Anomalies in quantum mechanics: The $1/r^2$ potential

Sidney A. Coon, and Barry R. Holstein

Citation: American Journal of Physics **70**, 513 (2002); doi: 10.1119/1.1456071
View online: https://doi.org/10.1119/1.1456071
View Table of Contents: http://aapt.scitation.org/toc/ajp/70/5
Published by the American Association of Physics Teachers

Anomalies in quantum mechanics: The $1/r^2$ potential

Sidney A. Coon[a)]
Physics Division, National Science Foundation, 4201 Wilson Boulevard, Arlington, Virginia 22230 and Physics Department, New Mexico State University, Las Cruces, New Mexico 88003

Barry R. Holstein[b)]
Institut für Kernphysik, Forschungszentrum Jülich, D-52425 Jülich, Germany and Department of Physics, University of Massachusetts, Amherst, Massachusetts 01003

(Received 28 September 2001; accepted 8 January 2002)

An anomaly is said to occur when a symmetry that is valid classically becomes broken as a result of quantization. Although most manifestations of this phenomenon are in the context of quantum field theory, there are at least two cases in quantum mechanics—the two-dimensional delta function interaction and the $1/r^2$ potential. The former has been treated in this journal; in this article we discuss the physics of the latter together with experimental consequences.
[DOI: 10.1119/1.1456071]

I. INTRODUCTION

The use of symmetry to enhance our understanding of physical systems is well known. An important manifestation of this simplification is Nöther's theorem, which guarantees the correspondence between symmetries and conservation laws.[1] Examples of the consequences of this theorem include

(a) translation invariance ↔ momentum conservation,
(b) time translation invariance ↔ energy conservation,
(c) rotational invariance ↔ angular momentum conservation.

These particular symmetries and conservation laws are exact. Far more common are cases for which the invariance is only approximate and is broken in some fashion. Nevertheless, the symmetry still represents a useful way of analyzing the system, and it is important to understand the mechanisms by which this symmetry breaking can take place. Despite the many physical situations involving symmetry violation, there exist just three mechanisms by which this violation can take place.

(i) Explicit symmetry breaking, wherein the breaking occurs explicitly in the Lagrangian. Familiar examples include particle physics, where the heavier mass of the strange quark compared to its up, down counterparts violates the underlying SU(3) invariance;[2] nuclear physics, where the up–down quark mass difference together with electromagnetic effects are responsible for small deviations from isotopic spin invariance;[3] gravitational physics, where general relativity together with small perturbations from the outer planets lead to deviations from the underlying O(4) invariance associated with a pure $1/r$ interaction and hence to the precession of Mercury's perihelion.[4]

(ii) Spontaneous (or hidden) symmetry breaking, wherein the Lagrangian remains invariant, but the symmetry is not present in the ground state. Familiar examples include condensed matter physics, where the spontaneous violation of rotational invariance by the creation of spin-correlated domains in materials such as iron leads to the phenomenon of ferromagnetism,[5] and the spontaneous violation of local gauge invariance by the condensation of spin- and momentum-correlated electron pairs in low temperature systems leads to superconductivity;[6] classical physics, where (as first studied by Jacobi) the rapid rotation of a gravitationally bound sphere leads to a lowest energy state not possessing the expected axial invariance.[7]

(iii) Anomalous (or quantum mechanical) symmetry breaking, wherein the symmetry is present at the classical level, but is broken by quantization.[8] Examples include elementary particle physics, where the two photon decay of the neutral pion verifies the anomalous breaking of axial SU(2) invariance,[9] and in elementary particle physics, where the so-called trace anomaly leads to a substantial component of the nucleon mass being due to its gluon substructure.[9]

Although these manifestations of explicit and spontaneous symmetry breaking are textbook examples and are well known and available to most physicists, the realization of anomalous symmetry breaking is generally presented only within the context of quantum field theory and is subsequently somewhat inaccessible to all but the experts. Yet this inaccessibility need not be the case. Indeed in previous contributions to this journal, it has been shown that the anomaly is manifested in ordinary quantum mechanics in two spatial dimensions with a delta function interaction.[10] However, this example is not realized in nature. In this note we point out that an additional example of anomalous symmetry breaking occurs in the real world of three spatial dimensions in the presence of a $1/r^2$ potential and that the resultant predictions have been experimentally verified in atomic physics. The discussion is at the level appropriate for a graduate quantum mechanics course.

After a brief review of the previously mentioned two-dimensional delta function potential, we show in Sec. II how the $1/r^2$ interaction can be analyzed using either cutoff regularization in both the bound state and scattering regimes. In Sec. III we demonstrate how this situation can be realized experimentally and discuss the confrontation of theory with recent experiments. Our results are summarized in Sec. IV.

