$t_r \equiv \frac{\text{mass of gas in the combustion chamber}}{\text{mass flow rate through the combustion chamber}} = \frac{M_g}{\dot{m}} = \frac{\rho_g V}{\rho^* U^* A^*} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{P_{t2}V}{\left(\frac{R_u}{M_w}T_{t2}\right)} \frac{\left(\gamma \frac{R_u}{M_w}T_{t2}\right)^{1/2}}{\gamma P_{t2} A^* f(M)}$$

$$t_r = \frac{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}{\left(\gamma \frac{R_u}{M_w}T_{t2}\right)^{1/2}} \frac{V}{A^*} = \frac{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}{\left(\gamma \frac{R_u}{M_w}T_{t2}\right)^{1/2}} L^*$$

$$L^* \equiv \frac{V}{A^*} = \frac{V}{A}\left(\frac{A}{A^*}\right) \cong 3\frac{V}{A} \cong 3L$$

Propellant Combination	L*, cm
Nitric acid/hydrazine-base fuel	76-89
Nitrogen tetroxide/hydrazine-base fuel	76-89
Hydrogen peroxide/RP-1 (including catalyst bed)	152-178
Liquid oxygen/RP-1	102-127
Liquid oxygen/ammonia	76-102
Liquid oxygen/liquid hydrogen (GH_2 injection)	56-71
Liquid oxygen/liquid hydrogen (LH_2 injection)	76-102
Liquid fluorine/liquid hydrogen (GH_2 injection)	56-66
Liquid fluorine/liquid hydrogen (LH_2 injection)	64-76
Liquid fluorine/hydrazine	61-71
Chlorine trifluoride/hydrazine-base fuel	51-89

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On density effects and large structure in turbulent mixing layers

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(Received 15 January 1974)

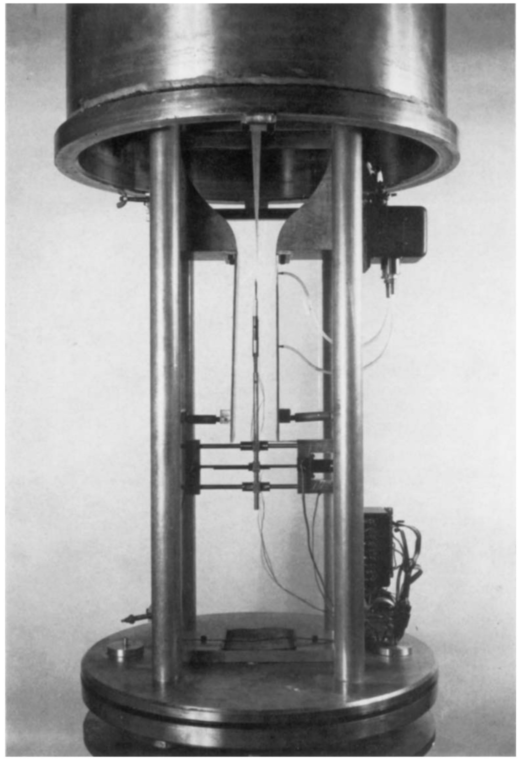


FIGURE 2. Mixing layer apparatus. The wedge-shaped splitter plate separates the two half-nozzles. A probe is mounted in the test section from a horizontal traverse. The pressure cylinder in the upper part of the photo can be pulled down over the seal on the circular plate in the lower part.

