Elastic Compression of Spheres and Cylinders at Point and Line Contact

By M. J. Puttock and E. G. Thwaite

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By M.J. Puttock* and E.G. Thwaite*

Summary

The purpose of this paper is primarily to present in a convenient form the formulae and data for the calculation of the compression effects which occur in the measurement and use of spheres and cylinders in dimensional metrology.

Only Hertzian compression effects are considered in the present paper and these assume that the surfaces in contact are perfectly smooth, that the elastic limits of the materials are not exceeded, that the materials are homogeneous, and that there are no frictional forces within the contact area. These conditions are closely met with materials and applied forces normally encountered in precise dimensional metrology, and with the *surfaces finely lapped.*

In the case of surfaces that are not finely lapped the actual compression effects may differ by up to 10% from those calculated using the formulae in this paper. Contributory factors include frictional forces and microstructure variations in the surface leading to variations in elastic modulii. Berndt (1928) has derived modified formulae to take into account frictional forces arising from non-smooth surfaces and these formulae, in general, lead to compression effects differing from those in this paper by approximately 5%.

It is considered that the formulae given in this paper are sufficiently precise for all practical purposes in precise dimensional metrology.

This paper is in two parts. Part 1 is a series of data sheets giving the appropriate formulae for various specific cases, together with appropriate tables and graphs. Part 2 gives the mathematical derivation of the formulae in a consistent notation and is primarily intended for students with an interest in the subject.

Where the formulae have been partially evaluated for steel the elastic constants used have been those for 1% carbon steel.

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SYMBOLS

			Units	
			mm	in
α	=	the total elastic compression at the point or line of contact of two bodies, measured along the line of the applied force		
P	=	total applied force	gf	lbf
D	=	diameter of body	mm	in
E	=	Young's modulus of material of body	gf/mm^2	lbf/in^2
σ	=	Poisson's ratio = $E/2G - 1$		
G	=	modulus of rigidity of material of body	gf/mm^2	lbf/in^2
V	=	$(1 - \sigma^2)/\pi E$	mm^2/gf	in^2/lbf
Q	=	$\frac{3}{4}(V_1 + V_2)$ for bodies of different materials	mm^2/gf	in^2/lbf
	=	$\frac{3}{2} V$ for bodies of the same material		
e	=	eccentricity of ellipse of contact		
	=	$(1 - b^2/a^2)^{1/2}$		

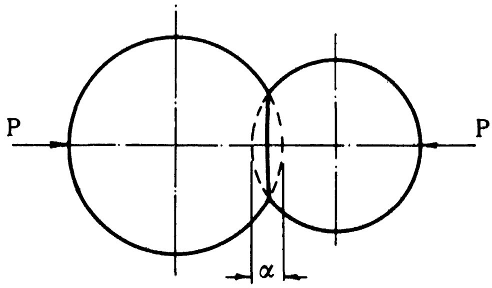
K and E* are the complete elliptic integrals of the first and second class respectively with modulus e

*Not to be confused with E, the Young's modulus.

PART 1

COMPRESSION FORMULAE

Case 1. Two Spheres in Contact



The suffixes 1 and 2 relate to spheres 1 and 2 respectively.

General Case

$$\alpha = \frac{(3\pi)^{2/3}}{2} \,.\, P^{2/3} \,.\, (V_1 + V_2)^{2/3} \,.\, (\frac{1}{D_1} + \frac{1}{D_2})^{1/3}.$$

Spheres of Same Material

$$\alpha = (\frac{9}{2})^{1/3} \,.\, (\frac{1 - \sigma^2}{E})^{2/3} \,.\, P^{2/3} \,.\, (\frac{1}{D_1} + \frac{1}{D_2})^{1/3}.$$

Spheres Both of Steel

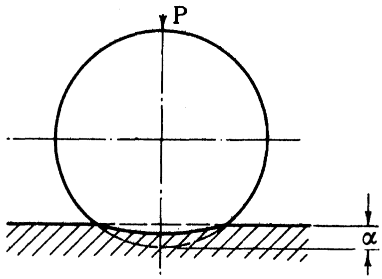
Metric Units: P in gf, D in mm

$$\alpha = 0.000\ 020 \,.\, P^{2/3} \,.\, (\frac{1}{D_1} + \frac{1}{D_2})^{1/3} \text{ mm.}$$

Inch Units: P in lbf, D in inch

$$\alpha = 0.000\ 016 \,.\, P^{2/3} \,.\, (\frac{1}{D_1} + \frac{1}{D_2})^{1/3} \text{ inch.}$$

Case 2. Sphere in Contact with a Plane



General Case

$$\alpha = \frac{(3\pi)^{2/3}}{2} \,.\, P^{2/3} \,.\, (V_1 + V_2)^{2/3} \,.\, (\frac{1}{D})^{1/3}.$$

Sphere and Plane of Same Material

$$\alpha = (\frac{9}{2})^{1/3} \,.\, (\frac{1 - \sigma^2}{E})^{2/3} \,.\, P^{2/3} \,.\, (\frac{1}{D})^{1/3}.$$

Sphere and Plane Both of Steel

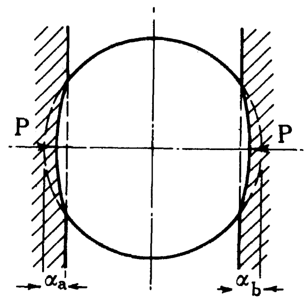
Metric Units: P in gf, D in mm

$$\alpha = 0.000\ 020 \,.\, P^{2/3} \,.\, (\frac{1}{D})^{1/3} \text{ mm.}$$

Inch Units: P in lbf, D in inch

$$\alpha = 0.000\ 016 \,.\, P^{2/3} \,.\, (\frac{1}{D})^{1/3} \text{ inch.}$$

Case 3. Sphere Between Two Parallel Planes



Total compression $\alpha_T = \alpha_a + \alpha_b$ where α_a and α_b are calculated as in Case 2.

If the two planes are of the same material then

$$\alpha_a = \alpha_b$$

and the total compression may be written as

$$\alpha_T = 2\alpha.$$

Sphere and Planes All of Steel

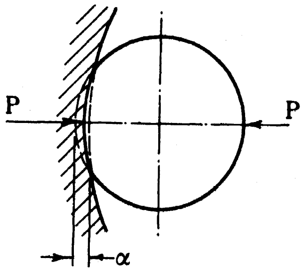
Metric Units: P in gf, D in mm

$$\alpha_T = 0.000\ 040\ .\ P^{2/3}\ .\ (\frac{1}{D})^{1/3}\ \text{mm.}$$

Inch Units: P in lbf, D in inch

$$\alpha_T = 0.000\ 032\ .\ P^{2/3}\ .\ (\frac{1}{D})^{1/3}\ \text{inch.}$$

Case 4. Sphere in Contact with an Internal Spherical Surface



Let diameter of internal spherical surface $= D_1$,
diameter of smaller sphere $= D_2$

General Case

$$\alpha = \frac{(3\pi)^{2/3}}{2} \cdot P^{2/3} \cdot (V_1 + V_2)^{2/3} \cdot \left(\frac{1}{D_2} - \frac{1}{D_1}\right)^{1/3}.$$

Spheres of Same Material

$$\alpha = \left(\frac{9}{2}\right)^{1/3} \cdot \left(\frac{1 - \sigma^2}{E}\right)^{2/3} \cdot P^{2/3} \cdot \left(\frac{1}{D_2} - \frac{1}{D_1}\right)^{1/3}.$$

Spheres Both of Steel

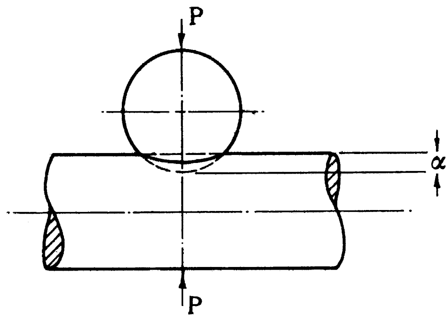
Metric Units: P in gf, D in mm

$$\alpha = 0.000\,020 \cdot P^{2/3} \cdot \left(\frac{1}{D_2} - \frac{1}{D_1}\right)^{1/3} \text{ mm.}$$

Inch Units: P in lbf, D in inch

$$\alpha = 0.000\,016 \cdot P^{2/3} \cdot \left(\frac{1}{D_2} - \frac{1}{D_1}\right)^{1/3} \text{ inch.}$$

Case 5. Equal Diameter Cylinders Crossed with Their Axes at Right Angles



General Case

$$\alpha = \frac{(3\pi)^{2/3}}{2} \cdot P^{2/3} \cdot (V_1 + V_2)^{2/3} \cdot (\frac{1}{D})^{1/3}.$$

Cylinders Both of Same Material

$$\alpha = (\frac{9}{2})^{1/3} \cdot (\frac{1 - \sigma^2}{E})^{2/3} \cdot P^{2/3} \cdot (\frac{1}{D})^{1/3}.$$

Cylinders Both of Steel

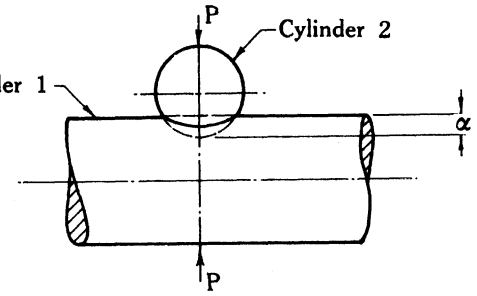
Metric Units: P in gf, D in mm

$$\alpha = 0.000\ 020 \cdot P^{2/3} \cdot (\frac{1}{D})^{1/3} \text{ mm.}$$

Inch Units: P in lbf, D in inch

$$\alpha = 0.000\ 016 \cdot P^{2/3} \cdot (\frac{1}{D})^{1/3} \text{ inch.}$$

Case 6. Unequal Diameter Cylinders Crossed with Their Axes at Right Angles



The suffix 1 refers to the larger diameter cylinder, the suffix 2 to the smaller.

General Case

$$\alpha = 2\mathrm{K} \cdot (P \cdot Q)^{2/3} \cdot \left(\frac{1}{2D_1 \cdot (-\frac{1}{e}\frac{\mathrm{dE}}{de})}\right)^{1/3},$$

where K and $-\frac{1}{e}\frac{\mathrm{dE}}{de}$ are functions of $\frac{A}{B} = \frac{D_2}{D_1}$,

$Q = \frac{3}{4}(V_1 + V_2)$ for dissimilar materials,

and $Q = \frac{3}{2} V$ when cylinders are of the same material.

For any given value of $\frac{A}{B} = \frac{D_2}{D_1}$ in the range 1.00 to 0.000 000 1 the corresponding values of K and $-\frac{1}{e}\frac{\mathrm{dE}}{de}$ may be obtained from Tables 3-6 or Figure 6.

Both Cylinders of Steel

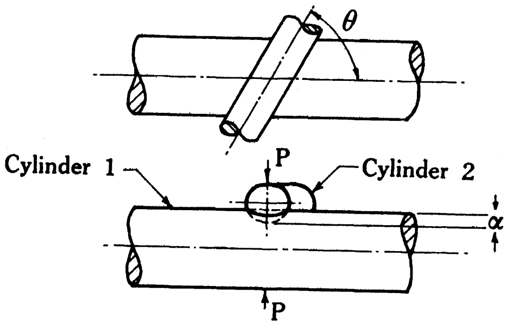
Metric Units: P in gf, D in mm

$$\alpha = 0.000\,015 \cdot \mathrm{K} \cdot P^{2/3} \cdot \left(\frac{1}{2D_1 \cdot (-\frac{1}{e}\frac{\mathrm{dE}}{de})}\right)^{1/3} \text{ mm.}$$

Inch Units: P in lbf, D in inch

$$\alpha = 0.000\,012 \cdot \mathrm{K} \cdot P^{2/3} \cdot \left(\frac{1}{2D_1 \cdot (-\frac{1}{e}\frac{\mathrm{dE}}{de})}\right)^{1/3} \text{ inch.}$$

Case 7. Unequal Diameter Cylinders Crossed with Their Axes at Any Angle



The suffix 1 refers to the larger diameter cylinder, the suffix 2 to the smaller.

Let the axes be inclined at an acute angle θ to one another.

It is first necessary to obtain the ratio A/B by solving the following equations, for A and B.

$$A + B = \frac{1}{D_1} + \frac{1}{D_2},$$

$$(A - B)^2 = \left(\frac{1}{D_1}\right)^2 + \left(\frac{1}{D_2}\right)^2 + \frac{2\cos 2\theta}{D_1 D_2}$$

From the calculated value of A/B the values of K and $-\frac{1}{e}\frac{dE}{de}$ may be obtained from Tables 3-6 or Figure 6.