II. ANOMALIES IN QUANTUM MECHANICS

To understand how an anomaly is realized in quantum mechanics, we first review the partial wave formalism, in which the solution to the time-independent Schrödinger equation is expanded in Legendre polynomials,

$$\psi(\mathbf{r})=\sum_l a_l P_l(\cos\theta)\frac{1}{r}R_l(r) . \tag{1}$$

The radial functions $R_l(r)$ obey the differential equation (henceforth we employ $\hbar=c=1$)

$$\left[-\frac{1}{2m}\frac{d^2}{dr^2}+\frac{l(l+1)}{2mr^2}+V(r)\right]R_l(r)=ER_l(r). \tag{2}$$

A. Free particle

For a free particle, $V(r)=0$ and $E\equiv\mathbf{k}^2/2m$, and we have the plane wave solution

$$\psi(\mathbf{r})=\exp(ikz)=\sum_l (2l+1)i^l j_l(kr)P_l(\cos\theta), \tag{3}$$

that is, $a_l=i^l(2l+1)/k$, $R_l(r)=krj_l(kr)$ in the notation of Eq. (1). By using the asymptotic behavior

$$j_l(kr)\overset{r\to\infty}{\to}\frac{1}{kr}\sin\left(kr-l\frac{\pi}{2}\right), \tag{4}$$

we can write Eq. (3) in the form

$$\exp(ikz)\overset{r\to\infty}{\to}\frac{1}{2ikr}\sum_l (2l+1)P_l(\cos\theta)(e^{ikr}-e^{-i(kr-l\pi)}). \tag{5}$$

Equation (5) is a linear combination of incoming, e^{-ikr}, and outgoing, e^{ikr}, spherical waves with a phase shift $l\pi$ between them in the channel having angular momentum l, due to the centrifugal potential term, $l(l+1)/2mr^2$ in Eq. (2). The existence of this *energy-independent* phase between incoming and outgoing spherical wave components can be understood from the invariance of Eq. (2) under the scale transformation $r\to\mu r$, $k\to k/\mu$, which requires that the solution be a function of the product kr, which is scale invariant.[11] This condition is obviously satisfied by the spherical Bessel functions $j_l(kr)$, and would be violated by the existence of an energy-dependent phase.

In the presence of a potential, the asymptotic form of the scattering solutions to the Schrödinger equation becomes

$$\begin{aligned}\psi^{(+)}(\mathbf{r})&\overset{r\to\infty}{\to}\frac{1}{2ikr}\sum_l (2l+1)\\ &\quad\times P_l(\cos\theta)[e^{i(kr+2\delta_l(k))}-e^{-i(kr-l\pi)}]\\ &=e^{ikz}+\frac{e^{ikr}}{r}f(\theta),\end{aligned} \tag{6}$$

where

$$f(\theta)=\sum_l (2l+1)\frac{e^{i2\delta_l(k)}-1}{2ik}P_l(\cos\theta) \tag{7}$$

is the scattering amplitude, and $\delta_l(k)$ is the scattering phase shift introduced by the potential. Because the presence of the potential breaks the scale invariance, the appearance of an energy-dependent phase is permitted.

B. $\delta^2(\mathbf{r})$ potential

Now consider what happens in two spatial dimensions, for which the asymptotic form of the scattering solution is[12]

$$\psi^{(+)}(\mathbf{r})\overset{r\to\infty}{\to}e^{ikx}+\frac{e^{i(kr+\pi/4)}}{\sqrt{r}}f(\theta) \tag{8}$$

with

$$f(\theta)=-i\sum_{n=-\infty}^{\infty}\frac{e^{i2\delta_n(k)}-1}{\sqrt{2\pi k}}e^{in\theta} . \tag{9}$$

In this case if we include the potential energy

$$V(\mathbf{r})=\lambda\,\delta^2(\mathbf{r}), \tag{10}$$

the scale invariance is maintained—indeed, at the classical level, there is no scattering because no particles are deflected. However, the absence of scattering no longer holds once the theory is quantized. Actually this anomalous behavior should be expected because of the wave-like nature of the particles—an incident beam even with zero impact parameter can sense the presence of the potential spike at $\mathbf{r}=0$. Of course, this scattering will occur only in the $n=0$ channel, because wave functions with $n\neq0$ must vanish at the origin.

To see how the scattering arises, we examine a scattering solution. In momentum space the Schrödinger equation becomes

$$\frac{1}{2m}(\mathbf{p}^2-\mathbf{k}^2)\phi^{(+)}(\mathbf{p})=-\lambda\,\psi^{(+)}(\mathbf{r}=0), \tag{11}$$

where

$$\phi^{(+)}(\mathbf{p})=\int d^2r\,e^{-i\mathbf{r}\cdot\mathbf{p}}\psi^{(+)}(\mathbf{r}) \tag{12}$$

is the Fourier transform of the scattering wave function. The solution to Eq. (11) is

$$\phi^{(+)}(\mathbf{p})=(2\pi)^2\delta^2(\mathbf{p}-\mathbf{k})-\frac{2m\lambda\,\psi^{(+)}(\mathbf{r}=0)}{\mathbf{p}^2-\mathbf{k}^2-i\epsilon}. \tag{13}$$

For consistency, we require

$$\begin{aligned}\psi^{(+)}(\mathbf{r}=0)&=\int\frac{d^2p}{(2\pi)^2}\phi^{(+)}(\mathbf{p})\\ &=1-2m\lambda\,\psi^{(+)}(\mathbf{r}=0)\frac{1}{4\pi}\log-\frac{\Lambda^2}{k^2},\end{aligned} \tag{14}$$

where we have regulated the otherwise divergent momentum space integration by introducing a cutoff parameter Λ. Then

$$\psi^{(+)}(\mathbf{r}=0)=\frac{1}{1+(\lambda m/2\pi)(\log\Lambda^2/k^2+i\pi)}. \tag{15}$$