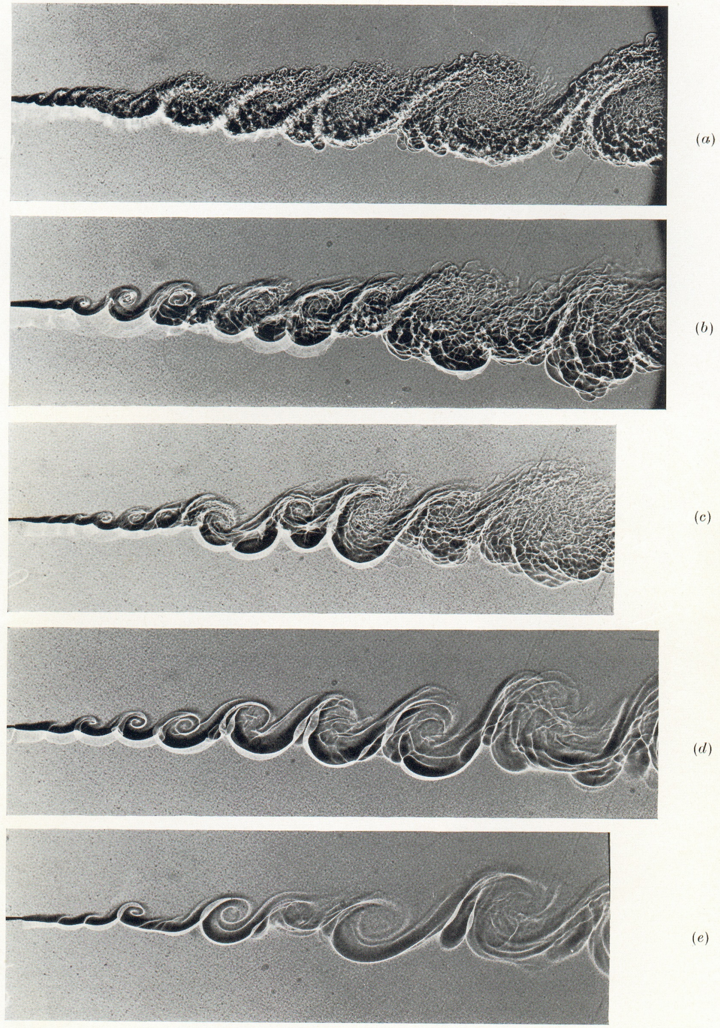
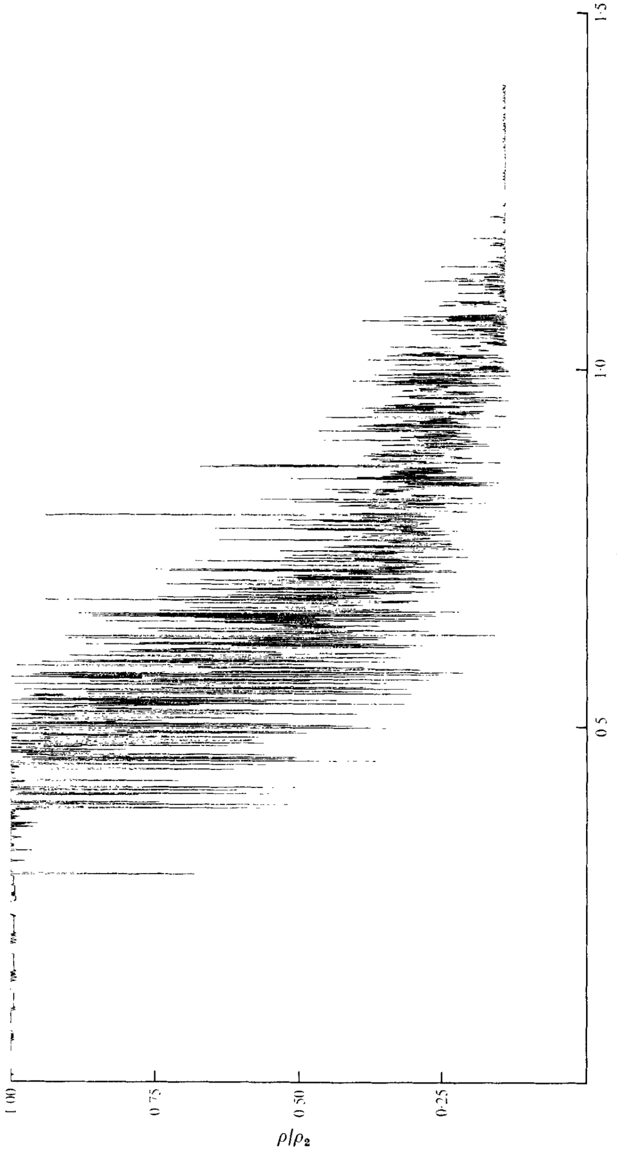


FIGURE 20. Effects of Reynolds number. Mixing layers between helium (upper) and nitrogen (lower) with $\rho_2 U_2^2 = \rho_1 U_1^2$. (*a*) Reynolds number is proportional to 8 × 10 (pressure = 8 atm, $U_1 = 10\,\mathrm{m\,s^{-1}}$); (*b*) 8 × 5; (*c*) 4 × 10; (*d*) 4 × 5; (*e*) 2 × 10.