General Case

$$\alpha = 2K \cdot (P \cdot Q)^{2/3} \cdot \left[\frac{A}{2 \cdot \left(-\frac{1}{e}\frac{dE}{de}\right)}\right]^{1/3},$$

where $Q = \frac{3}{4}(V_1 + V_2)$ for dissimilar materials

and $Q = \frac{3}{2}V$ when cylinders are of the same material.

Both Cylinders of Steel

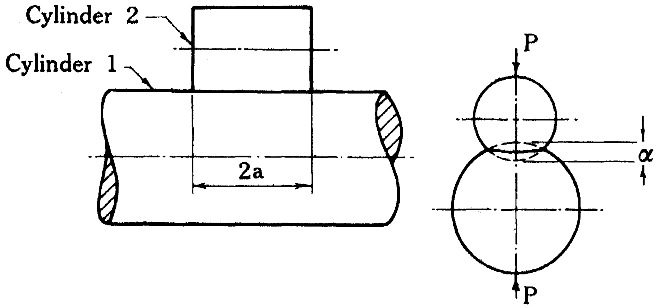
Metric Units: P in gf, D in mm

$$\alpha = 0.000\,015 \cdot K \cdot P^{2/3} \cdot \left[\frac{A}{2 \cdot \left(-\frac{1}{e}\frac{dE}{de}\right)}\right]^{1/3} \text{ mm.}$$

Inch Units: P in lbf, D in inch

$$\alpha = 0.000\,012 \cdot K \cdot P^{2/3} \cdot \left[\frac{A}{2 \cdot \left(-\frac{1}{e}\frac{dE}{de}\right)}\right]^{1/3} \text{ inch.}$$

Case 8. Two Cylinders in Contact with Axes Parallel



General Case

$$\alpha = \bar{P} \cdot (V_1 + V_2) \cdot \left[1 + \ln\left\{\frac{8a^2}{(V_1 + V_2)\bar{P}} \cdot \left(\frac{1}{D_1} + \frac{1}{D_2}\right)\right\}\right].$$

Same Materials

$$\alpha = 2\bar{P} \cdot V \cdot \left[1 + \ln\left\{\frac{4a^2}{V \cdot \bar{P}} \cdot \left(\frac{1}{D_1} + \frac{1}{D_2}\right)\right\}\right].$$

Both Diameters Equal

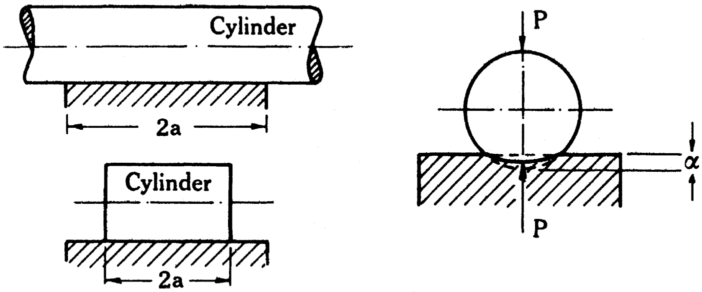
$$\alpha = \bar{P} \cdot (V_1 + V_2) \cdot \left[1 + \ln\left\{\frac{16a^2}{(V_1 + V_2) \cdot \bar{P} \cdot D}\right\}\right].$$

Same Materials

$$\alpha = 2\bar{P} \cdot V \cdot \left[1 + \ln\left\{\frac{8a^2}{V \cdot \bar{P} \cdot D}\right\}\right].$$

$\bar{P} = P/2a$ = force per unit length.
ln = natural logarithm.
$2a$ = length of contact.

Case 9. Cylinder in Contact with a Plane



General Case

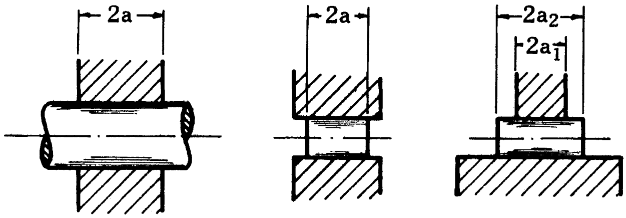
$$\alpha = \bar{P} \cdot (V_1 + V_2) \cdot \left[1 + \ln\left\{\frac{8a^2}{(V_1 + V_2) \cdot \bar{P} \cdot D}\right\}\right].$$

Same Materials

$$\alpha = 2\bar{P} \cdot V \cdot \left[1 + \ln\left\{\frac{4a^2}{V \cdot \bar{P} \cdot D}\right\}\right].$$

$\bar{P} = P/2a$ = force per unit length.
ln = natural logarithm.
$2a$ = length of contact.

Case 10. Cylinder Between Two Parallel Planes



(a) (b) (c)

The calculations are as for Case 9.

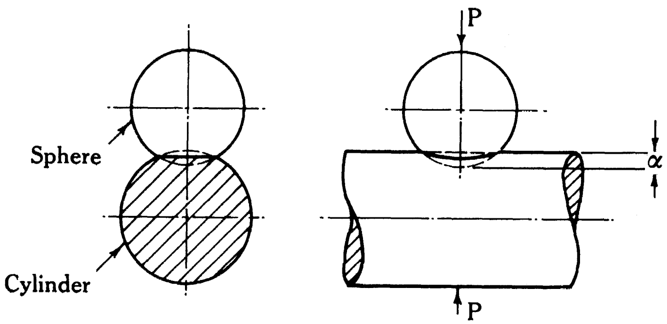
If the lengths of the lines of contact are the same (as in (*a*) and (*b*)), and the material of the two planes is also the same, then the total compression is twice the compression for a single contact, i.e.

$$\alpha_T = 2\alpha.$$

If the lengths of the lines of contact are not equal (such as in (*c*)), *or* the materials of the two planes are different, then the compressions at each contact must be calculated independently, i.e.

$$\alpha_T = \alpha_1 + \alpha_2.$$

Case 11. Sphere in Contact with a Cylinder (External)



Let diameter of sphere $= D_1$
and diameter of cylinder $= D_2$

First obtain the ratio A/B and the value of $1/A$ from the following equations:

$$\frac{A}{B} = \frac{\frac{1}{D_1}}{\frac{1}{D_1} + \frac{1}{D_2}},$$

$$\frac{1}{A} = D_1.$$

From the calculated value of A/B, obtain the appropriate values of K and $-\frac{1}{e}\frac{dE}{de}$ from Tables 3-6 or Figure 6.

Calculate a from the following equation

$$a^3 = \frac{2QP}{A} \cdot -\frac{1}{e}\frac{dE}{de} \cdot$$

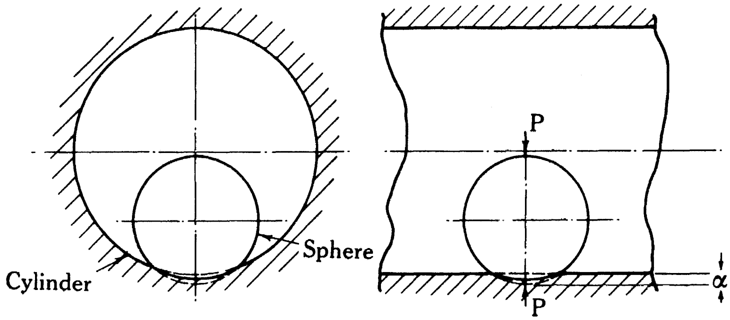
Then calculate the compression α from the equation

$$\alpha = \frac{2QP}{a} \cdot \mathrm{K},$$

where $Q = \frac{3}{4}(V_1 + V_2)$ for dissimilar materials,

$Q = \frac{3}{2} V$ for similar materials.

Case 12. Sphere in Contact with a Cylinder (Internal)



Let diameter of sphere $= D_1$
and diameter of cylinder $= D_2$

First obtain the ratio A/B and the value of $1/A$ from the following equations:

$$\frac{A}{B} = \frac{\frac{1}{D_1} - \frac{1}{D_2}}{\frac{1}{D_1}} ,$$

$$\frac{1}{A} = \frac{1}{\frac{1}{D_1} - \frac{1}{D_2}} .$$

From the calculated value of A/B obtain the appropriate values of K and $-\frac{1}{e}\frac{dE}{de}$ from Tables 3-6 or Figure 6.

Calculate a from the following equation

$$a^3 = \frac{2QP}{A} \cdot -\frac{1}{e}\frac{dE}{de} .$$

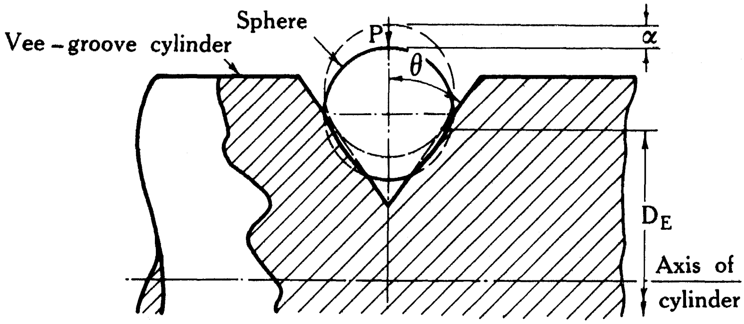
Then calculate the compression α from the equation:

$$\alpha = \frac{2QP}{a} \cdot K,$$

where $Q = \frac{3}{4}(V_1 + V_2)$ for dissimilar materials,

$Q = \frac{3}{2} V$ for similar materials.

Case 13. Sphere in Contact with a Cylindrical Vee Groove, the Vee Groove Being Symmetrical with Respect to a Normal to the Axis of the Cylinder



Let diameter of sphere = D,
diameter of cylinder at point of contact = D_E,
semi-angle of vee groove = θ.

(1) Calculate the value of A/B from

$$\frac{A}{B} = \frac{\frac{1}{D}}{\frac{1}{D} + \frac{1}{D_E \operatorname{cosec} \theta}} .$$

(2) Obtain appropriate values of K and $-\frac{1}{e}\frac{dE}{de}$ from Tables 3-6 or Figure 6.

(3) Calculate a from the equation

$$a^3 = QP \operatorname{cosec} \theta \,.\, D \,.\, -\frac{1}{e}\frac{dE}{de} .$$

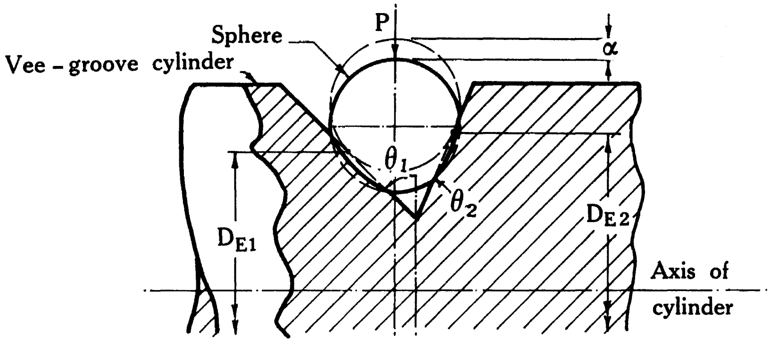
(4) Calculate the total compression α normal to the axis of the cylinder from the equation

$$\alpha = \frac{QP \operatorname{cosec}^2\theta}{a} \,.\, K,$$

where $Q = \frac{3}{4}(V_1 + V_2)$ for dissimilar materials,

and $Q = \frac{3}{2} V$ for sphere and cylindrical vee groove of the same material.

Case 14. Sphere in Contact with a Cylindrical Vee Groove, the Vee Groove Being Asymmetrical with Respect to a Normal to the Axis of the Cylinder



Let diameter of sphere = D,
angles vee groove flanks make with normal to vee cylinder axis = θ_1 and θ_2,
diameters of vee cylinder at points of contact = D_{E1} and D_{E2} respectively.

Initially each contact point must be treated separately.

(1) Calculate the values of A/B from equations:

$$\left(\frac{A}{B}\right)_1 = \frac{\frac{1}{D}}{\frac{1}{D} + \frac{1}{D_{E1}\ \mathrm{cosec}\,\theta_1}}, \quad \left(\frac{A}{B}\right)_2 = \frac{\frac{1}{D}}{\frac{1}{D} + \frac{1}{D_{E2}\ \mathrm{cosec}\,\theta_2}}.$$

(2) Obtain appropriate values of K and $-\frac{1}{e}\frac{dE}{de}$ for each case from Tables 3-6 or Figure 6.