We can obtain the scattering amplitude by taking the inverse Fourier transform and find

$$\psi^{(+)}(\mathbf{r})=e^{ikx}-2m\lambda\,\psi^{(+)}(\mathbf{r}=0)\frac{i}{4}H_0^{(1)}(kr), \tag{16}$$

where

$$\begin{aligned}\frac{i}{4}H_0^{(1)}(kr)&=\int\frac{d^2p}{(2\pi)^2}e^{i\mathbf{p}\cdot\mathbf{r}}\frac{1}{\mathbf{p}^2-\mathbf{k}^2-i\epsilon}\\ &\overset{r\to\infty}{\to}\frac{1}{2\sqrt{2\pi kr}}e^{i(kr+\pi/4)}\end{aligned} \tag{17}$$

is the two-dimensional Green's function. If we compare Eq. (16) with the asymptotic form Eq. (8), we identify the scattering amplitude as

$$f(\theta)=-\frac{1}{\sqrt{2\pi k}}\frac{1}{\frac{1}{m\lambda}+\frac{1}{2\pi}\log(\Lambda^2/k^2)+i/2}. \tag{18}$$

To eliminate the dependence on the cutoff, we note that the scattering amplitude has a pole at

$$E_{\rm bs}=-\frac{\Lambda^2}{2m}e^{2\pi/\lambda m}, \tag{19}$$

indicating the presence of a (single) bound state. Using this binding energy as a parameter, we can rewrite Eq. (18) as

$$f(\theta)=\sqrt{\frac{2}{\pi k}}\frac{\pi}{\log\left(\frac{k^2}{-2mE_{\rm bs}}\right)-i\pi}, \tag{20}$$

which is now expressed only in terms of experimental quantities. Equivalently, we can characterize the scattering amplitude in terms of a phase shift

$$\delta_n(k)=\delta_{n,0}\cot^{-1}\left(\frac{1}{\pi}\right)\log\left(\frac{k^2}{-2mE_{\rm bs}}\right), \tag{21}$$

or in terms of the differential scattering cross section

$$\frac{d\sigma}{d\Omega}=\frac{2}{\pi k}\frac{1}{1+\pi^{-2}\log^2\left(\frac{k^2}{-2mE_{\rm bs}}\right)}. \tag{22}$$

The existence of this energy-dependent phase in the $n=0$ channel or of the fixed energy bound state is clear evidence of anomalous symmetry breaking.

The manifestation of anomalous symmetry breaking in quantum mechanics for the two-dimensional delta function interaction has been explored previously[10] and was reviewed to set the context for our primary topic—the $1/r^2$ potential. As we shall see, there is an anomaly in this case also, and although there are similarities to the $\delta^2(\mathbf{r})$ case, there are also important differences, one of which is the fact that there are experimental consequences.

C. $1/r^2$ potential

Consider the potential

$$V(r)=\frac{\lambda}{r^2} \tag{23}$$

in three spatial dimensions. We define the partial wave amplitude $R_l(r)\equiv u_l(r)/r$ and find

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)+2m\lambda}{r^2}-k^2\right]R_l(r)=0, \tag{24}$$

so that invariance under the scale transformation $r\to\mu r$, $k\to k/\mu$ again holds. It is clear that the solutions to Eq. (24) can be written in terms of Bessel functions

$$u_l(r)\sim\sqrt{kr}J_{\rho+1/2}(kr),\quad \sqrt{kr}N_{\rho+1/2}(kr), \tag{25}$$

where we have defined

$$\rho(\rho+1)\equiv l(l+1)+2m\lambda. \tag{26}$$

Because ρ is defined via a quadratic equation, it is apparent that the character of the solutions must change when the discriminant

$$D_l=(l+\tfrac{1}{2})^2+2m\lambda \tag{27}$$

becomes negative. For repulsive interactions, $\lambda>0$, there is nothing unusual. In general, the orders of the Bessel functions become irrational, but the problem can be solved as usual. Because the wave function must be regular at the origin, we must have

$$u_l(r)/r\propto\frac{1}{\sqrt{kr}}J_{\rho+1/2}(kr)\overset{r\to\infty}{\to}\frac{1}{kr}\sin\left(kr-\rho\frac{\pi}{2}\right), \tag{28}$$

and the scattering phase shift can be read off as

$$\delta_l=\left(l+\frac{1}{2}-\sqrt{\left(l+\frac{1}{2}\right)^2+2m\lambda}\right)\frac{\pi}{2}. \tag{29}$$

Because δ_l is independent of k, this result is consistent with the expected scale invariance.