7.8 The Moore - Tsiolkovsky rocket equation

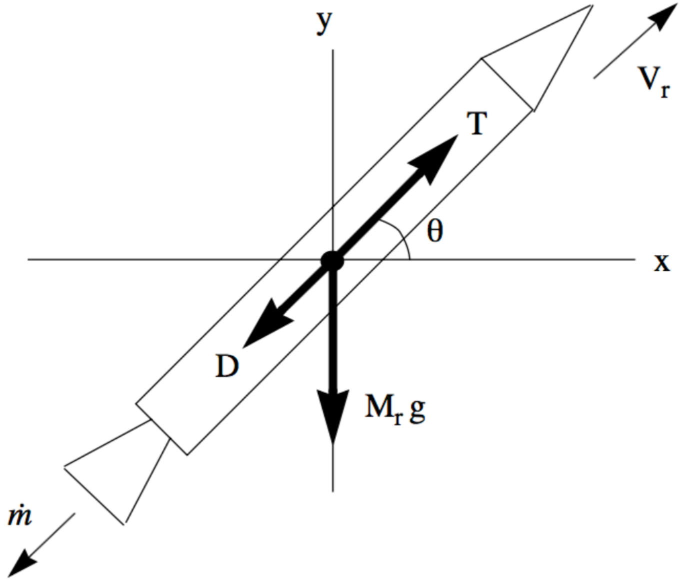
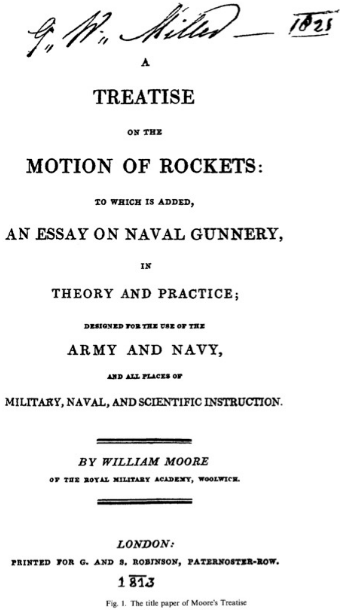


Figure 7.6: *Rocket free body diagram.*

$T =$ *vehicle thrust*
$D =$ *vehicle aerodynamic drag*
$V_r =$ *vehicle velocity*
$\theta =$ *angle with respect to the horizontal*
$\dot{m} =$ *nozzle mass flow*
$M_r =$ *vehicle mass*
$g =$ *gravitational acceleration*



A

TREATISE

ON THE

MOTION OF ROCKETS:

TO WHICH IS ADDED,

AN ESSAY ON NAVAL GUNNERY,

IN

THEORY AND PRACTICE;

DESIGNED FOR THE USE OF THE

ARMY AND NAVY,

AND ALL PLACES OF

MILITARY, NAVAL, AND SCIENTIFIC INSTRUCTION.

BY WILLIAM MOORE

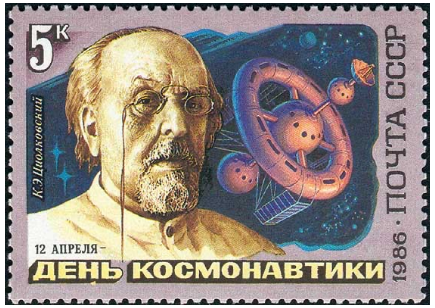
OF THE ROYAL MILITARY ACADEMY, WOOLWICH.

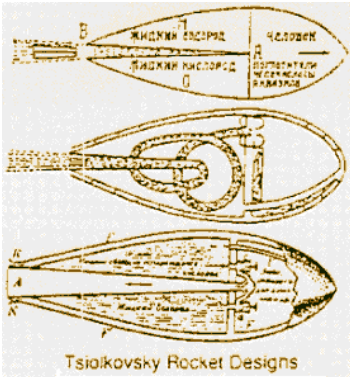
LONDON:

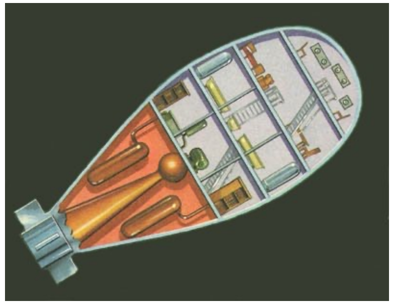
PRINTED FOR G. AND S. ROBINSON, PATERNOSTER-ROW.