(3) Calculate relevant values of a from the equations:

$$a_1^3 = \frac{2Q\ .\ P\ .\ D}{\tan\theta_2\cos\theta_1 + \sin\theta_1}\ .\ \left(-\frac{1}{e}\frac{dE}{de}\right)_1,$$

$$a_2^3 = \frac{2Q\ .\ P\ .\ D}{\tan\theta_1\cos\theta_2 + \sin\theta_2}\ .\ \left(-\frac{1}{e}\frac{dE}{de}\right)_2.$$

(4) Calculate the relevant compressions normal to the vee groove flanks from the equations:

$$\alpha_1 = \frac{2QP}{a_1(\tan\theta_2 \cos\theta_1 + \sin\theta_1)} \cdot K_1$$

and

$$\alpha_2 = \frac{2QP}{a_2(\tan\theta_1 \cos\theta_2 + \sin\theta_2)} \cdot K_2,$$

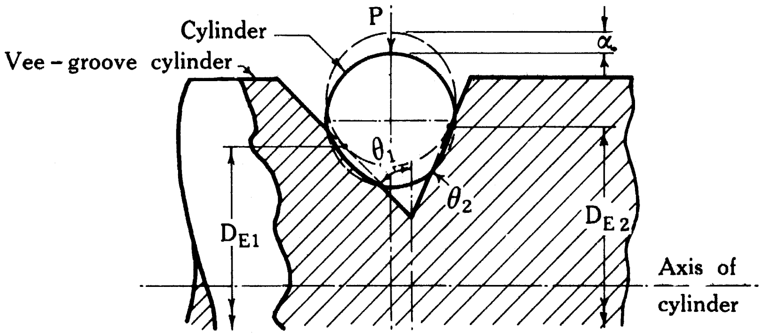
where $Q = \frac{3}{4}(V_1 + V_2)$ for dissimilar materials,

$Q = \frac{3}{2} V$ where both sphere and cylindrical vee groove are of the same material.

(5) Calculate total compression effect α normal to vee cylinder axis from the equation

$$\alpha = (\alpha_1 \cos\theta_2 + \alpha_2 \cos\theta_1)\operatorname{cosec}(\theta_1 + \theta_2).$$

Case 15. Cylinder in Contact with a Cylindrical Vee Groove, the Vee Groove Being Asymmetrical with Respect to a Normal to the Vee Cylinder



Let diameter of cylinder = D,
angles vee groove flanks make with normal to vee cylinder axis = θ_1 and θ_2,
diameters of vee cylinder at points of contact = D_{E1} and D_{E2} respectively.

Initially each contact must be treated separately.

(1) Calculate values of A/B from the equations:

$$\left(\frac{A}{B}\right)_1 = \frac{D}{D_{E1} \operatorname{cosec} \theta_1}, \quad \left(\frac{A}{B}\right)_2 = \frac{D}{D_{E2} \operatorname{cosec} \theta_2}.$$

(2) Obtain appropriate values of K and $-\frac{1}{e}\frac{dE}{de}$ for each case from Tables 3-6 or Figure 6.

(3) Calculate relevant values of a from the equations:

$$a_1^3 = \frac{2QPD_{E1} \operatorname{cosec} \theta_1}{\tan \theta_2 \cos \theta_1 + \sin \theta_1} \cdot \left(-\frac{1}{e}\frac{dE}{de}\right)_1,$$

$$a_2^3 = \frac{2QPD_{E2} \operatorname{cosec} \theta_2}{\tan \theta_1 \cos \theta_2 + \sin \theta_2} \cdot \left(-\frac{1}{e}\frac{dE}{de}\right)_2.$$

(4) Calculate the relevant compressions normal to the vee groove from the equations:

$$\alpha_1 = \frac{2QP}{a_1(\tan\theta_2\cos\theta_1 + \sin\theta_1)} \cdot K_1,$$

$$\alpha_2 = \frac{2QP}{a_2(\tan\theta_1\cos\theta_2 + \sin\theta_2)} \cdot K_2,$$

where $Q = \frac{3}{4}(V_1 + V_2)$ for dissimilar materials,

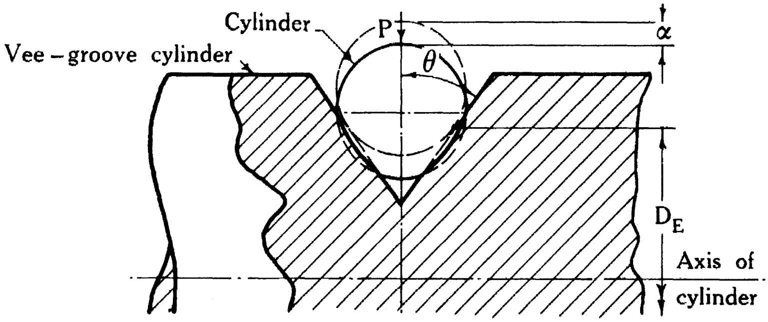
$Q = \frac{3}{2}V$ for both cylinders of same material.

(5) Calculate total compression effect α normal to vee cylinder axis from the equation

$$\alpha = (\alpha_1 \cos\theta_2 + \alpha_2 \cos\theta_1)\operatorname{cosec}(\theta_1 + \theta_2).$$

Note: In the above, the assumption has been made that both $D_{E1} \operatorname{cosec}\theta_1$ and $D_{E2} \operatorname{cosec}\theta_2$ are greater than D; this is so in all practical cases.

Case 16. Cylinder in Contact with a Cylindrical Vee Groove, the Vee Groove Being Symmetrical with Respect to a Normal to the Axis of the Vee Cylinder



Let diameter of cylinder = D,
diameter of grooved cylinder at point of contact = D_E,
semi-angle of vee groove = θ.

(1) Calculate the value of A/B from $\frac{A}{B} = \frac{D}{D_E \operatorname{cosec}\theta}$ and obtain appropriate values of K and $-\frac{1}{e}\frac{dE}{de}$ from Tables 3-6 or Figure 6.

(2) Calculate a from the equation

$$a^3 = QPD_E \operatorname{cosec}^2\theta \;.\; -\frac{1}{e}\frac{dE}{de}.$$

(3) Calculate total compression α normal to axis of vee grooved cylinder from the equation

$$\alpha = \frac{QP \operatorname{cosec}^2\theta}{a} \;.\; K,$$

where $Q = \frac{3}{4}(V_1 + V_2)$ for dissimilar materials,

$Q = \frac{3}{2}V$ where both cylinders are of the same material.

Note: In the above the assumption has been made that $D_E \operatorname{cosec}\theta$ is greater than D; this is so in all practical cases.

PART 2

THEORY

I. INTRODUCTION

The mathematical theory for the general three-dimensional contact problem was first given by Hertz (1881). There is an extensive literature dealing with the contact problem and a review of the Hertzian theory which includes both stress and strain analysis together with a comprehensive bibliography has been published by Lubkin (1962). Among the works of particular interest are those of A.E.H. Love (1892), Prescott (1924), Landau and Lifshitz (1959), Shtaerman (1949), and Lur'e (1964). The work of Shtaerman is a complete treatise on the contact problem.

The following derivations are given in a consistent notation and are sufficiently detailed to be readily followed by students. The theory stems from the general body of elasticity theory dealing with the relation of the displacement at a point on a plane surface due to a pressure at another point. This is the approach given in the classical work of A.E.H. Love (1892) and adopted in a large part of the literature and would seem to be the appropriate treatment for this work.

The two-dimensional line contact problem is in general more difficult theoretically than the three-dimensional one and it is not possible to derive an explicit relation for the two-dimensional case in a direct manner from the three-dimensional. The derivation for the two-dimensional problem, cylinders in contact with their axes parallel, given here has its roots in works by Thomas and Hoersch (1930), Prescott (1924) and E.R. Love (1942). The derivation is to a degree a parallel argument to the three-dimensional case and thus preserves a unity in the theory.

The usefulness of compression formulae depends, of course, on their experimental verification and, while for large forces there is a large body of information available, for forces in the range used in length metrology the data are not so extensive. Reference can be made, however, to the work of Rolt and Grant (1921), Pérard and Maudet (1927), Berndt (1928), Poole (1962), and to brief information in the National Physical Laboratory (Teddington) Annual Reports for 1921 and 1923. Verification in the two-dimensional case, like its theory, presents particular problems, which are mainly due to the high degree of geometric perfection required in the apparatus. Measurements with a resolution of the order 0.003 μm of the compression of a roller on a flat, for the load range 0.05 to 0.4 kgf/mm, recently made at the National Standards Laboratory, Australia, agree within practical limits with the formulae given here for the two-dimensional case.*

*"A Precise Determination of the Compression of a Cylinder in Contact with a Flat Surface" to be published in Journal of Scientific Instruments (Journal of Physics E) 1969 Series 2 Volume 2.

II. GENERAL THEORY

(a) *General*

The assumption is made that the surfaces in contact are perfectly smooth, that the bodies are isotropic and linearly elastic, that the elastic limits of the material are not exceeded, and that there are no frictional forces in action. The finite force keeping them together will then be distributed over the common area of contact. For our purpose, the surfaces of bodies in contact may be assumed to be of the second degree, and the following theory is based on this assumption.

(b) *Geometry of the Unstressed Surface in the Region of Contact*

Suppose that two bodies are in mathematical contact (i.e. unstressed and undeformed) so that the common normal is parallel to the applied force; the common tangent plane is the plane xy and the common normal is the axis z (see Fig. 1).

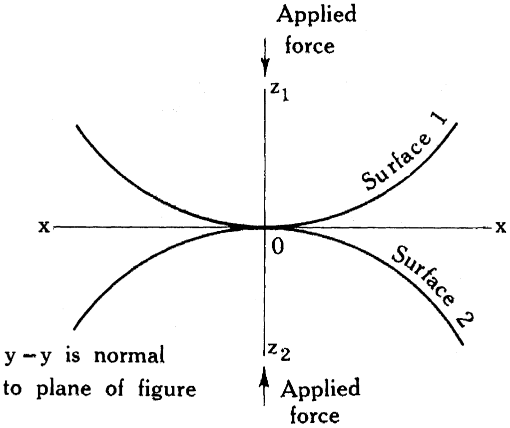


Fig. 1

The general equation for a surface of the second degree is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0. \quad (1)$$

At the origin, $x = 0$, $y = 0$, $z = 0$. Therefore $d = 0$. Differentiating equation (1) with respect to x,

$$2ax + 2cz\frac{\partial z}{\partial x} + 2fy\frac{\partial z}{\partial x} + 2gz + 2gx\frac{\partial z}{\partial x} + 2hy + 2u + 2w\frac{\partial z}{\partial x} = 0. \quad (2)$$

Again, at the origin, $x = 0$, $y = 0$, $z = 0$, $\frac{\partial z}{\partial x} = 0$ (tangent plane). Therefore $u = 0$.

Similarly, by differentiating with respect to y it can be shown that $v = 0$.

The precise equation can therefore be written as

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2wz = 0. \tag{3}$$

To obtain an approximation to this equation which will be adequate for our purpose, we make use of Taylor's series, namely,

$$\begin{aligned} f[(x + \delta x), (y + \delta y)] &= f(x,y) + \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} \\ &+ \frac{1}{2!}\left[\delta x^2 \frac{\partial^2 f}{\partial x^2} + 2\delta x\delta y \frac{\partial^2 f}{\partial x\partial y} + \delta y^2 \frac{\partial f^2}{\partial y^2}\right] \\ &+ \text{higher order terms (neglected).} \end{aligned} \tag{4}$$

Differentiating equation (2) again with respect to x,

$$\begin{aligned} 2a + 2cz \frac{\partial^2 z}{\partial x^2} + 2c\left(\frac{\partial z}{\partial x}\right)^2 + 2fy \frac{\partial^2 z}{\partial x^2} + 2g \frac{\partial z}{\partial x} + 2g \frac{\partial z}{\partial x} \\ + 2gx \frac{\partial^2 z}{\partial x^2} + 2w \frac{\partial^2 z}{\partial x^2} = 0. \end{aligned} \tag{5}$$

Again, at the origin, $x = 0$, $y = 0$, $z = 0$, $\frac{\partial z}{\partial x} = 0$ and, substituting in equation (5), this gives:

$$2a + 2w \frac{\partial z^2}{\partial x^2} = 0,$$

$$\frac{\partial z^2}{\partial x^2} = -\frac{a}{w}.$$

Similarly, if we differentiate equation (1) twice with respect to y,

$$\frac{\partial^2 z}{\partial y^2} = -\frac{b}{w}.$$

Differentiating equation (2) with respect to y,

$$2c\,\frac{\partial z}{\partial x}\,\frac{\partial z}{\partial y} + 2cz\,\frac{\partial^2 z}{\partial y \partial x} + 2f\,\frac{\partial z}{\partial x} + 2fy\,\frac{\partial^2 z}{\partial y \partial x} + 2g\,\frac{\partial z}{\partial y}$$
$$+ 2gx\,\frac{\partial^2 z}{\partial y \partial x} + 2h + 2w\,\frac{\partial^2 z}{\partial y \partial x} = 0. \tag{6}$$

At the origin, $x = 0$, $y = 0$, $z = 0$, $\partial z/\partial y = 0$, and $\partial z/\partial x = 0$ and substituting in equation (6),

$$2h + 2w\,\frac{\partial^2 z}{\partial y \partial x} = 0,$$
$$\frac{\partial^2 z}{\partial y \partial x} = -\frac{h}{w}\,.$$

Substituting now in Taylor's series, equation (4), and regarding z as $f(x,y)$,

$$f[(x + \delta x),\ (y + \delta y)] = z = f(0,0) + x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y}$$
$$+ \frac{1}{2!}\left[\frac{x^2\partial^2 z}{\partial x^2} + \frac{2xy\partial^2 z}{\partial x \partial y} + y\quad\frac{\partial^2 z}{\partial y^2}\right],$$

hence

$$z = \frac{1}{2}\left(-\,\frac{a}{w}\,x^2 - \frac{2hxy}{w} - \frac{by^2}{w}\right)$$

and may then be written as

$$z = Ex^2 + Fy^2 + 2Hxy. \tag{7}$$

If z is constant (i.e. in any given plane parallel to the xy plane), equation (7) is an ellipse with its principal axes rotated with respect to the coordinate axes (see Fig. 2). If now the coordinate axes are aligned with the principal axes the xy term will vanish.