Now consider the case of an attractive potential, $\lambda<0$. As long as the discriminant is positive, things go through as before. However, once $D_l<0$, the potential has overcome the centrifugal barrier, and the order ρ of the Bessel function becomes imaginary.[13] First consider the case of a bound state, in which case the boundary condition as $r\to\infty$ demands that the solution be constructed in terms of the Bessel function $K_{i\Xi_l}(\mu r)$, where we have defined $\sqrt{D_l}\equiv i\Xi_l$ and the binding energy as

$$E_{\rm bs}\equiv-\frac{\mu^2}{2m}. \tag{30}$$

The asymptotic behavior is then

$$u_l(r)\propto\sqrt{\mu r}K_{i\Xi_l}(\mu r)\overset{r\to\infty}{\to}\sqrt{\frac{\pi}{2}}e^{-\mu r}. \tag{31}$$

The allowed values of μ (and thereby of the binding energy) are determined by the boundary condition that the wave function vanishes at the origin. From the behavior

$$K_{i\Xi_l}(\mu r)\overset{r\to 0}{\to}-\sqrt{\frac{\pi}{\Xi_l\sinh(\pi\Xi_l)}}\sin\left[\Xi_l\log\left(\frac{\mu r}{2}\right)-\arg[\Gamma(1+i\Xi_l)]\right], \tag{32}$$

where arg indicates the phase of the following complex number, we see that the wave function goes through infinitely many zeroes as $r\to 0$. As a consequence, the spectrum becomes continuous and unbounded from below, implying that, despite its appearance, the Hamiltonian is not self-adjoint. Similarly, the phase shift of the scattering wave function as $r\to 0$ is undetermined.

One mathematically attractive solution to these problems is to define a so-called self-adjoint extension of the Hamiltonian by specifying a particular boundary condition at $r=0$.[12] For example, each member of the continuum of self-adjoint extensions of this Hamiltonian can be characterized by the (energy-independent) scattering phase shift as $r\to 0$.[14] We briefly discuss self-adjoint extensions in the applications of the $1/r^2$ potential, but this regularization method does not directly illustrate anomalous symmetry breaking.

We instead opt for an alternative route and introduce a short distance cutoff a and demand that the wave function vanish at this point, $u_l(r=a)=0$, and that the physics be independent of the choice of cutoff parameter.[13] This prescription yields the points

$$\mu_n a = 2\,\exp\left(\frac{\arg[\Gamma(1+i\Xi_l)]-n\pi}{\Xi_l}\right). \tag{33}$$

In order that $\mu_n a \to 0$, we require $\Xi_l \ll 1$ and, because $\arg[\Gamma(1+i\Xi_l)] = -\gamma\Xi_l + \mathcal{O}(\Xi_l)^2$, the energy becomes

$$E_{n,l} = -\frac{1}{2m}\left(\frac{2e^{-\gamma}}{a}\right)^2 e^{-2\pi n/\Xi_l}. \tag{34}$$

In order that the ground state $(n=1, l=0)$ energy remains finite and well-defined as $a\to 0$, we require $\Xi_{gs}=\Xi_{gs}(a)\to 0^+$, which demands the scaling behavior

$$\frac{2\pi}{\Xi_{gs}(a)} = -2\log\left(\frac{\mu a}{2}\right) - 2\gamma. \tag{35}$$

This relation does not predict the value of μ, but rather defines the scaling of $\Xi_{gs}(a)$ as $a\to 0$ in terms of the *experimental* value of $\mu=\sqrt{2mE_{gs}}$. The corresponding ground state wave function is

$$\Psi_{gs}(r) = \frac{\mu}{\sqrt{2\pi r}} K_0(\mu r)\,. \tag{36}$$

The very existence of a bound state implies the presence of an energy scale and the breaking of scale invariance as a result of quantization, just as in the case of the two-dimensional delta function—a quantum mechanical example of an anomaly!

Another similarity with the two-dimensional delta function potential is that there is but a single bound state. This similarity can be seen from the fact that because $\Xi_{gs}\to 0^+$ and $D_l > D_0$, the discriminant in any but the s-wave channel must be positive so that no anomaly (and no bound state) can occur. Similarly in the $l=0$ case but with $n>1$, we see from Eq. (34) that such states cannot have nonzero binding energies, because

$$\frac{E_{n,0}}{E_{1,0}} = e^{-2\pi(n-1)/\Xi_{gs}} \overset{\Xi_{gs}\to 0^+}{\longrightarrow} 0. \tag{37}$$

We summarize this discussion with the observation that from Eq. (27) and the following, there exists a critical value of the coupling constant, $2m\lambda = -\frac{1}{4}$, below which there exists a single bound state and above which there are no bound states.

We can also consider scattering in the presence of an anomaly. In this case the solutions of the partial wave equation must be linear combinations of $H_\rho^{(1)}(kr)$ and $H_\rho^{(2)}(kr)$:

$$u_l(r)/\sqrt{r} = A_1 H_\rho^{(1)}(kr) + A_2 H_\rho^{(2)}(kr). \tag{38}$$

From the asymptotic dependence

$$u_l(r)/\sqrt{r} \overset{r\to\infty}{\to} \sqrt{\frac{2}{\pi kr}}\,(A_1 e^{i(kr-(1/2)\rho\pi-\pi/4)} + A_2 e^{-i(kr-(1/2)\rho\pi-\pi/4)}), \tag{39}$$

then for the s-wave channel in which $\Xi_0\to 0^+$, we identify the scattering phase shift via