1813

Fig. 1. The title paper of Moore's Treatise







Balance of forces along the direction of flight

$$M_r \frac{dV_r}{dt} = T - M_r g Sin(\theta) - D \qquad (7.56)$$

Substitute the thrust equation expressed in terms of the effective exhaust velocity.

$C > 0$

$$M_r \frac{dV_r}{dt} = -C \frac{dM_r}{dt} - M_r g Sin(\theta) - D \qquad (7.57)$$

Divide through by the current mass of the vehicle.

$$\frac{dV_r}{dt} = -C \frac{d(\ln M_r)}{dt} - g Sin(\theta) - \frac{D}{M_r} \qquad (7.58)$$

Let.

$$M_{ri} = initial\ mass\ at\ t = 0$$
$$M_{rf} = final\ mass\ at\ t = t_b \qquad (7.59)$$
$$t_b = time\ of\ burnout.$$

The velocity change of the vehicle is.

$$\Delta V_r = V_{rb} - V_{r0} = \Delta V_r|_{ideal} - \Delta V_r|_{gravitational} - \Delta V_r|_{drag} \qquad (7.60)$$

where

$$\Delta V_r|_{gravitational} = \int_0^{t_b} gSin\left(\theta\right)dt$$
$$(7.61)$$
$$\Delta V_r|_{drag} = \int_0^{t_b} \left(\frac{D}{M_r}\right)dt.$$

To minimize drag losses the rocket should be long and slender.

$$\Delta V_r|_{drag} = \int_0^{t_b} \left(\frac{D}{M_r}\right) dt = \int_0^{t_b} \frac{1}{2} \frac{\rho {V_r}^2 A C_D}{M_{ri}} \left(\frac{M_{ri}}{M_r}\right) dt = \frac{A}{2M_{ri}} \int_0^{t_b} \left(\rho {V_r}^2 C_D\right) \left(\frac{M_{ri}}{M_r}\right) dt \qquad (7.63)$$

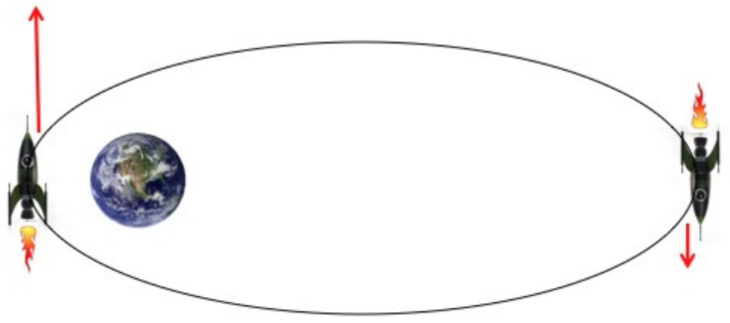
$$M_{ri} \cong density_{rocket} \times Volume_{rocket}$$

$$\Delta V_{rocket}|_{drag} \cong \frac{FrontalArea_{rocket}}{2density_{rocket} Volume_{rocket}} \int_0^{t_b} \left(\rho {V_{rocket}}^2 C_D\right) \left(\frac{M_{rocket_i}}{M_{rocket}}\right) dt \sim \frac{1}{Length_{rocket}} \qquad (7.64)$$

<u>The Oberth effect</u>

When adding energy to a spacecraft in a gravitational field the largest energy increase for a given mass of propellant burned occurs when the spacecraft is at its highest speed.

$$\Delta V_r\big|_{ideal} = C \ln\left(\frac{M_{ri}}{M_{r_f}}\right)$$



Thrust $T = M_r \dfrac{dV_r}{dt} = -C\dfrac{dM_r}{dt}$

Power applied to the vehicle in the center-of-mass frame

$$P = T\,V_r = -C\frac{dM_r}{dt}V_r$$

7.10 The thrust coefficient

$$C_F = \frac{T}{P_{t2}A^*} = \frac{\dot{m}C}{P_{t2}A^*} = \frac{C}{C^*} = \frac{\dot{m}U_e + \left(P_e - P_0\right)A_e}{\dfrac{\dot{m}}{\gamma}\left(\dfrac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\sqrt{\gamma\dfrac{R_u}{M_w}T_{t2}}} \qquad (7.65)$$