To do this, make the transformation:

$$x = X\cos\theta - Y\sin\theta,$$

$$y = X\sin\theta + Y\cos\theta,$$

where the angle θ is given by $\tan 2\theta = 2H/(E - F)$.

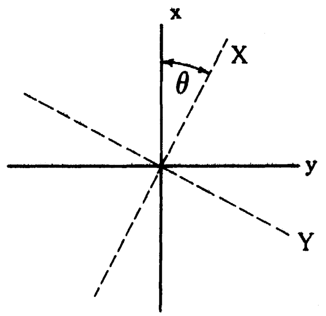


Fig. 2

Substituting in equation (7):

$$
\begin{aligned}
z &= E(X\cos\theta - Y\sin\theta)^2 + F(X\sin\theta + Y\cos\theta)^2 \\
&\qquad + 2H(X\cos\theta - Y\sin\theta) \times (X\sin\theta + Y\cos\theta) \\
&= X^2(E\cos^2\theta + F\sin^2\theta + 2H\cos\theta\sin\theta) \\
&\qquad + Y^2(E\sin^2\theta + F\cos^2\theta - 2H\cos\theta\sin\theta) \\
&\qquad + XY(-2E\sin\theta\cos\theta + 2F\sin\theta\cos\theta \\
&\qquad + 2H\cos^2\theta - 2H\sin^2\theta) \\
&= \text{Constant} \times X^2 + \text{Constant} \times Y^2 \\
&\qquad + XY(-(E - F)\sin 2\theta + 2H\cos 2\theta).
\end{aligned}
$$

When $\tan 2\theta = 2H/(E - F)$, the xy term vanishes and the equation with respect to the new coordinate axes is

$$z = \text{Constant} \times X^2 + \text{Constant} \times Y^2. \tag{8}$$

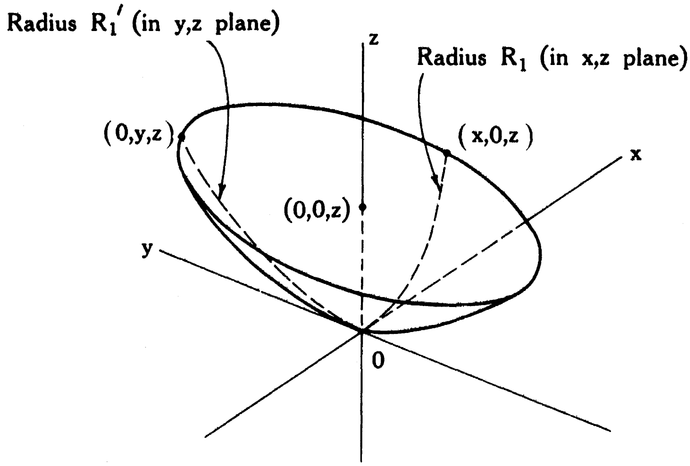


Fig. 3

It is now necessary to determine these constants in equation (8) in terms of a dimension or dimensions of the respective bodies.

Let R_1 and R_1' be the principle radii of curvature of one of the bodies (see Fig. 3); writing equation (8) in the form

$$z = Ax^2 + By^2. \tag{9}$$

Then, in the plane $y = 0$, we have $Ax^2 = z$. Assuming circular curvature in the plane $y = 0$, which is permissible in view of the magnitude of z, then,

$$x^2 = 2R_1 z - z^2.$$

Ignoring the second-order term of the small quantity z,

$$z = \frac{x^2}{2R_1} .$$

Since also

$$z = Ax^2,$$

$$A = \frac{1}{2R_1} .$$

Similarly,

$$B = \frac{1}{2R_1'} .$$

We can, therefore, now write the equations for the two bodies by substituting in equation (9):

$$z_1 = A_1 x_1^2 + B_1 y_1^2 = \frac{x_1^2}{D_1} + \frac{y_1^2}{D_1'} , \tag{10}$$

and

$$z_2 = A_2 x_2^2 + B_2 y_2^2 = \frac{x_2^2}{D_2} + \frac{y_2^2}{D_2'} , \tag{11}$$

D_1, D_1' and D_2, D_2' being twice the principal radii of curvature of the two bodies respectively.

To obtain the compression effect between the two bodies, i.e. their mutual approach under an applied force, it is necessary to transform the coordinate axes of the two bodies (which so far have been treated as independent) to a single coordinate system with different signs for the z-axes and then combine equations (10) and (11).

Let the new common coordinate axes (normal to the z-axes) be (X,Y), making angles β_1 and β_2 with the independent axes x_1 and x_2 respectively, such that $\beta_1 + \beta_2 = w$ (see Fig. 4).

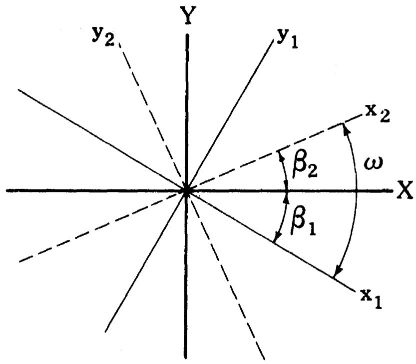


Fig. 4

Then the transformation of coordinates is given by the equations:

$$x_1 = X\cos\beta_1 - Y\sin\beta_1,$$

$$y_1 = X\sin\beta_1 + Y\cos\beta_1,$$

$$x_2 = X\cos\beta_2 + Y\sin\beta_2,$$

$$y_2 = -X\sin\beta_2 + Y\cos\beta_2.$$

Substituting in equations (10) and (11) we now have, in the coordinate system (X,Y),

$$z_1 = A_1(X\cos\beta_1 - Y\sin\beta_1)^2 + B_1(X\sin\beta_1 + Y\cos\beta_1)^2, \quad (12)$$

$$z_2 = A_2(X\cos\beta_2 + Y\sin\beta_2)^2 + B_2(-X\sin\beta_2 + Y\cos\beta_2)^2. \quad (13)$$

These two equations may be combined with a single equation, as all these coordinate systems (x_1y_1), (x_2y_2), and (XY) have a common z-axis but with different signs. Adding equations (12) and (13) and expanding the bracket terms gives

$$z_1 + z_2 = X^2(A_1 \cos^2\beta_1 + A_2 \cos^2\beta_2 + B_1 \sin^2\beta_1 + B_2 \sin^2\beta_2)$$
$$+ 2XY(-A_1 \cos\beta_1 \sin\beta_1 + B_1 \sin\beta_1 \cos\beta_1 + A_2 \cos\beta_2 \sin\beta_2 - B_2 \sin\beta_2 \cos\beta_2)$$
$$+ Y^2(A_1 \sin^2\beta_1 + A_2 \sin^2\beta_2 + B_1 \cos^2\beta_1 + B_2 \cos^2\beta_2). \tag{14}$$

Now writing the coefficients of X^2 and Y^2 as A and B:

$$A = (A_1 \cos^2\beta_1 + A_2 \cos^2\beta_2 + B_1 \sin^2\beta_1 + B_2 \sin^2\beta_2)$$

and

$$B = (A_1 \sin^2\beta_1 + A_2 \sin^2\beta_2 + B_1 \cos^2\beta_1 + B_2 \cos^2\beta_2).$$

Adding,

$$A + B = A_1 + A_2 + B_1 + B_2. \tag{15}$$

Subtracting,

$$A - B = A_1 \cos 2\beta_1 - B_1 \cos 2\beta_1 + A_2 \cos 2\beta_2 - B_2 \cos 2\beta_2$$
$$= (A_1 - B_1)\cos 2\beta_1 + (A_2 - B_2)\cos 2\beta_2. \tag{16}$$

Equation (14) would be further simplified if the cross-product term in XY could be made to vanish. This will be achieved if the coefficient of XY is equal to zero, namely,

$$-A_1 \cos\beta_1 \sin\beta_1 + B_1 \sin\beta_1 \cos\beta_1 + A_2 \cos\beta_2 \sin\beta_2 - B_2 \sin\beta_2 \cos\beta_2 = 0,$$

i.e. $$-(A_1 - B_1)\sin 2\beta_1 + (A_2 - B_2)\sin 2\beta_2 = 0.$$

Squaring this equation gives

$$(A_1 - B_1)^2\sin^2 2\beta_1 - 2(A_1 - B_1)(A_2 - B_2)\sin 2\beta_1 \sin 2\beta_2 + (A_2 - B_2)^2\sin^2 2\beta_2 = 0. \tag{17}$$

If we square equation (16) we have

$$(A - B)^2 = (A_1 - B_1)^2\cos^2 2\beta_1 + 2(A_1 - B_1)(A_2 - B_2)\cos 2\beta_1 \cos 2\beta_2 + (A_2 - B_2)^2\cos^2 2\beta_2. \quad (18)$$

Adding equations (17) and (18) then gives

$$(A - B)^2 = (A_1 - B_1)^2 + (A_2 - B_2)^2 + 2(A_1 - B_1)(A_2 - B_2)\cos 2\omega, \quad (19)$$

since $(2\beta_1 + 2\beta_2) = 2\omega$.

Equation (14) may therefore be rewritten as

$$z_1 + z_2 = AX^2 + BY^2, \quad (20)$$

where

$$A + B = \frac{1}{D_1} + \frac{1}{D_1'} + \frac{1}{D_2} + \frac{1}{D_2'}, \quad (21)$$

$$(A - B)^2 = \left(\frac{1}{D_1} - \frac{1}{D_1'}\right)^2 + \left(\frac{1}{D_2} - \frac{1}{D_2'}\right)^2 + 2\left(\frac{1}{D_1} - \frac{1}{D_1'}\right)\left(\frac{1}{D_2} - \frac{1}{D_2'}\right)\cos 2\omega, \quad (22)$$

ω = angle between the original x-axes of the two bodies.

(c) *Equations for Area of Contact, Pressure Distribution, and Compression**

When two bodies are pressed together, displacements will occur in both: in this case, we are considering forces operating parallel to the z-axis, and displacements along this axis.

If the displacements at a point are w_1 and w_2, then for points inside the area of contact, since the bodies touch over this area,

$$(z_1 + w_1) + (z_2 + w_2) = \alpha,$$

while, outside the area of contact,

$$(z_1 + w_1) + (z_2 + w_2) > \alpha,$$

*The argument here is essentially that of Landau and Lifshitz (1959).

α being the value of $(w_1 + w_2)$ at the origin; i.e. α is the compression we are seeking. The distribution of the bodies is illustrated by Figure 5.