$$\frac{A_1}{A_2} = e^{i(2\delta_0 - \pi/2)}. \tag{40}$$

On the other hand, for small values of r, we find

$$\begin{aligned} u_0(r)/\sqrt{r} &= A_1(J_{i\Xi_0}(kr) + iN_{i\Xi_0}(kr)) + A_2(J_{i\Xi_0}(kr) - iN_{i\Xi_0}(kr)) \\ &\overset{r\to 0}{\to} A_1\left(\frac{(\frac{1}{2}kr)^{i\Xi_0}}{\Gamma(1+i\Xi_0)}\left(1+\frac{\cosh\pi\Xi_0}{\sinh\pi\Xi_0}\right) - \frac{1}{\sinh\pi\Xi_0}\,\frac{(\frac{1}{2}kr)^{-i\Xi_0}}{\Gamma(1-i\Xi_0)}\right) \\ &\quad + A_2\left(\frac{(\frac{1}{2}kr)^{i\Xi_0}}{\Gamma(1+i\Xi_0)}\left(1-\frac{\cosh\pi\Xi_0}{\sinh\pi\Xi_0}\right) + \frac{1}{\sinh\pi\Xi_0}\,\frac{(\frac{1}{2}kr)^{-i\Xi_0}}{\Gamma(1-i\Xi_0)}\right). \end{aligned} \tag{41}$$

The requirement that $u_0(r=a)=0$ yields

$$\begin{aligned} \frac{A_1}{A_2} &= \frac{e^{i\sigma}(1+i\xi)e^{-\pi\Xi_0} - e^{-i\sigma}(1-i\xi)}{e^{i\sigma}(1+i\xi)e^{\pi\Xi_0} - e^{-i\sigma}(1-i\xi)} \\ &= \frac{\xi + \tan\sigma + i\tanh\frac{1}{2}\pi\Xi_0(1-\xi\tan\sigma)}{\xi+\tan\sigma - i\tanh\frac{1}{2}\pi\Xi_0(1-\xi\tan\sigma)}, \end{aligned} \tag{42}$$

where we have defined

$$\begin{aligned} i\xi &= \frac{\Gamma(1-i\Xi_0)-\Gamma(1+i\Xi_0)}{\Gamma(1-i\Xi_0)+\Gamma(1+i\Xi_0)}, \\ e^{i\sigma} &= (\tfrac{1}{2}ka)^{i\Xi_0} = \exp(i\Xi_0\log\tfrac{1}{2}ka). \end{aligned} \tag{43}$$

If we compare Eq. (43) with Eq. (40), we can identify

$$\delta_0(k) - \frac{\pi}{4} = \tan^{-1}\frac{\tanh\frac{1}{2}\pi\Xi_0(1-\xi\tan\sigma)}{\xi+\tan\sigma}. \tag{44}$$

In the limit that $\Xi_0\to 0^+$, we then have

$$\tan\left(\delta_0(k)-\frac{\pi}{4}\right) \approx \frac{1}{2}\pi\Xi_0\,\frac{1-\gamma\Xi_0\log\frac{1}{2}ka}{\gamma\Xi_0 + \tan\Xi_0\log\frac{1}{2}ka}. \tag{45}$$

Using the scaling behavior

$$\Xi_0(a) = \frac{-\pi}{\log\frac{1}{2}\mu a} + \gamma, \tag{46}$$

we find that

$$\tan\left(\delta_0(k)-\frac{\pi}{4}\right) = \frac{1-\cot\delta_0(k)}{1+\cot\delta_0(k)} \approx \frac{\pi}{\log k^2/\mu^2}, \tag{47}$$

which yields the partial wave amplitude

$$ka_0(k) = \frac{1}{\cot\delta_0(k) - i} = \frac{1}{\left(\dfrac{\log\dfrac{k^2}{\mu^2}-\pi}{\log\dfrac{k^2}{\mu^2}+\pi}\right) - i}. \tag{48}$$

As a check we verify that that Eq. (48) has a pole at the bound state energy $k^2=2mE_{\rm bs}=-2m\mu^2$.

Because

$$\Xi_l=\sqrt{(l+\tfrac{1}{2})^2-2m|\lambda|}>\Xi_0=\sqrt{\tfrac{1}{4}-2m|\lambda|}\to 0, \tag{49}$$

we see that there is no anomaly in the non-s-wave states so that the scattering phase shift is given by its nonanomalous value given in Eq. (29). The form of the scattering amplitude is

$$f(\theta)=\frac{1}{k}\sum_{l=1}^{\infty}(2l+1)i^l\exp(i\delta_l\sin\delta_l)P_l(\cos\theta)+\frac{1}{k}\frac{1}{\left(\log\frac{k^2}{\mu^2}-\pi\Big/\log\frac{k^2}{\mu^2}+\pi\right)-i}, \tag{50}$$

where δ_l is given by Eq. (29).