This can be expressed in terms of the exit Mach number and ambient pressure.

$$C_F = \frac{T}{P_{t2}A^*} = \frac{C}{C^*} = \frac{\left(\gamma M_e^2 + 1 - \dfrac{P_0}{P_e}\right)}{\left(\dfrac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} M_e\left(1+\dfrac{\gamma-1}{2}M_e^2\right)^{1/2}} \qquad (7.66)$$

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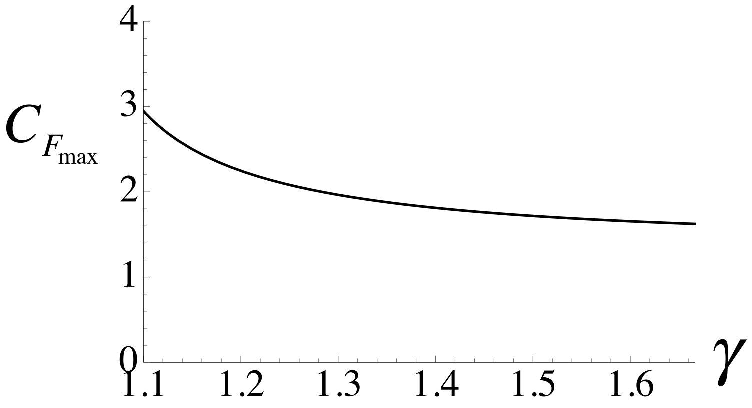
For a rocket operating with a very large expansion ratio to vacuum the exit Mach number becomes large and the thrust coefficient has an upper limit.

$$C_{F_{max}} = \frac{\gamma}{\left(\frac{\gamma-1}{2}\right)^{1/2}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \qquad (7.67)$$