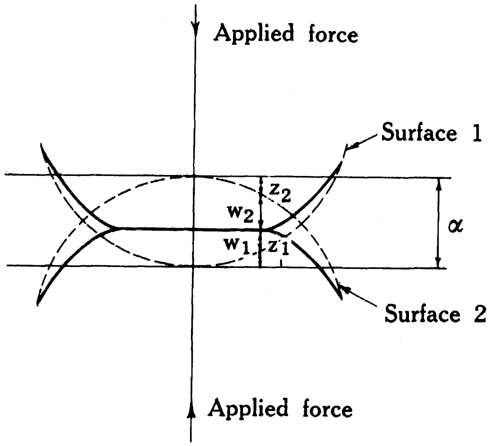


Fig. 5

Having chosen the axis such that

$$z_1 + z_2 = Ax^2 + By^2,$$

(i.e. equation (20)), it follows that

$$Ax^2 + By^2 + (w_1 + w_2) = \alpha. \tag{23}$$

Let the component of the pressure at a point (x',y') on the surface of contact be $p(x',y')$. It can be shown (see for example Prescott 1924, pp. 623-7) that, assuming the surface to be plane, the deformation at a point (x,y) owing to this pressure is given by

$$w(x,y) = \frac{1 - \sigma^2}{\pi E} \cdot \frac{p(x',y')}{r} \, dx' \, dy',$$

where r is the distance from the point (x,y) to the point (x',y'). Further, using the superposition theorem, the displacement at a point (x,y) due to the distribution of pressure over an area A is given by

$$w(x,y) = \frac{1 - \sigma^2}{\pi E} \iint_A \frac{p(x',y')}{r} \, dx' \, dy'. \tag{24}$$

Substituting equation (24) in (23) gives

$$\left(\frac{1-\sigma_1^2}{\pi E_1}+\frac{1-\sigma_2^2}{\pi E_2}\right)\iint_A \frac{p(x',y')}{r}\,\mathrm{d}x'\,\mathrm{d}y' = \alpha - Ax^2 - By^2, \tag{25}$$

where the subscripts 1 and 2 designate the elastic constants for the two bodies.

It also follows that

$$w_1/w_2 = \left(\frac{1-\sigma_1^2}{E_1}\right)\Big/\left(\frac{1-\sigma_2^2}{E_2}\right).$$

A solution of equation (25) yields expressions for the area of contact, the pressure distribution over the area, and the compression. This solution can be found by analogy with a problem in potential theory.

If an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ has a uniform volume charge of density ρ, then it can be shown (Kellogg 1929, p. 192) that the potential for points inside the ellipsoid is given by

$$\phi(x,y,z) = \pi\rho\, abc\int_0^\infty\left(1-\frac{x^2}{a^2+\psi}-\frac{y^2}{b^2+\psi}-\frac{z^2}{c^2+\psi}\right)\cdot\frac{\mathrm{d}\psi}{\left((a^2+\psi)(b^2+\psi)(c^2+\psi)\right)^{1/2}}\,.$$

If the ellipsoid is very much flattened, so that c becomes very small, the contribution from the integral

$$\int_0^\infty \frac{z^2}{c^2+\psi}\cdot\frac{\mathrm{d}\psi}{\left((a^2+\psi)(b^2+\psi)(c^2+\psi)\right)^{1/2}}$$

becomes negligible and we may write

$$\phi(x,y) = \pi\rho\, abc\iint_0^\infty\left(1-\frac{x^2}{a^2+\psi}-\frac{y^2}{b^2+\psi}\right)\cdot\frac{\mathrm{d}\psi}{\left((a^2+\psi)(b^2+\psi)\psi\right)^{1/2}}\,. \tag{26}$$

The potential can also be expressed in a more elementary way as

$$\phi(x,y,z) = \iiint_v \frac{\rho\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{1/2}},$$

integrating over the volume of the ellipsoid.

If, in this last expression, z and z' are written as zero, and the resulting expression is integrated with respect to z' over the limits $\pm c\sqrt{1-(x'^2/a^2)-(y'^2/b^2)}$, then

$$\phi(x,y) = 2\rho c \iint \left(1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right)^{1/2} \cdot \frac{dx'\ dy'}{r}, \tag{27}$$

where $r = \left((x - x')^2 + (y - y')^2\right)^{1/2}$.

Equation (27) then refers, as does equation (26), to the case of an ellipsoid very much flattened in the z-direction, and the two may be equated, giving

$$\iint \left(1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right)^{1/2} \cdot \frac{dx'\ dy'}{r}$$

$$= \frac{1}{2}\pi ab \int_0^\infty \left(1 - \frac{x^2}{a^2+\psi} - \frac{y^2}{b^2+\psi}\right) \cdot \frac{d\psi}{\left((a^2+\psi)(b^2+\psi)\psi\right)^{1/2}}. \tag{28}$$

Comparing equations (25) and (28), it will be seen that, if the right-hand sides are viewed as quadratics in x and y, they have identical forms, while the left-hand sides are integrals of the same form. It follows that the area of contact is bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$, and that the pressure distribution over the area of contact is given by

$$p(x,y) = k\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2}.$$

Equating the integral $\int_S p(x,y)\ dx\ dy$ to the total force, P, tending to compress the two bodies,

$$k = 3P/2\pi ab$$

and

$$p(x,y) = \left(3P/2\pi ab\right)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2}. \tag{29}$$

Substituting equation (29) in equation (25), and using equation (28)

$$\frac{3}{4} P(V_1 + V_2) \int_0^\infty \left(1 - \frac{x^2}{a^2+\psi} - \frac{y^2}{b^2+\psi}\right) \cdot \frac{d\psi}{\left((a^2+\psi)(b^2+\psi)\psi\right)^{1/2}}$$

$$= \alpha - Ax^2 - By^2, \tag{30}$$

where $V_1 = (1 - \sigma_1^2)/\pi E_1$ and $V_2 = (1 - \sigma_2^2)/\pi E_2$.

As this expression must hold for all values of x and y within the contact ellipse, expressions for α, A, and B can be obtained by equating coefficients on both sides of (30), leading to:

$$\alpha = \frac{3}{4} P(V_1 + V_2) \int_0^\infty \frac{d\psi}{\left((a^2+\psi)(b^2+\psi)\psi\right)^{1/2}} , \tag{31}$$

$$A = \frac{3}{4} P(V_1 + V_2) \int_0^\infty \frac{d\psi}{(a^2+\psi)\left((a^2+\psi)(b^2+\psi)\psi\right)^{1/2}} , \tag{32}$$

$$B = \frac{3}{4} P(V_1 + V_2) \int_0^\infty \frac{d\psi}{(b^2+\psi)\left((a^2+\psi)(b^2+\psi)\psi\right)^{1/2}} . \tag{33}$$

The quantities a and b appear in the expression for α as parameters and are in general unknown, and are determined from equations (32) and (33). These expressions are then used to obtain a and b from known values of A and B.

III. SPECIAL CASES

(a) *Two Spheres in Contact*

If the spheres have diameters D_1 and D_2 respectively, then from equations (10) and (11), we have:

$$z_1 = \frac{x^2}{D_1} + \frac{y^2}{D_1} ,$$

$$z_2 = \frac{x^2}{D_2} + \frac{y^2}{D_2} ,$$

and the area of contact is a circle (very flattened sphere) and $a = b$. By adding the above equations, we have

$$z_1 + z_2 = x^2\left(\frac{1}{D_1} + \frac{1}{D_2}\right) + y^2\left(\frac{1}{D_1} + \frac{1}{D_2}\right).$$

Comparing this equation with equation (20), it follows that

$$A = B = \frac{1}{D_1} + \frac{1}{D_2} .$$

Then equations (32) and (33) become identical and may be written

$$A = B = \frac{3}{4} P(V_1 + V_2) \int_0^\infty \frac{d\psi}{(a^2+\psi)^2\psi^{1/2}} .$$

Putting $\psi^{1/2} = \rho$, we can write

$$A = B = \left(\frac{1}{D_1} + \frac{1}{D_2}\right) = \frac{3}{4} P(V_1 + V_2) \int_0^\infty \frac{2d\rho}{(a^2 + \rho^2)^2}$$

$$= \frac{3\pi}{8a^3} \cdot P(V_1 + V_2).$$

Therefore

$$a^3 = \frac{3\pi}{8} \cdot P(V_1 + V_2)\left(\frac{1}{D_1} + \frac{1}{D_2}\right)^{-1}.$$

Equation (31) gives the total compression in this case as

$$\alpha = \frac{3}{4} P(V_1 + V_2) \int_0^\infty \frac{d\psi}{(a^2+\psi)\psi^{1/2}} \;.$$

Again putting $\psi^{1/2} = \rho$, we can write

$$\alpha = \frac{3}{4} P(V_1 + V_2) \int_0^\infty \frac{2d\rho}{a^2 + \rho^2}$$

$$= \frac{3\pi}{4a} P(V_1 + V_2).$$

Substituting for a, we then have

$$\alpha = \frac{(3\pi)^{2/3}}{2} \cdot P^{2/3} \cdot (V_1 + V_2)^{2/3} \cdot \left(\frac{1}{D_1} + \frac{1}{D_2}\right)^{1/3}.$$

(b) Sphere in Contact with a Plane

This can be considered as two spheres in contact, the diameter of one sphere being infinite.

The formula for α then becomes

$$\alpha = \frac{(3\pi)^{2/3}}{2} \cdot P^{2/3} \cdot (V_1 + V_2)^{2/3} \cdot \left(\frac{1}{D}\right)^{1/3}.$$

(c) Sphere in Contact with an Internal Sphere

If the diameter of the internal sphere is D_1 and the diameter of the small sphere is D_2 then the situation is similar to that in III(*b*) except that in the coordinate system adopted, the diameter of the internal sphere becomes negative, giving

$$\alpha = \frac{(3\pi)^{2/3}}{2} \cdot P^{2/3} \cdot (V_1 + V_2)^{2/3} \cdot \left(\frac{1}{D_2} - \frac{1}{D_1}\right)^{1/3}.$$

(d) Equal Cylinders Crossed at Right Angles

Since two of the curvatures are equal and two are infinite, we can write equations (10) and (11) as:

$$z_1 = \frac{x^2}{D} + 0,$$

$$z_2 = 0 + \frac{y^2}{D},$$

$$z_1 + z_2 = \frac{x^2}{D} + \frac{y^2}{D} = Ax^2 + By^2,$$

i.e.

$$A = B = \frac{1}{D}.$$

From similar derivation to that given in III(*b*), we therefore have

$$\alpha = \frac{(3\pi)^{2/3}}{2} \cdot P^{2/3} \cdot (V_1 + V_2)^{2/3} \cdot (\frac{1}{D})^{1/3}.$$

(e) Unequal Cylinders Crossed at Right Angles

If the diameters of the two cylinders be D_1 and D_2 respectively, then, if their axes are at right angles, equations (10) and (11) become:

$$z_1 = \frac{x^2}{D_1} + 0,$$

$$z_2 = 0 + \frac{y^2}{D_2},$$

$$z_1 + z_2 = \frac{x^2}{D_1} + \frac{y^2}{D_2},$$

i.e.

$$A = \frac{1}{D_1}, \quad B = \frac{1}{D_2}.$$

Now the equations connecting stress and strain, i.e. equations (31), (32), and (33), can be expressed in terms of the eccentricity e of the ellipse of contact $1 - e^2 = b^2/a^2$.

Considering equation (32), if we multiply the top and bottom lines of the integral by $(1/a^2)^{5/2}$ we have

$$A = \frac{3}{4} P(V_1 + V_2) \int_0^\infty \frac{(\frac{1}{a^2})^{5/2} \, . \, d\psi}{\left(1 + \frac{\psi}{a^2}\right)^{3/2} \left(\frac{b^2}{a^2} + \frac{\psi}{a^2}\right)^{1/2} \left(\frac{\psi}{a^2}\right)^{1/2}} \, .$$

Writing $\psi/a^2 = \zeta$ the equation becomes

$$A = \frac{3}{4} P(V_1 + V_2) \int_0^\infty \frac{d\zeta}{a^3(1 + \zeta)^{3/2}(1 - e^2 + \zeta)^{1/2}\zeta^{1/2}} \, ,$$

i.e.

$$Aa^3 = \frac{3}{4} P(V_1 + V_2) \int_0^\infty \frac{d\zeta}{(1 + \zeta)^{3/2}(1 - e^2 + \zeta)^{1/2}\zeta^{1/2}} \, .$$

Similarly, it can be shown that equations (31) and (33) may be written:

$$Ba^3 = \frac{3}{4} P(V_1 + V_2) \int_0^\infty \frac{d\zeta}{(1 + \zeta)^{1/2}(1 - e^2 + \zeta)^{3/2}\zeta^{1/2}} \, ,$$

$$\alpha = \frac{P}{a} \, . \, (V_1 + V_2) \int_0^\infty \frac{d\zeta}{(1 + \zeta)^{1/2}(1 - e^2 + \zeta)^{1/2}\zeta^{1/2}} \, .$$