D. An aside

Before proceeding to applications of this formalism, it is interesting (but unrelated to considerations of the anomaly) that there is another curious feature of the $1/r^2$ interaction, as previously pointed out by Kayser.[15] If one solves for the classical scattering angle for motion in the presence of such a potential, the result is

$$\theta_{\rm cl}=\pi\left(1-\frac{L}{\sqrt{L^2+2m\lambda}}\right), \tag{51}$$

where L is the angular momentum. Taking the interaction to be repulsive ($\lambda>0$) and solving for the cross section, we find

$$\frac{d\sigma_{\rm cl}}{d\Omega}=\frac{1}{p^2}\frac{L}{\sin\theta}\left|\frac{dL}{d\theta}\right|=\frac{\lambda}{2\pi E}\frac{1-x}{x^2(2-x)^2\sin\pi x}, \tag{52}$$

where $p=\sqrt{2mE}$ is the incoming momentum, and we have defined $x=\theta/\pi$. Thus the cross section is *linear* in the coupling constant, in apparent violation of the simple Born approximation result when the potential becomes weak. In Ref. 15 it is shown that the same cross section results from a quantum mechanical evaluation. The resolution of the apparent paradox lies in the fact that the classical scattering condition in Eq. (51) can be satisfied only when $2m\lambda\gg 1$. This condition can be seen from the result $L\theta\gg 1$ which implies that

$$2m\lambda\gg\frac{1}{\pi^2}\frac{2-x}{x(1-x)^2}. \tag{53}$$

Thus the classical result corresponds to the strong coupling regime, where the Born approximation is not appropriate.

III. APPLICATIONS

A remarkable feature of the above analysis of the $1/r^2$ potential is that this interaction is realized in nature. As has recently been emphasized, one such application is to that of a charge interacting with a point dipole.[16] Because the potential outside an electric dipole $\boldsymbol{p}$ is given by

$$\phi(\boldsymbol{r})=\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi r^3}, \tag{54}$$

the potential energy for a charge e in the vicinity is given by

$$V(\boldsymbol{r})=\frac{e\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi r^3}=\frac{\sigma\cos\theta}{r^2}, \tag{55}$$

which has the desired $1/r^2$ dependence. Point dipoles are not physical, but a good approximation is provided by a polar molecule such as water and the point charge can be taken to be a nearby electron. To analyze this system by means of the formalism developed in Sec. II, we write the bound state solution as in Ref. 16:

$$\psi(\boldsymbol{r})=\frac{1}{r}u(r)\Theta(\theta). \tag{56}$$

Then the equations determining the radial and angular dependence are given by

$$\left(-\frac{1}{2m}\frac{d^2}{dr^2}+\frac{\gamma}{2mr^2}\right)u(r)=Eu(r), \tag{57}$$

$$(\hat{L}^2+2m\sigma\cos\theta)\Theta(\theta)=\gamma\Theta(\theta), \tag{58}$$

and the separation constant γ is related to the actual coupling σ of the point-dipole potential by Eq. (58). If we use the normalized Legendre polynomials

$$\sqrt{\frac{(2l+1)}{2}}P_l(\cos\theta), \tag{59}$$

as a basis, Eq. (58b) can be written as a matrix equation

$$M_{ll'}\Theta_{l'}=0, \tag{60}$$

with

$$M_{ll'}=\delta_{ll'}(l(l+1)-\gamma)+2m\sigma\left[\frac{l}{\sqrt{(2l-1)(2l+1)}}\delta_{l-1,l'}+\frac{l+1}{\sqrt{(2l+1)(2l+3)}}\delta_{l+1,l'}\right]. \tag{61}$$

Equation (61) is an eigenvalue equation, for which the existence of a solution requires that

$$\det M_{ll'}=\det\begin{vmatrix}-\gamma & \frac{2m\sigma}{\sqrt{3}} & 0 & \cdots\\ \frac{2m\sigma}{\sqrt{3}} & 2-\gamma & \frac{4m\sigma}{\sqrt{15}} & \cdots\\ 0 & \frac{4m\sigma}{\sqrt{15}} & \cdots & \\ \cdots & \cdots & \cdots & \cdots\end{vmatrix}=0. \tag{62}$$

Equation (62) can be easily solved numerically by successively evaluating the $n\times n$ determinant, which converges extremely rapidly:

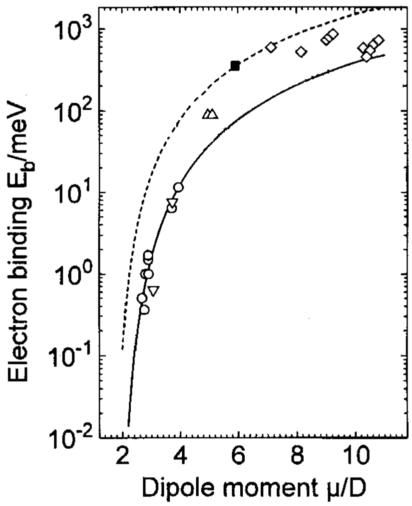

Fig. 1. Shown are data reported in Ref. 17 on experimental anion binding energies. Note that there seems to exist a limiting value $p_{\rm crit}\simeq 2D$, where $D=1$ debye$\equiv 1\times 10^{-18}$ esu cm. Because $ea_0\simeq 2.4$ D, we have $p_{\rm crit}\simeq 0.86 ea_0$.

$$n=2:\ (2m\sigma)^2=-3\gamma(2-\gamma),$$
$$n=3:\ (2m\sigma)^2=-3\gamma(2-\gamma)\left(\frac{1}{1-\frac{4\gamma}{5(6-\gamma)}}\right). \tag{63}$$

From our study of the $1/r^2$ potential in Sec. II C, we know that there exists a critical value of the coupling constant, $2m\lambda=-\frac{1}{4}$, below which there exists a bound state and above which there does not. Hence from Eq. (62), we observe that there must exist a critical value of the dipole moment σ^*, defined via

$$\det M_{ll'}(\gamma=-\tfrac{1}{4},2m\sigma^*)=0, \tag{64}$$

such that a bound electron-polar molecule state, an anion, can (cannot) exist for values of the dipole moment larger (smaller) than this critical value. From Eq. (63), we find $2m\sigma^*=1.279\ldots$, that is,

$$p^*=\frac{4\pi}{e}\sigma^*=0.640 ea_0, \tag{65}$$

where $a_0=1/m\alpha$ is the Bohr radius.