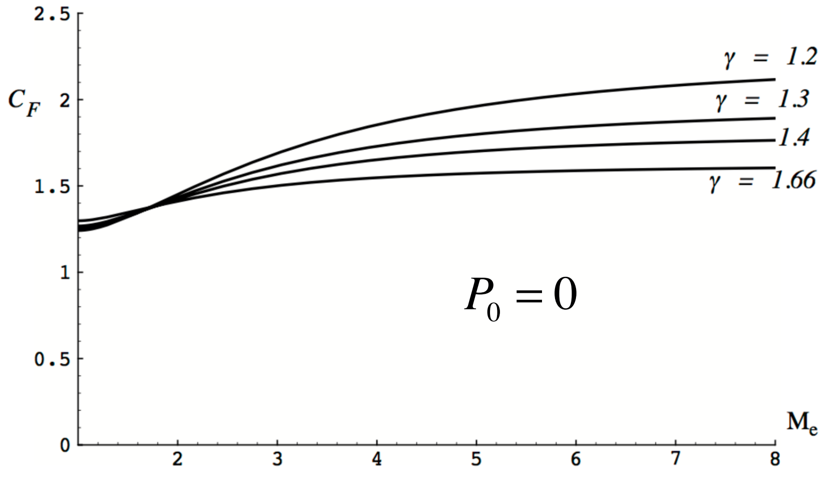


Figure 7.7: *Thrust coefficient versus Mach number.*

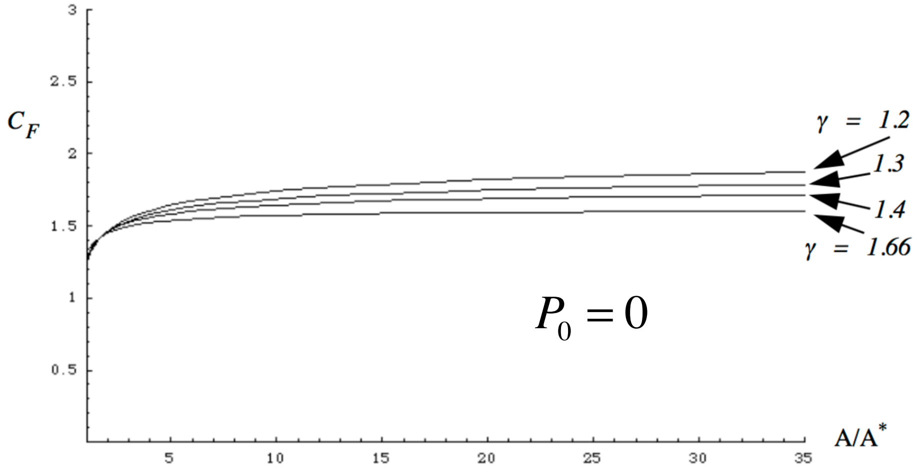
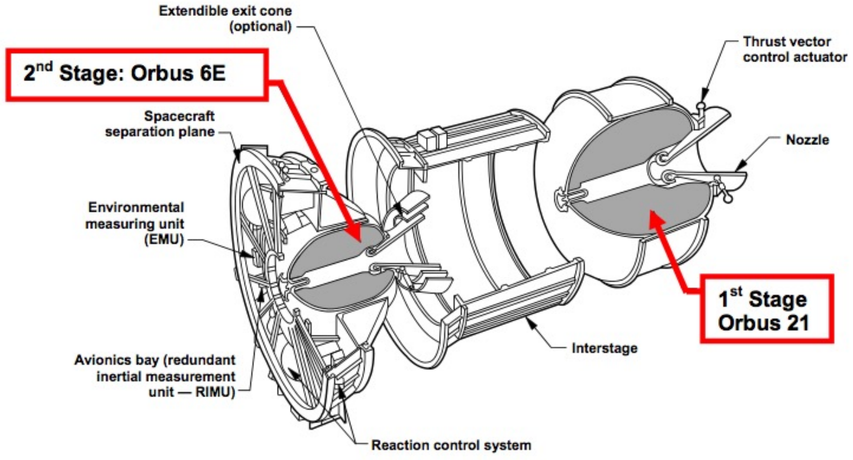


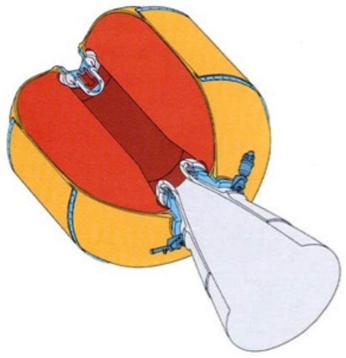
Figure 7.8: *Thrust coefficient versus area ratio.*

Manufacturers will go to great lengths to achieve improvements of even a few seconds of specific impulse.

Boeing – CSD Inertial Upper Stage

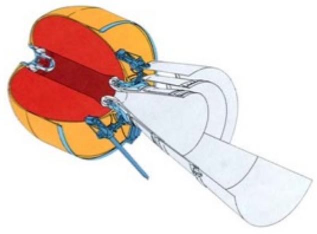


Air Force/NASA IUS, built by Boeing, a 2-Stage Space Vehicle using CSD's Orbus 21 and Orbus 6E Solid Propellant Rockets. It was Configured to Fly off both the Shuttle and Titan Launch Vehicles