These equations can be simplified by a further change of variable, namely, by putting $\zeta = \cot^2\theta$ where θ goes from $\frac{1}{2}\pi$ to 0 as ζ goes from 0 to ∞. Then

$$d\zeta = -2 \cot\theta \, \operatorname{cosec}^2\theta \, . \, d\theta.$$

Substituting in the equation for Aa^3 we then have:

$$Aa^3 = \frac{3}{4} P(V_1 + V_2) \int_{\frac{\pi}{2}}^{0} \frac{-2 \cot\theta \,.\, \mathrm{cosec}^2\theta \,.\, d\theta}{(1 + \cot^2\theta)^{3/2}(1 - e^2 + \cot^2\theta)^{1/2}(\cot^2\theta)^{1/2}}$$

$$= \frac{3}{4} P(V_1 + V_2) \int_{\frac{\pi}{2}}^{0} \frac{-2\, \mathrm{cosec}^2\theta \,.\, d\theta}{\mathrm{cosec}^3\theta(\mathrm{cosec}^2\theta - e^2)^{1/2}}$$

$$= \frac{3}{4} P(V_1 + V_2) \int_{\frac{\pi}{2}}^{0} \frac{-2d\theta}{\mathrm{cosec}^2\theta\left(1 - \dfrac{e^2}{\mathrm{cosec}^2\theta}\right)^{1/2}}$$

$$= \frac{3}{4} P(V_1 + V_2) \int_{\frac{\pi}{2}}^{0} \frac{-2 \sin^2\theta\, d\theta}{(1 - e^2 \sin^2\theta)^{1/2}}$$

and, by reversing limits and sign,

$$Aa^3 = \frac{3}{2} P(V_1 + V_2) \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\, d\theta}{(1 - e^2 \sin^2\theta)^{1/2}} .$$

Similarly it can be shown that

$$Ba^3 = \frac{3}{2} P(V_1 + V_2) \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\, d\theta}{(1 - e^2 \sin^2\theta)^{3/2}} ,$$

$$\alpha = \frac{3}{2} \frac{P}{a} (V_1 + V_2) \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - e^2 \sin^2\theta)^{1/2}} .$$

Now the complete elliptic integral of the first class, K, is

$$\mathrm{K} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - e^2 \sin^2\theta)^{1/2}}$$

and

$$\frac{d\mathrm{K}}{de} = e \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\, d\theta}{(1 - e^2 \sin^2\theta)^{3/2}} .$$

Also the complete elliptic integral of the second class, E, is

$$\mathrm{E} = \int_0^{\frac{\pi}{2}} (1 - e^2 \sin^2\theta)^{1/2} d\theta$$

and

$$\frac{d\mathrm{E}}{de} = -e \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \, d\theta}{(1 - e^2 \sin^2\theta)^{1/2}} \; .$$

The equations can therefore be written in terms of the complete elliptic integrals thus:

$$Aa^3 = -\frac{2QP}{e} \cdot \frac{d\mathrm{E}}{de} \, ,$$

$$Ba^3 = \frac{2QP}{e} \cdot \frac{d\mathrm{K}}{de} \, ,$$

$$\alpha = \frac{2QP}{a} \cdot \mathrm{K},$$

where $Q = \frac{3}{4} (V_1 + V_2)$.

These equations may be combined to give a compression equation independent of a, namely,

$$\alpha = 2\mathrm{K}(PQ)^{2/3} \left(\frac{1}{2D_1 (- \frac{1}{e} \frac{d\mathrm{E}}{de})} \right)^{1/3} .$$

Now the relationships connecting E and K are:

$$\frac{d\mathrm{E}}{de} = \frac{1}{e} (\mathrm{E} - \mathrm{K})$$

and

$$\frac{d\mathrm{K}}{de} = \frac{1}{e(1 - e^2)} \left(\mathrm{E} - (1 - e^2)\mathrm{K} \right),$$

from which we have

$$\frac{1}{e}\frac{dE}{de} = \frac{1}{e^2}(E - K)$$

and

$$\frac{A}{B} = \frac{-\dfrac{dE}{de}}{\dfrac{dK}{de}} = \frac{-(1 - e^2)(E - K)}{E - (1 - e^2)K} .$$

Therefore, for any chosen value of e we can give the corresponding values of $\frac{A}{B}$, K, and $-\frac{1}{e}\frac{dE}{de}$.

Sets of such values are given in Appendix II.

(*f*) *Unequal Diameter Cylinders Crossed with Their Axes at Any Angle*

This case differs from that of III(*e*) only in that the angle between the axes of the cylinders, θ, is some other value than 90°.

It is therefore necessary to obtain the ratio A/B by solving the following equations (cf. equations (21) and (22)) for A and B:

$$A + B = \frac{1}{D_1} + \frac{1}{D_2} ,$$

$$(A - B)^2 = \left(\frac{1}{D_1}\right)^2 + \left(\frac{1}{D_2}\right)^2 + \frac{2\cos 2\theta}{D_1 D_2}$$

θ being the *acute* angle between the cylinder axes, and D_1 and D_2 being the diameters of the larger and smaller cylinders respectively. The general formula for the compression is

$$\alpha = 2K(PQ)^{2/3}\left(\frac{A}{2 \cdot -\frac{1}{e}\frac{dE}{de}}\right)^{1/3}$$

(*g*) *Sphere on a Cylinder*

Since one diameter of the cylinder has become infinite, equations (10) and (11) become:

for the sphere, $$z_1 = \frac{x^2}{D_1} + \frac{y^2}{D_1} ,$$

for the cylinder, $$z_2 = 0 + \frac{y^2}{D_2} ,$$

where D_1 = diameter of the sphere,
D_2 = diameter of the cylinder,

$$z_1 + z_2 = \frac{1}{D_1} x^2 + (\frac{1}{D_1} + \frac{1}{D_2}) y^2,$$

$$A = \frac{1}{D_1} ,$$

and

$$\frac{A}{B} = \frac{\frac{1}{D_1}}{\frac{1}{D_1} + \frac{1}{D_2}} .$$

From these values of A and A/B, it is necessary to calculate the value of a from

$$a^3 = \frac{2QP}{A} . - \frac{1}{e}\frac{\mathrm{dE}}{de}$$

and thence

$$\alpha = \frac{2QP}{a} . \mathrm{K}.$$

(h) Sphere Inside a Cylinder

This case is similar to that of III(f) except for the change in sign necessitated by the internal form of the cylinder. We therefore have:

for the sphere, $$z_1 = \frac{x^2}{D_1} + \frac{y^2}{D_1} ,$$

for the cylinder, $$z_2 = - \frac{x^2}{D_2} + 0,$$

$$A = \frac{1}{D_1} - \frac{1}{D_2} ,$$

and

$$\frac{A}{B} = \frac{\frac{1}{D_1} - \frac{1}{D_2}}{\frac{1}{D_1}} .$$

The calculation of a from

$$a^3 = \frac{2QP}{A} . - \frac{1}{e}\frac{\mathrm{dE}}{de}$$

and α from

$$\alpha = \frac{2QP}{a} \cdot K$$

then follow.

(i) Cylinders in Contact along a Line Parallel to Their Axes and a Cylinder on a Plane

It is not possible to obtain the solution by direct use of the expressions already derived by allowing one axis of the ellipse of contact to become infinite as the solution itself then becomes infinite. This may appear surprising at first but the reason lies in the fact that the analysis requires the bodies to be fixed at infinity and this leads to an infinite displacement. Prescott (1924) has likened the situation to a load applied to an infinitely long string fixed at one end. The extension of such a string on the application of any load would be infinite.

In determining the pressure distribution and the breadth of the area of contact we shall make use of expressions obtained by allowing one axis of an ellipse of contact to become infinite. For the remainder, the contact area will be taken as being a finite rectangle but with one side very much longer than the other.

The derivation given will be for the case of a pair of cylinders in contact with their axes parallel. The solution for a cylinder on a plane is then obtained by allowing the radius of one of the cylinders to become infinite.

It will be remembered that the pressure distribution over the ellipse of contact in the three-dimensional case is given by

$$p(x,y) = \frac{3P}{2\pi ab}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2} .$$

The integrated pressure across the minor axis of the ellipse in the plane $x = 0$ is then

$$\begin{aligned}\bar{P} &= (3P/2\pi \cdot ab)\int_{-b}^{+b}\left(1 - \frac{y^2}{b^2}\right)^{1/2} dy \\ &= \frac{3P}{2\pi ab} \cdot \frac{\pi b}{2} = \frac{3}{4} \cdot \frac{P}{a} .\end{aligned}$$

If both a and P approach infinity in such a way that P/a remains finite, $\bar{P}$ is the force per unit length along the area of contact, which is now rectangular with one side infinite.

It follows that

$$p(y) = (3P/2\pi ab)\left(1 - \frac{y^2}{b^2}\right)^{1/2} = \frac{2\bar{P}}{\pi b}\left(1 - \frac{y^2}{b^2}\right)^{1/2}$$

In the region of the original line of contact the cylinders are adequately represented by the surfaces:

$$z_1 = B_1 y^2,$$
$$z_2 = B_2 y^2.$$

The cylinders are initially in contact over a line of length $2a$.

Applying equation (25),

$$\left(\frac{1 - \sigma_1^2}{\pi E_1} + \frac{1 - \sigma_2^2}{\pi E_2}\right) \iint_A \frac{p(y')}{r}\, dx'\, dy' = \alpha - By^2, \qquad (34)$$

where $r^2 = (y - y')^2 + x'^2$ and the integral extends over the region of contact, which is taken to be a finite rectangle but with one side very much longer than the other. We are considering here only points lying along the y-axis. The assumption that one side of the rectangle, $2a$, is very much longer than the other, $2b$, allows the integral in the left-hand side of (34) to be evaluated.

Write

$$\begin{aligned}
\phi(0,y) &= \iint_A p(y')/r\, dx'\, dy' \\
&= \int_{-b}^{+b} \int_{-a}^{+a} p(y')/\left(x'^2 + (y - y')^2\right)^{1/2} dx'\, dy' \\
&= \int_{-b}^{+b} 2p(y') \int_0^a 1/\left(x'^2 + (y - y')^2\right)^{1/2} dx'\, dy' \\
&= \int_{-b}^{+b} 2p(y') \ln\left(\frac{a + \left((y - y')^2 + a^2\right)^{1/2}}{|y - y'|}\right) dy'.
\end{aligned}$$

If now a is considered to be large in comparison with $(y - y')$

$$\phi(0,y) = \int_{-b}^{+b} 2p(y') \ln\left(2a/|y - y'|\right) dy'$$

and

$$\phi(0,0) = 2(\ln 2a)\int_{-b}^{+b} p(y')\,dy' - \int_{-b}^{+b} p(y')\ln (y')^2 dy',$$

where we are now restricted to the point (0,0).

Now $\int_{-b}^{+b} p(y')\,dy' = \bar{P}$, the force per unit length and

$$\phi(0,0) = 2\bar{P}\ln 2a - (2\bar{P}/\pi b)\int_{-b}^{+b}\left(1 - \frac{y'^2}{b^2}\right)^{1/2}\ln (y')^2 dy'.$$

It remains then to determine the breadth of the area of contact and to evaluate the integral.

From equations (25) and (28),

$$B = (V_1 + V_2)\bar{P}\int_0^{\infty}\frac{d\psi}{(b^2+\psi)^{3/2}\left(1+\frac{\psi}{a^2}\right)^{1/2}\psi^{1/2}},$$

using $p(y) = (2\bar{P}/\pi b)(1 - y^2/b^2)^{1/2}$, which for a infinite gives

$$B = (V_1 + V_2)\bar{P}\int_0^{\infty}\frac{d\psi}{(b^2+\psi)^{3/2}\psi^{1/2}}$$
$$= 2(V_1 + V_2)\bar{P}/b^2,$$

i.e.

$$b^2 = 2(V_1 + V_2)\,.\,\bar{P}/B.$$

Turning now to the evaluation of the integral

$$I = \int_{-b}^{+b}\left(1 - \frac{y^2}{b^2}\right)^{1/2}\ln y^2 dy.$$

Write $y/b = \sin\theta$, then $dy = b \cos\theta \, d\theta$ giving

$$I = b \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2\theta \ln (b^2 \sin^2\theta) d\theta$$

$$= 2b \ln b \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2\theta \, d\theta + 2b \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2\theta \ln |\sin\theta| d\theta.$$

Now

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2\theta \, d\theta = \frac{\pi}{2}$$

and

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \ln |\sin\theta| d\theta = -\frac{\pi}{4} (1 + \ln 4),^*$$

so that

$$I = \pi b \left(\ln b - \frac{1 + \ln 4}{2} \right).$$

Substitution then leads to

$$\phi(0,0) = 2\bar{P}\left(\ln 2a + (1 + \ln 4)/2 - \ln b\right),$$

which in turn gives

$$\alpha = 2\bar{P}(V_1 + V_2)[(1 + \ln 4)/2 + \ln 2a - \ln b].$$

The form of the expression for the compression of a pair of cylinders with their axes parallel and for a cylinder on a plane is identical. The compressions are then given by substituting the appropriate value for b in each case.

*Birens de Haan (1957), [305][8].