This prediction can be checked experimentally, and studies have been conducted attempting to measure the binding energies of anion systems.[17] It has indeed been found that a minimum dipole moment exists, as shown in Fig. 1; there does seem to exist a minimum dipole moment $p^*_{\rm exp}\simeq 0.86 ea_0$, as would be expected from the above considerations. Note that there is a bit of a subtlety here in that a real molecule is *not* a point dipole. Rather the potential for a finite dipole can be written as

$$V(\boldsymbol{r})=\frac{qe}{4\pi}\left(\frac{1}{R_+}-\frac{1}{R_-}\right)=\frac{ep\cos\theta}{4\pi r^2}+U_{\rm br}(\boldsymbol{r}), \tag{66}$$

which is the superposition of a pure point dipole interaction with a scale symmetry-breaking interaction $U_{\rm br}(\boldsymbol{r})$. Because experimentally such an explicitly symmetry breaking term cannot be eliminated, the observed critical moment is due to a combination of the anomalous and explicit symmetry breaking effects. It is clear that the presence of explicit breaking does not affect the existence of a critical moment, as expected from the quantum mechanical anomaly. However, the size of p^* *is* affected, as can be seen from the comparison of experimental and theoretical sizes of the critical moment. This difference is perhaps not surprising, because in QCD, the combination of explicit and anomalous symmetry breaking results in a value for the $\pi^0\to\gamma\gamma$ decay rate very near that predicted by the anomaly.[9]

It is interesting that a second application of the $1/r^2$ potential has recently been discussed. In this case it involves the interpretation of an experiment involving the interaction of a neutral but polarizable atom with a charged wire.[18] Because the electric field generated by the wire falls off linearly with distance, the induced electric dipole moment has the form

$$\boldsymbol{p}=4\pi\alpha_E\boldsymbol{E}\propto\frac{\alpha_E}{r}, \tag{67}$$

where α_E is the electric polarizability. The corresponding interaction potential is

$$U(r)=-\frac{1}{2}\,\boldsymbol{p}\cdot\boldsymbol{E}\propto-\frac{\alpha_E}{r^2}, \tag{68}$$

and falls off as $1/r^2$. The issue here is not the experimental appearance of a critical value of the coupling constant because the attractive $1/r^2$ potential is always critical in this cylindrical *two-dimensional* problem,[13] no matter what the strength of the electric field produced by the charged wire. The intriguing aspect of this experiment is that the atoms are observed to disappear from the system with an absorption cross section which can be fitted by a classical argument.[18] An analogous treatment of the two-dimensional $1/r^2$ potential, regularized and renormalized as done here, would have only bound state and elastic scattering solutions, and no absorption. Bawin and Coon utilized a method suggested by Radin[19] to sum over the infinite number of elastic scattering solutions of the self-adjoint extensions of the $1/r^2$ interaction to obtain a quantum mechanical expression that displayed absorption.[20] As the parameters of this initial experiment corresponded to the classical limit established by Kayser,[15] the classical limit of their quantum mechanical treatment recovered the experimental situation. However, further discussion is beyond the scope of this paper.

IV. CONCLUSIONS

The phenomenon of symmetry breaking is universal in physics, but although explicit and spontaneous violations are well known and are generally treated in introductory courses, the same cannot be said of quantum mechanical or anomalous symmetry breaking, where a symmetry is valid at the classical level but is violated due to quantization of the theory. This omission is presumably due to the fact that most familiar manifestations of the anomaly, for example, the two-photon decay of the neutral pion, occur in the context of quantum field theory and belong in the realm of elementary particle physics. However, this need not be the case. We have argued here that one can present this subject within the realm of ordinary quantum mechanics by studying the violation of scale symmetry, $r\to\mu r$, $k\to k/\mu$. In earlier papers in this journal, the case of a two-dimensional delta function was examined.[10] However, this case is purely an intellectual exercise and has no experimental consequences. In this paper, we have examined the $1/r^2$ potential, which is anomalous and which *does* have experimental ramifications in the existence of a critical dipole moment allowing the binding of anions in atomic physics[17] and in the recent study of the interaction of a polarizable atom with a charged wire.[18]

Finally, it has long been known that the main features of the quantum mechanical three-body bound state and some striking aspects of nuclear three-body scattering are due to an effective $1/r^2$ potential, built from the relative distances between the particles, including mass factors where appropriate.[21] This effective potential is displayed most clearly in recent effective field theory discussions of three-body systems, where it also exhibits scale invariance violation,[22] but this discussion is clearly beyond the level of our paper. In any case we believe that the above discussion is at a level that is appropriate in an advanced quantum mechanics course, and it is hoped that this paper will encourage the introduction of this fascinating topic into such a venue.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge very interesting discussions with Boris Kayser and with Michel Bawin about this material. This work was supported in part by the National Science Foundation under Award Nos. NSF-PHY-98-01875 and NSF-PHY-00-70938.