Orbus 21: IUS 1st Stage

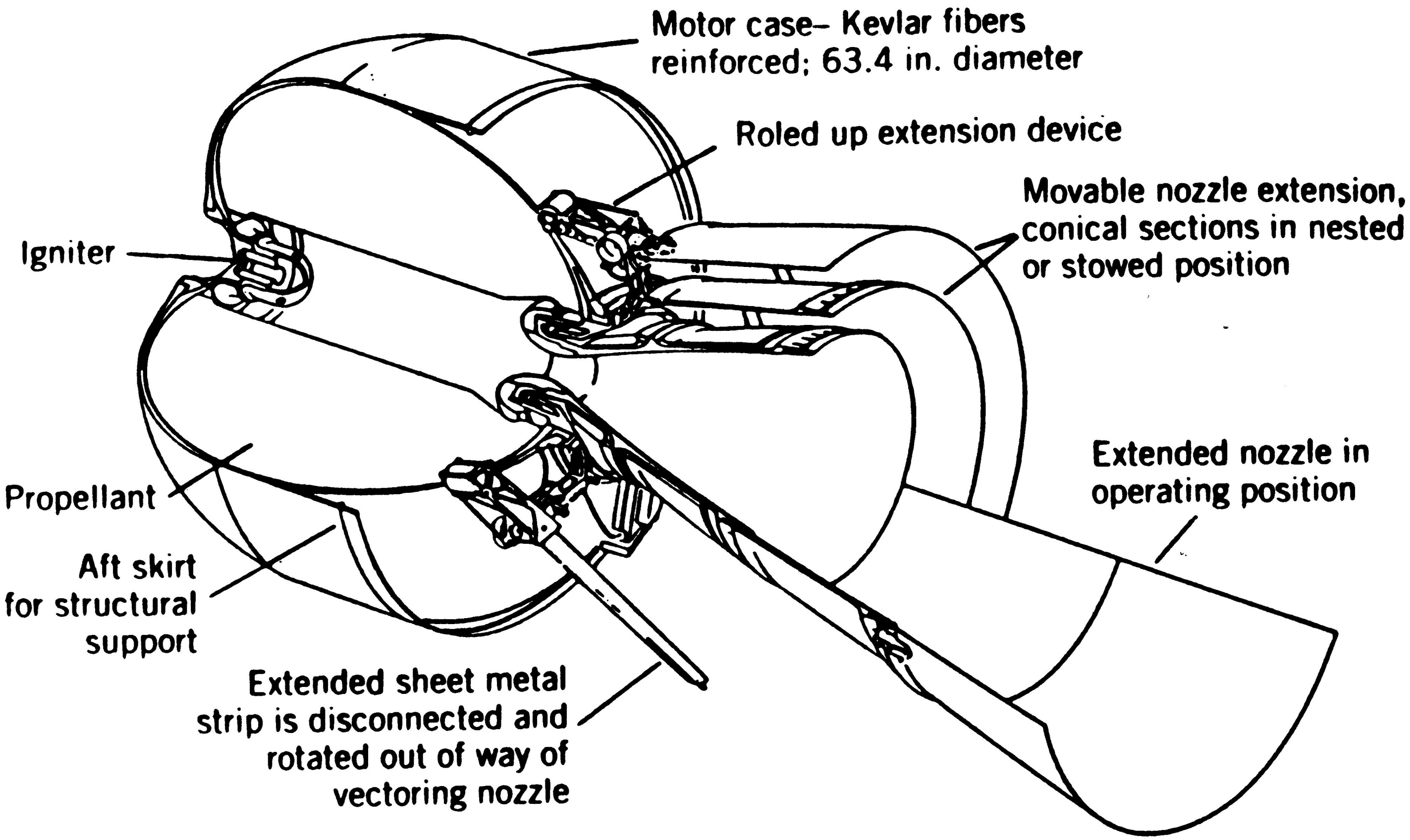
Diameter = 92-in
Wp = 21,400-lb



Orbus 6E: IUS 2nd Stage

Diameter = 63-in
Wp = 6,000-lb

Boeing inertial upper stage (IUS) with extensible vectored nozzle.
Nozzle area ratio could change from 49.3 to 181 increasing specific impulse by 14 seconds while allowing the motor to fit within a smaller length when the nozzle was stowed. The extensible nozzle added $1M to the cost of the motor.



7.11 Problems

Problem 1 - A monopropellant thruster using Argon gas at 100 $psia$ and 1500 K exhausts through a large area ratio convergent-divergent nozzle to the vacuum of space. Determine the energy per unit mass of a parcel of gas at three locations: in the plenum, at the nozzle throat, and at the end of the expansion where the gas pressure approaches vacuum. What mechanism is responsible for the change of energy from one position to the next? How does your answer change if the gas is changed to Helium?

Problem 2 - The designer of a spacecraft maneuvering system needs to choose between Argon (atomic weight 40) and Helium (atomic weight 4) as propellants for a monopropellant thruster. The gas pressure and temperature in the propellant tank are $5 \times 10^6\ N/m^2$ and 300 K respectively. The propellant tank volume is 1.0 m^3 and the empty mass of the vehicle is 10 kg.

1) Which propellant gas will give the largest velocity change to the vehicle? Estimate the vehicle velocity change for each gas?

2) Suppose the vehicle mass is 1000 kg, which propellant would deliver the largest velocity change?

Problem 3 - Consider two different systems used for space propulsion. System A uses propellants with an average density of 2 gm/cm^3 and specific impulse of 200 seconds while system B uses propellants with an average density of 1 gm/cm^3 and specific impulse 300 seconds. The ideal velocity increment generated by either system is given by

$$\Delta V = I_{sp} g_0 \ln \left(\frac{m_{initial}}{m_{final}} \right) \tag{7.68}$$

where $g_0 = 9.8\ m/\sec^2$. Two missions are under consideration.