Pair of Cylinders with Their Axes Parallel:

$$z_1 = \frac{1}{D_1} y^2,$$
$$z_2 = \frac{1}{D_2} y^2.$$

Therefore

$$B = \frac{1}{D_1} + \frac{1}{D_2} ,$$

giving

$$\ln b = \frac{1}{2} \ln \left[2(V_1 + V_2)\bar{P} \cdot \frac{D_1 D_2}{D_1 + D_2} \right].$$

Cylinder on a Flat:

$$z_1 = \frac{1}{D} y^2$$
$$z_2 = 0,$$

giving

$$B = \frac{1}{D} ,$$

so that

$$\ln b = \frac{1}{2} \ln \left[2(V_1 + V_2)\bar{P}D \right] .$$

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APPENDIX I

Tables of Elastic Constants and Derived Quantities

The values for the elastic constants given in Tables 1 and 2 are intended as a guide to the values to be expected. The actual values of the constants for a material are dependent on its precise composition and past history and are affected by such things as heat treatment and the method of fabrication. The values given, however, should be adequate for the calculation of compressions in most practical cases, as the percentage error in a calculated compression due to an error in a constant is of the same order as the percentage error in the constant. The formulae derived here do not necessarily apply to anisotropic materials, in particular to crystals where the elastic properties may be significantly different for different axes.

TABLE 1

Elastic Constants in Metric Units

Material	Young's Modulus E (10^{10} Newtons/m^2)	Young's Modulus E (10^6 gf/mm^2)	Poisson's Ratio σ	Source*
Aluminium	7.05	7.19	0.345	K & L
Copper	13.0	13.24	0.343	K & L
Gold	7.8	8.0	0.440	K & L
Platinum	16.8	17.13	0.377	K & L
Silver	8.28	8.43	0.367	K & L
Tungsten carbide				
% Co				
6	72.4	73.8	0.280	A.S.M.
10	60.0	61.2	0.200	
16	52.4	53.4	0.220	
Chromium carbide				
(Carmet CA-815G)	33.9	34.6	0.280	Carmet
Steel				
1% C	20.9	21.4	0.293	K & L
Mild	21.0	21.4	0.291	K & L
Glass				
Pyrex	6.2	6.3	0.24	A.I.P.
Heavy silicate flint	5.35	5.46	0.224	A.I.P.
Light borate crown	4.61	4.70	0.274	A.I.P.
Brass				
70% Cu, 30% Zn	10.4	10.6	0.374	A.I.P.
Silica (fused)	7.29	7.43	0.17	A.I.P.

*A.I.P. : American Institute of Physics Handbook: 2nd Edition.
K & L : Kaye & Laby "Physical & Chemical Constants": 12th Edition (1959).
A.S.M. : A.S.M. Handbook: 8th Edition, 1961, p. 664.
Carmet : Allegheny Ludlum Steel Corporation.

TABLE 1 (Cont'd)

Elastic Constants in Metric Units

Material	in terms of gf/mm^2			
	$(1-\sigma^2)/E$ (10^{-7})	$\left[(1-\sigma^2)/E\right]^{2/3}$ (10^{-5})	$V = (1-\sigma^2)/\pi E$ (10^{-8})	$Q^{2/3} = (\frac{3}{2} V)^{2/3}$ (10^{-5})
Aluminium	1.225	2.467	3.901	1.507
Copper	0.667	1.644	2.122	1.004
Gold	1.014	2.174	3.227	1.328
Platinum	0.501	1.359	1.594	0.830
Silver	1.026	2.192	3.266	1.339
Tungsten carbide				
% Co				
6	0.125	0.538	0.397	0.329
10	0.157	0.627	0.500	0.383
16	0.178	0.682	0.567	0.417
Chromium carbide				
(Carmet CA-815G)	0.266	0.892	0.848	0.545
Steel				
1% C	0.427	1.221	1.359	0.746
Mild	0.427	1.222	1.361	0.747
Glass				
Pyrex	1.491	2.811	4.745	1.717
Heavy silicate flint	1.741	3.118	5.542	1.905
Light borate crown	1.968	3.383	6.263	2.067
Brass				
70% Cu, 30% Zn	0.811	1.874	2.582	1.145
Silica (fused)	1.306	2.575	4.158	1.573

TABLE 2

Elastic Constants in lbf/in^2

Material	Young's Modulus E (10^6 lbf/in^2)	Poisson's Ratio σ	Source*
Aluminium	10.22	0.345	K & L
Copper	18.83	0.343	K & L
Gold	11.3	0.44	K & L
Platinum	24.37	0.377	K & L
Silver	12.00	0.367	K & L
Tungsten carbide			
% Co			
6	105	0.28	A.S.M.
10	87	0.20	
16	76	0.22	
Chromium carbide			
(Carmet CA-815G)	49.2	0.28	Carmet
Steel			
1% C	30.5	0.293	K & L
Mild	30.5	0.291	K & L
Glass			
Pyrex	9.0	0.24	A.I.P.
Heavy silicate flint	7.76	0.224	A.I.P.
Light borate crown	6.69	0.274	A.I.P.
Brass			
70% Cu, 30% Zn	15.1	0.374	A.I.P.
Silica (fused)	10.57	0.17	A.I.P.

*A.I.P. : American Institute of Physics Handbook: 2nd Edition.
K & L : Kaye & Laby "Physical & Chemical Constants": 12th Edition (1959).
A.S.M. : A.S.M. Handbook: 8th Edition, 1961, p. 664.
Carmet : Allegheny Ludlum Steel Corporation.

TABLE 2 (Cont'd)

Elastic Constants in lbf/in^2

Material	$(1-\sigma^2)/E$ (10^{-8})	$\left((1-\sigma^2)/E\right)^{2/3}$ (10^{-5})	$V = (1-\sigma^2)/\pi E$ (10^{-8})	$Q^{2/3} = (\frac{3}{2} V)^{2/3}$ (10^{-5})
Aluminium	8.616	1.951	2.742	1.192
Copper	4.687	1.300	1.492	0.794
Gold	7.128	1.719	2.269	1.050
Platinum	3.521	1.074	1.121	0.656
Silver	7.214	1.733	2.296	1.059
Tungsten carbide				
% Co				
6	0.878	0.426	0.279	0.260
10	1.103	0.496	0.351	0.303
16	1.252	0.539	0.399	0.329
Chromium carbide				
(Carmet CA-815G)	1.873	0.705	0.596	0.431
Steel				
1% C	3.001	0.966	0.955	0.590
Mild	3.005	0.967	0.957	0.590
Glass				
Pyrex	10.48	2.223	3.336	1.358
Heavy silicate flint	12.24	2.465	3.896	1.506
Light borate crown	13.83	2.675	4.403	1.634
Brass				
70% Cu, 30% Zn	5.702	1.481	1.815	0.905
Silica (fused)	9.184	2.036	2.923	1.244

APPENDIX II

Values of K, $-\frac{1}{e}\frac{dE}{de}$, *and Eccentricities for Arguments A/B*

The values given for the complete elliptical integral of the first type, K, and the quantity $-\frac{1}{e}\frac{dE}{de}$ have been derived from a number of sources. Tables 3-6 for *A/B* in the range 0.01 to 1.00 are due to Rolt and Grant (1921), while the values given in Figure 6 have been derived using expressions given by Airey (1935). Both these series of values have been point-checked against a digital computer program based on the method of the arithmetic-geometric mean. The curve of $(1 - e^2)$, Figure 7, has been derived from the relationships that exist between *A/B*, K, and E.

TABLE 3

A/B : [1.00(0.01)0.50]

$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$	$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$
1.00	1.5708	0.7854	0.75	1.7249	0.9037
0.99	1.5761	.7894	74	1.7322	.9095
98	1.5814	.7934	73	1.7397	.9153
97	1.5868	.7974	72	1.7472	.9213
96	1.5922	.8015	71	1.7549	.9274
0.95	1.5978	0.8057	0.70	1.7628	0.9336
94	1.6034	.8100	69	1.7707	.9399
93	1.6090	.8142	68	1.7788	.9463
92	1.6148	.8186	67	1.7870	.9529
91	1.6206	.8230	66	1.7953	.9595
0.90	1.6264	0.8275	0.65	1.8038	0.9664
89	1.6324	.8320	64	1.8125	.9733
88	1.6384	.8367	63	1.8213	.9804
87	1.6445	.8413	62	1.8302	.9876
86	1.6507	.8461	61	1.8393	.9949
0.85	1.6570	0.8509	0.60	1.8486	1.0025
84	1.6634	.8558	59	1.8581	1.0101
83	1.6698	.8608	58	1.8677	1.0180
82	1.6764	.8659	57	1.8775	1.0260
81	1.6830	.8710	56	1.8876	1.0341
0.80	1.6897	0.8762	0.55	1.8978	1.0425
79	1.6965	.8815	54	1.9082	1.0511
78	1.7035	.8869	53	1.9188	1.0598
77	1.7105	.8924	52	1.9297	1.0688
76	1.7176	.8980	51	1.9408	1.0779
			0.50	1.9521	1.0874

TABLE 3 (Cont'd)

A/B : [0.500(0.005)0.200]

$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$	$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$
0.500	1.9521	1.0874	0.350	2.1595	1.2632
495	1.9579	1.0921	345	2.1680	1.2705
490	1.9637	1.0970	340	2.1766	1.2780
485	1.9696	1.1019	335	2.1853	1.2856
480	1.9755	1.1068	330	2.1942	1.2933
0.475	1.9816	1.1119	0.325	2.2032	1.3012
470	1.9877	1.1170	320	2.2124	1.3092
465	1.9938	1.1221	315	2.2218	1.3173
460	2.0001	1.1273	310	2.2312	1.3256
455	2.0064	1.1326	305	2.2409	1.3341
0.450	2.0128	1.1380	0.300	2.2507	1.3427
445	2.0192	1.1434	295	2.2607	1.3515
440	2.0258	1.1490	290	2.2709	1.3604
435	2.0324	1.1546	285	2.2812	1.3696
430	2.0391	1.1602	280	2.2918	1.3789
0.425	2.0459	1.1660	0.275	2.3025	1.3884
420	2.0528	1.1718	270	2.3135	1.3981
415	2.0597	1.1777	265	2.3247	1.4080
410	2.0668	1.1837	260	2.3361	1.4181
405	2.0739	1.1898	255	2.3477	1.4285
0.400	2.0812	1.1960	0.250	2.3595	1.4391
395	2.0885	1.2022	245	2.3716	1.4499
390	2.0960	1.2086	240	2.3840	1.4609
385	2.1035	1.2150	235	2.3966	1.4723
380	2.1112	1.2216	230	2.4096	1.4839
0.375	2.1189	1.2282	0.225	2.4228	1.4958
370	2.1268	1.2350	220	2.4363	1.5080
365	2.1348	1.2419	215	2.4501	1.5205
360	2.1429	1.2489	210	2.4643	1.5333
355	2.1511	1.2560	205	2.4788	1.5465
			0.200	2.4937	1.5600

TABLE 4

A/B : [0.200(0.001)0.100]

$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$	$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$
0.200	2.4937	1.5600	0.175	2.5745	1.6337
199	2.4968	1.5627	174	2.5779	1.6369
198	2.4998	1.5655	173	2.5814	1.6401
197	2.5029	1.5683	172	2.5849	1.6433
196	2.5059	1.5711	171	2.5885	1.6465
0.195	2.5090	1.5739	0.170	2.5920	1.6498
194	2.5121	1.5767	169	2.5956	1.6531
193	2.5152	1.5796	168	2.5992	1.6564
192	2.5184	1.5824	167	2.6028	1.6597
191	2.5215	1.5853	166	2.6064	1.6631
0.190	2.5247	1.5882	0.165	2.6101	1.6664
189	2.5279	1.5911	164	2.6138	1.6698
188	2.5311	1.5940	163	2.6175	1.6733
187	2.5343	1.5970	162	2.6212	1.6767
186	2.5376	1.5999	161	2.6250	1.6802
0.185	2.5408	1.6029	0.160	2.6287	1.6836
184	2.5441	1.6059	159	2.6325	1.6871
183	2.5474	1.6089	158	2.6364	1.6907
182	2.5507	1.6119	157	2.6402	1.6942
181	2.5541	1.6150	156	2.6441	1.6978
0.180	2.5574	1.6181	0.155	2.6480	1.7014
179	2.5608	1.6211	154	2.6519	1.7051
178	2.5642	1.6243	153	2.6559	1.7087
177	2.5676	1.6274	152	2.6598	1.7124
176	2.5710	1.6305	151	2.6639	1.7161

TABLE 4 (Cont'd)

A/B : [0.200(0.001)0.100]