[a]Electronic mail: coon@barbara.nmsu.edu
[b]Electronic mail: holstein@physics.umass.edu

[1]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw–Hill, New York, 1985); N. Byers, ''E. Noether's discovery of the deep connection between symmetries and conservation laws,'' physics/9807044.
[2]See for example, M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (Benjamin, New York, 1964).
[3]See for example, B. R. Holstein, *Weak Interactions in Nuclei* (Princeton U.P., Princeton, 1989).
[4] H. Goldstein, ''Prehistory of the 'Runge–Lenz' vector,'' Am. J. Phys. **43**, 737–738 (1975); G. Baym, *Lectures in Quantum Mechanics* (Benjamin, Reading, MA, 1969).
[5]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison–Wesley, Reading, MA, 1964), Vol. II, Chap. 36.
[6]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison–Wesley, Reading, MA, 1964), Vol. III, Chap. 21.
[7]N. Straumann, ''Historical and other remarks on hidden symmetries,'' in *Proceedings of the Summer School on Hidden Symmetries and Higgs Phenomena*, edited by D. Graudenz (PSI, Villigen, Switzerland, 1998).
[8]See, for example, S. B. Treiman, R. Jackiw, B. Zumino, and E. Witten, *Current Algebras and Anomalies* (World Scientific, Singapore, 1985).
[9]See, for example, J. F. Donoghue, E. Golowich, and B. R. Holstein, *Dynamics of the Standard Model* (Cambridge U.P., New York, 1992), Chap. III.
[10]L. R. Mead and J. Godines, ''An analytic example of renormalization in two dimensional quantum mechanics,'' Am. J. Phys. **59**, 935 (1991); B. R. Holstein, ''Anomalies for pedestrians,'' *ibid.* **61**, 142–147 (1993).
[11]R. Jackiw, ''Introducing scale symmetry,'' Phys. Today **25**(1), 23–27 (1972).
[12]R. Jackiw, ''Delta-function potentials in two- and three-dimensional quantum mechanics,'' in *M. A. B. Beq Memorial Volume*, edited by A. Ali and P. Hoodbhoy (World Scientific, Singapore, 1991), pp. 25–42; I. Lapidus, ''Quantum-mechanical scattering in two dimensions,'' Am. J. Phys. **50**, 45–47 (1982).
[13]Here we follow the approach of H. E. Camblong, L. N. Epele, H. Fanchiotti, and C. A. Garcia Canal, ''Renormalization of the inverse square potential,'' Phys. Rev. Lett. **85**, 1590–1593 (2000); ''Dimensional transmutation and dimensional regularization in quantum mechanics. I. General theory,'' Ann. Phys. (N.Y.) **287**, 14–56 (2001); ''Dimensional transmutation and dimensional regularization in quantum mechanics. II. Rotational invariance,'' **287**, 57–100 (2001).
[14]A. M. Perelomov and V.S. Popov, '' 'Fall to the Center' in quantum mechanics,'' Teor. Mat. Fiz. **4**, 48 (1970) [Theor. Math Phys. **4**, 664–677 (1970)].
[15]B. Kayser, ''Classical limit of scattering in a $1/r^2$ potential,'' Am. J. Phys. **42**, 960–964 (1974).
[16]H. E. Camblong, L. N. Epele, H. Fanchiotti, and C. A. Garcia Canal, ''Quantum anomaly in molecular physics,'' Phys. Rev. Lett. **87**, 220402 (2001).
[17]C. Desfrancois, H. Abdoul-Carime, N. Khelifa, and J. P. Schermann, ''From $1/r$ to $1/^2$ potentials: Electron exchange between Rydberg atoms and polar molecules,'' Phys. Rev. Lett. **73**, 2436–2439 (1994); J.-M. Levy-Leblond, ''Electron capture by polar molecules,'' Phys. Rev. **153**, 1–4 (1967).
[18]J. Denschlag, G. Umshaus, and J. Schmiedmayer, ''Probing a singular potential with cold atoms: A neutral atom and a charged wire,'' Phys. Rev. Lett. **81**, 737–741 (1998).
[19]C. Radin, ''Some remarks on the evolution of a Schrödinger particle in an attractive $1/r^2$ potential,'' J. Math. Phys. **16**, 544–547 (1975).
[20]M. Bawin and S. A. Coon, ''A neutral atom and a charged wire: From elastic scattering to absorption,'' Phys. Rev. A **63**, 1–3 (2001).
[21]For a review, see, for example, V. Efimov, ''Is a qualitative approach to the three-body problem useful?,'' Comments Nucl. Part. Phys. **19**, 271–294 (1990).
[22]P. F. Bedaque, H.-W. Hammer, and U. Van Kolck, ''The three-boson system with short-range interactions,'' Nucl. Phys. A **646**, 444–466 (1999); ''Effective theory of the triton,'' **676**, 357–370 (2000).