1) Mission I involves maneuvering of a large satellite where the satellite empty mass (mfinal) is 2000 kg and the required velocity increment is 100 m/sec.

2) Mission II involves a deep space mission where the vehicle empty mass (m_{final}) is 200 kg and the required velocity increment is 6000 $m/\sec$.

The design requirement in both cases is to keep the tank volume required for the propellant as small as possible. Which propellant choice is best for each mission?

Problem 4 - Recently one of the popular toys being sold was called a stomp rocket. The launcher consists of a flexible plastic bladder connected to a 1.5 cm diameter rigid plastic tube. The rocket is a slightly larger diameter rigid plastic tube, closed at the top end, about 20 cm long. The rocket weighs about 10 gm. The rocket slips over the tube as shown in Figure 7.9.

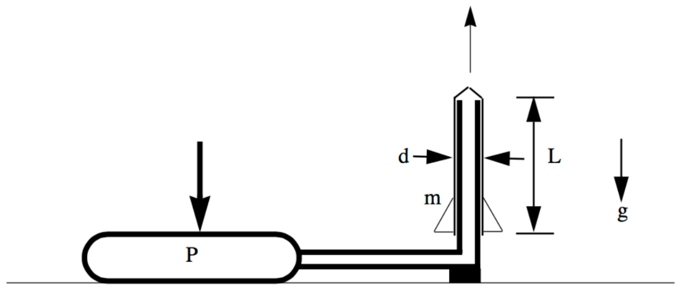


Figure 7.9: *Stomp rocket toy.*

Jumping on the bladder pressurizes the air inside and launches the rocket to a height which the manufacturer claims can exceed 50 m. The area of the bladder in contact with the ground is approximately 100 cm^2. Use basic principles of mechanics to roughly estimate how much a child would have to weigh to be able to achieve this height.

Problem 5 - Consider a class of monopropellant thrusters based on the use of the noble gases including Helium ($M_w = 4$), Neon ($M_w = 20$), Argon ($M_w = 40$), Krypton, ($M_w = 84$) and Xenon ($M_w = 131$). Radon ($M_w = 222$) is excluded because of its radioactivity. The thruster is comprised of a tank that exhausts through a simple convergent nozzle to the vacuum of space. Onboard heaters are used to maintain the gas in the tank at a constant stagnation temperature T_{t2} as it is exhausted.

1) The thrust is often expressed in terms of an effective exhaust velocity $T = \dot{m}C$. Show that the effective exhaust velocity of this system can be expressed as

$$C = \left(\frac{2(\gamma+1)}{\gamma}\left(\frac{R_u}{M_w}\right)T_{t2}\right)^{1/2}. \tag{7.69}$$

2) The mass of propellant contained in the tank is

$$M_{propellant} = \frac{P_{t2_{initial}} V_{\tan k} M_w}{R_u T_{t2}}. \tag{7.70}$$

The initial tank pressure is some rated value (a do-not-exceed pressure) independent of the type of gas used. The designer would like to choose the propellant gas so that the velocity increment produced by the propulsion system ΔV is as large as possible for fixed tank volume, initial pressure and gas temperature. The problem is to decide whether to choose a gas with low M_w, thus achieving a high value of C but low propellant mass, or a gas with high M_w reducing C but increasing propellant mass. By mixing two or more gases, any mean atomic mass between 4 and 131 can be selected by the designer. Note that γ is the same regardless of what gas or mixture of gases is used.

Show that the maximum ΔV occurs when the ratio $M_{propellant}/M_{structure}$ is approximately 4 (actually 3.922). In other words, once the tank volume, pressure and temperature are determined and the vehicle empty mass is known, show that for maximum ΔV the gas should be selected to have a mean atomic weight M_w such that

$$\frac{P_{t2_{initial}} V_{\tan k} M_w}{R_u T_{t2} M_{structure}} = 3.922. \tag{7.71}$$

Problem 6 - The space shuttle main engine has a nozzle throat diameter of $0.26\,m$ a nozzle area ratio of 77.5 and produces $1,859,264\,N$ thrust at lift-off from Cape Canaveral. Determine the engine thrust when it reaches the vacuum of space.