$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$	$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$
0.150	2.6679	1.7198	0.125	2.7786	1.8230
149	2.6719	1.7236	124	2.7835	1.8275
148	2.6760	1.7274	123	2.7884	1.8321
147	2.6801	1.7312	122	2.7934	1.8368
146	2.6843	1.7350	121	2.7984	1.8415
0.145	2.6885	1.7389	0.120	2.8034	1.8462
144	2.6927	1.7428	119	2.8085	1.8510
143	2.6969	1.7468	118	2.8136	1.8558
142	2.7012	1.7507	117	2.8188	1.8607
141	2.7054	1.7547	116	2.8240	1.8656
0.140	2.7098	1.7587	0.115	2.8293	1.8705
139	2.7141	1.7628	114	2.8346	1.8755
138	2.7185	1.7669	113	2.8399	1.8805
137	2.7229	1.7710	112	2.8453	1.8856
136	2.7274	1.7751	111	2.8508	1.8908
0.135	2.7319	1.7793	0.110	2.8563	1.8960
134	2.7364	1.7835	109	2.8618	1.9012
133	2.7409	1.7878	108	2.8674	1.9065
132	2.7455	1.7920	107	2.8731	1.9118
131	2.7501	1.7964	106	2.8788	1.9172
0.130	2.7548	1.8007	0.105	2.8846	1.9226
129	2.7595	1.8051	104	2.8904	1.9281
128	2.7642	1.8095	103	2.8962	1.9337
127	2.7690	1.8140	102	2.9022	1.9393
126	2.7738	1.8184	101	2.9082	1.9449
			0.100	2.9142	1.9507

TABLE 5

A/B : [0.100(0.001)0.050]

$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$	$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$
0.100	2.9142	1.9507	0.075	3.0889	2.1171
099	2.9203	1.9565	74	3.0970	2.1249
98	2.9265	1.9623	73	3.1053	2.1328
97	2.9327	1.9682	72	3.1136	2.1408
96	2.9390	1.9742	71	3.1221	2.1489
0.095	2.9454	1.9802	0.070	3.1307	2.1572
94	2.9518	1.9863	69	3.1394	2.1656
93	2.9583	1.9925	68	3.1483	2.1741
92	2.9649	1.9987	67	3.1573	2.1827
91	2.9715	2.0050	66	3.1664	2.1915
0.090	2.9782	2.0114	0.065	3.1756	2.2004
89	2.9850	2.0179	64	3.1850	2.2094
88	2.9919	2.0244	63	3.1945	2.2186
87	2.9988	2.0310	62	3.2042	2.2279
86	3.0058	2.0377	61	3.2141	2.2374
0.085	3.0129	2.0445	0.060	3.2241	2.2471
84	3.0201	2.0513	59	3.2342	2.2569
83	3.0274	2.0583	58	3.2446	2.2669
82	3.0347	2.0653	57	3.2551	2.2770
81	3.0422	2.0724	56	3.2658	2.2874
0.080	3.0497	2.0796	0.055	3.2767	2.2979
79	3.0574	2.0869	54	3.2877	2.3086
78	3.0651	2.0943	53	3.2990	2.3195
77	3.0729	2.1018	52	3.3105	2.3307
76	3.0809	2.1094	51	3.3222	2.3420
			0.050	3.3342	2.3536

TABLE 5 (Cont'd)

A/B : [0.0500(0.0005)0.0200]

$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$	$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$
0.0500	3.3342	2.3536	0.0350	3.5486	2.5626
495	3.3403	2.3595	345	3.5573	2.5710
490	3.3464	2.3655	340	3.5660	2.5796
485	3.3526	2.3715	335	3.5749	2.5883
480	3.3588	2.3775	330	3.5839	2.5971
0.0475	3.3651	2.3837	0.0325	3.5930	2.6060
470	3.3715	2.3899	320	3.6023	2.6151
465	3.3779	2.3961	315	3.6117	2.6244
460	3.3845	2.4025	310	3.6213	2.6337
455	3.3910	2.4089	305	3.6310	2.6433
0.0450	3.3977	2.4153	0.0300	3.6409	2.6530
445	3.4044	2.4219	295	3.6509	2.6628
440	3.4112	2.4285	290	3.6611	2.6728
435	3.4181	2.4352	285	3.6715	2.6830
430	3.4251	2.4420	280	3.6821	2.6934
0.0425	3.4321	2.4488	0.0275	3.6928	2.7039
420	3.4392	2.4558	270	3.7038	2.7147
415	3.4464	2.4628	265	3.7149	2.7256
410	3.4537	2.4699	260	3.7263	2.7368
405	3.4611	2.4771	255	3.7378	2.7482
0.0400	3.4685	2.4843	0.0250	3.7496	2.7598
395	3.4761	2.4917	245	3.7616	2.7716
390	3.4837	2.4992	240	3.7739	2.7837
385	3.4915	2.5067	235	3.7864	2.7960
380	3.4993	2.5144	230	3.7992	2.8086
0.0375	3.5073	2.5222	0.0225	3.8123	2.8215
370	3.5153	2.5300	220	3.8256	2.8346
365	3.5235	2.5380	215	3.8393	2.8481
360	3.5318	2.5461	210	3.8532	2.8618
355	3.5401	2.5543	205	3.8675	2.8759
			0.0200	3.8821	2.8903

TABLE 6

A/B : [0.0200(0.0001)0.0100]

$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$	$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$
0.0200	3.8821	2.8903	0.0175	3.9611	2.9683
199	3.8850	2.8932	174	3.9644	2.9716
198	3.8880	2.8962	173	3.9678	2.9750
197	3.8910	2.8991	172	3.9713	2.9784
196	3.8940	2.9021	171	3.9747	2.9818
0.0195	3.8971	2.9051	0.0170	3.9782	2.9852
194	3.9001	2.9081	169	3.9816	2.9886
193	3.9032	2.9111	168	3.9851	2.9921
192	3.9062	2.9142	167	3.9887	2.9956
191	3.9093	2.9172	166	3.9922	2.9991
0.0190	3.9124	2.9203	0.0165	3.9958	3.0026
189	3.9156	2.9234	164	3.9994	3.0062
188	3.9187	2.9264	163	4.0030	3.0097
187	3.9219	2.9296	162	4.0066	3.0133
186	3.9250	2.9327	161	4.0102	3.0169
0.0185	3.9282	2.9358	0.0160	4.0139	3.0205
184	3.9314	2.9390	159	4.0176	3.0242
183	3.9346	2.9422	158	4.0213	3.0279
182	3.9379	2.9454	157	4.0251	3.0316
181	3.9411	2.9486	156	4.0288	3.0353
0.0180	3.9444	2.9518	0.0155	4.0326	3.0391
179	3.9477	2.9551	154	4.0364	3.0428
178	3.9510	2.9584	153	4.0403	3.0466
177	3.9543	2.9617	152	4.0441	3.0504
176	3.9577	2.9650	151	4.0480	3.0543

TABLE 6 (Cont'd)

A/B : [0.0200(0.0001)0.0100]

$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$	$\frac{A}{B}$	K	$-\frac{1}{e}\frac{dE}{de}$
0.0150	4.0519	3.0582	0.0125	4.1590	3.1642
149	4.0558	3.0620	124	4.1637	3.1689
148	4.0598	3.0660	123	4.1684	3.1736
147	4.0638	3.0699	122	4.1732	3.1783
146	4.0678	3.0739	121	4.1780	3.1831
0.0145	4.0718	3.0779	0.0120	4.1829	3.1879
144	4.0759	3.0819	119	4.1878	3.1928
143	4.0800	3.0860	118	4.1927	3.1977
142	4.0841	3.0901	117	4.1977	3.2026
141	4.0883	3.0942	116	4.2027	3.2076
0.0140	4.0925	3.0983	0.0115	4.2078	3.2126
139	4.0967	3.1025	114	4.2129	3.2177
138	4.1009	3.1067	113	4.2181	3.2228
137	4.1052	3.1109	112	4.2233	3.2280
136	4.1095	3.1152	111	4.2285	3.2332
0.0135	4.1138	3.1195	0.0110	4.2338	3.2384
134	4.1182	3.1238	109	4.2391	3.2437
133	4.1226	3.1282	108	4.2445	3.2491
132	4.1270	3.1325	107	4.2499	3.2545
131	4.1315	3.1370	106	4.2554	3.2599
0.0130	4.1360	3.1414	0.0105	4.2610	3.2654
129	4.1405	3.1459	104	4.2666	3.2710
128	4.1451	3.1504	103	4.2722	3.2766
127	4.1497	3.1550	102	4.2779	3.2822
126	4.1543	3.1596	101	4.2837	3.2879
			0.0100	4.2895	3.2937

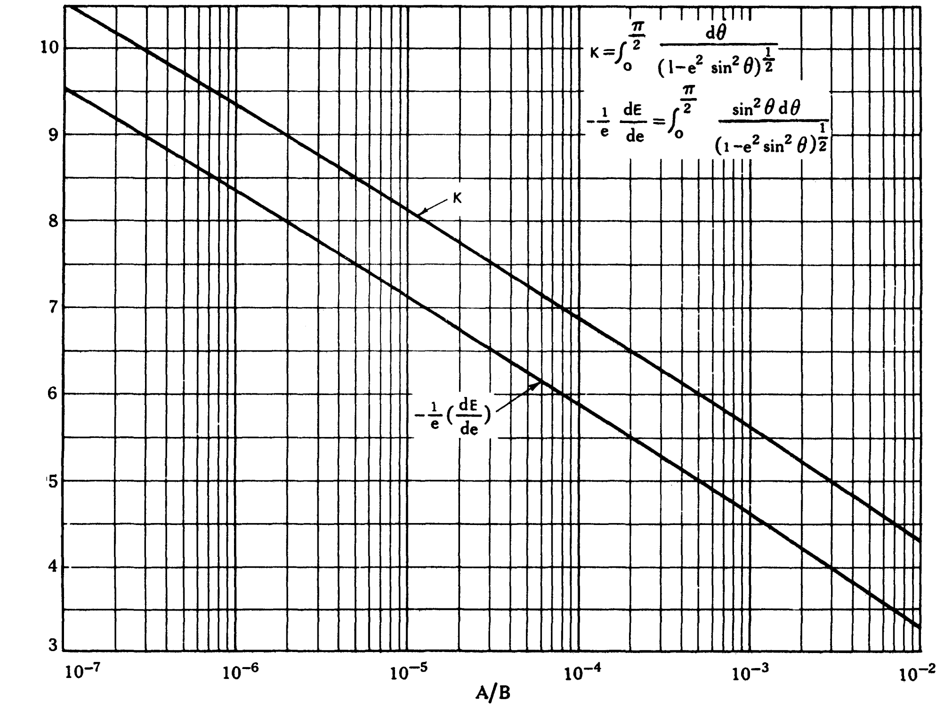

Fig. 6

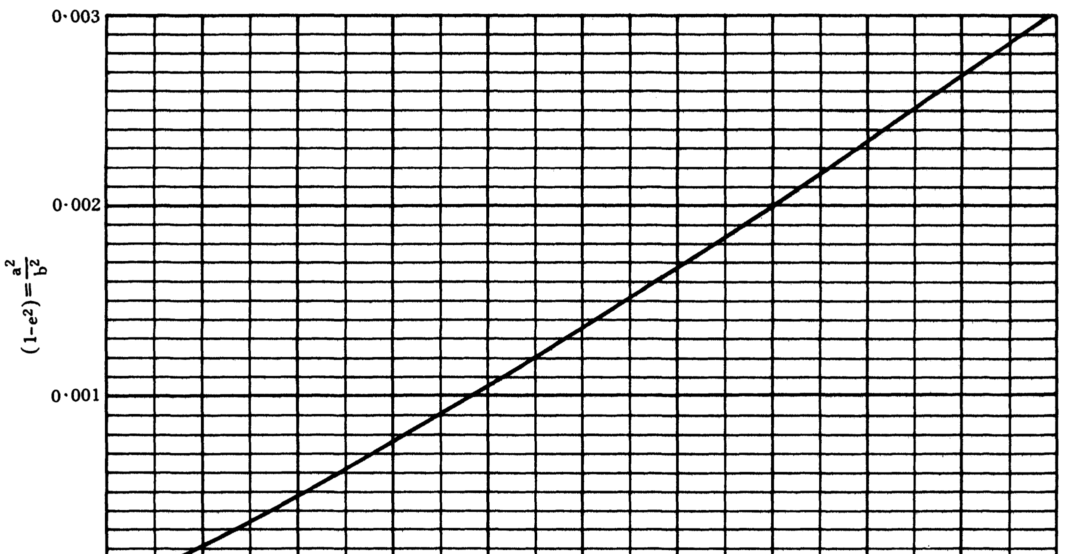

Fig. 7 - Eccentricity.