A second order system of PDEs in Boundary Layer Theory reducible to Abel's equation. A Lie Symmetry Analysis

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Abstract

We perform a Lie symmetry analysis of a second order system of partial differential equations known from the boundary layer theory in fluid mechanics. We first determine the infinitesimals of the underlying symmetries of the system, and then find its invariants. The solution of the original system is reduced into a nonlinear, third order ODE which in turn is being converted, through Nucci's reduction, into a second order nonlinear ODE. This last ODE is then reduced into Abel's second kind differential equation, accessible by Panayotounakos algorithm.

Keywords: Lie Symmetries, Systems of Partial Differential Equations, Exact Solutions, Boundary Layer Theory, Nucci Reduction, Abel's equation, Panayotounakos Algorithm.

1 Introduction

Lie symmetry method, introduced by the Norwegian mathematician S. Lie, is a powerfull method of solving differential equations and systems of diferential equations. It has been used in an enormous number of problems modern scince and engineering faces. Some references explaining the main ingredients of the method appear at the end of the present article.
The main purpose of this article is to find some explicit solutions to the system of PDEs

$$u_x + v_y = 0, \quad uu_x + vu_y = u_{yy} \tag{1.1}$$

which appears in Boundary Layer Theory of the Mechanics of Fluids, using the Lie symmetry method.
Using Lie symmetry analysis, we find that this system admits five symmetry generators. Introducing a potential function $w(x, y)$, we reduce the system into a second order nonlinear PDE. Considering the invariant surface conditions, we find the invariants and the reduce the system into a nonlinear, third order ODE which is the reduced, through Nucci's reduction into a second order ODE. This last ODE is then reduced (through Lie point symmetries) to an Abel equation which in turn can be solved using the **Panayotounakos algorithm**.
The paper is organized as follows:
In section 2 we review the invariance of second order systems of PDEs under Lie point symmetries and calculate explicitly the infinitesimal coefficients. In section 3 we determine the generators of Lie symmetries of the system and find a closed-form solution of the system. The Panayotounakos algorithm is listed in Appendix A. In Appendix B we provide some useful transformations needed to transform Abel's equation.

2 Systems of Partial Differential Equations of the Second Order. Preliminaries

We consider a system of PDEs of the second order

$$F_1(x, y, u, v, u_x, v_x, u_y, v_y, u_{xx}, ..., v_{yy}) = 0 \tag{2.1}$$

$$F_2(x, y, u, v, u_x, v_x, u_y, v_y, u_{xx}, ..., v_{yy}) = 0 \tag{2.2}$$

Invariance of the system under infinitesimal point symmetry transformations is being expressed by the conditions

$$\Gamma^{(2)}(F_1)|_{F_1=0,\,F_2=0} = 0, \quad \Gamma^{(2)}(F_2)|_{F_1=0,\,F_2=0} = 0 \tag{2.3}$$

where the operator Γ is defined by

$$\Gamma = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} \tag{2.4}$$

and $\Gamma^{(1)}$, $\Gamma^{(2)}$ are the extended operators of the first and second order, defined by

$$\Gamma^{(1)} = \Gamma + U^{[x]}\frac{\partial}{\partial u_x} + U^{[y]}\frac{\partial}{\partial u_y} + V^{[x]}\frac{\partial}{\partial v_x} + V^{[y]}\frac{\partial}{\partial v_y} \tag{2.5}$$

$$\begin{aligned}\Gamma^{(2)} = \Gamma^{(1)} &+ U^{[xx]}\frac{\partial}{\partial u_{xx}} + U^{[xy]}\frac{\partial}{\partial u_{xy}} + U^{[yy]}\frac{\partial}{\partial u_{yy}} \\ &+ V^{[xx]}\frac{\partial}{\partial v_{xx}} + V^{[xy]}\frac{\partial}{\partial v_{xy}} + V^{[yy]}\frac{\partial}{\partial v_{yy}}\end{aligned} \tag{2.6}$$

We also have the following expressions for the extended infinitesimals

$$U^{[x]} = D_x(U) - u_x D_x(X) - u_y D_x(Y) \tag{2.7}$$

$$U^{[y]} = D_y(U) - u_x D_y(X) - u_y D_y(Y) \tag{2.8}$$

$$V^{[x]} = D_x(V) - v_x D_x(X) - v_y D_x(Y) \tag{2.9}$$

$$V^{[y]} = D_y(V) - v_x D_y(X) - v_y D_y(Y) \tag{2.10}$$

and

$$U^{[xx]} = D_x(U^{[x]}) - u_{xx} D_x(X) - u_{xy} D_x(Y) \tag{2.11}$$

$$U^{[xy]} = D_x(U^{[y]}) - u_{xy} D_x(X) - u_{yy} D_x(Y) \tag{2.12}$$

$$U^{[yy]} = D_y(U^{[y]}) - u_{xy} D_y(X) - u_{yy} D_y(Y) \tag{2.13}$$

$$V^{[xx]} = D_x(V^{[x]}) - v_{xx} D_x(X) - v_{xy} D_x(Y) \tag{2.14}$$

$$V^{[xy]} = D_x(V^{[y]}) - v_{xy} D_x(X) - v_{yy} D_x(Y) \tag{2.15}$$

$$V^{[yy]} = D_y(V^{[y]}) - v_{xy} D_y(X) - v_{yy} D_y(Y) \tag{2.16}$$

The total differential operators D_x and D_y are defined by

$$\begin{aligned} D_x = {} & \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + \cdots + \\ & + v_x \frac{\partial}{\partial v} + v_{xx} \frac{\partial}{\partial v_x} + v_{xy} \frac{\partial}{\partial v_y} + \cdots \end{aligned} \tag{2.17}$$

$$\begin{aligned} D_y = {} & \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y} + \cdots + \\ & + v_y \frac{\partial}{\partial v} + v_{xy} \frac{\partial}{\partial v_x} + v_{yy} \frac{\partial}{\partial v_y} + \cdots \end{aligned} \tag{2.18}$$

We shall calculate explicitly some of the coefficients $U^{[x]}$, $V^{[x]}$, etc. We have for example

$$\begin{aligned} & U^{[x]} = D_x(U) - u_x D_x(X) - u_y D_x(Y) \\ & = \Big(\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v}\Big) U - u_x \Big(\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v}\Big) X \\ & - u_y \Big(\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v}\Big) Y \\ & = U_x + (U_u - X_x) u_x - (u_x)^2 X_u - u_x v_x X_v - u_x u_y Y_u \\ & \quad - u_y v_x Y_v - u_y Y_x - u_x u_t T_u - u_t v_x T_v - u_t T_x + v_x U_v \end{aligned} \tag{2.19}$$

$$\begin{aligned} & U^{[y]} = D_y(U) - u_x D_y(X) - u_y D_y(Y) \\ & = \Big(\frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v}\Big) U - u_x \Big(\frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v}\Big) X \\ & - u_y \Big(\frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v}\Big) Y \\ & = U_y + (U_u - Y_y) u_y - (u_y)^2 Y_u - u_t T_y + (U_v - u_t T_v) v_y \\ & \quad - (u_y X_u + v_y X_v + X_y) u_x - (v_y Y_v + u_t T_u) u_y \end{aligned} \tag{2.20}$$

$$\begin{aligned} & V^{[y]} = D_y(V) - v_x D_y(X) - v_y D_y(Y) \\ & = \Big(\frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v}\Big) V - v_x \Big(\frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v}\Big) X \\ & - v_y \Big(\frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v}\Big) Y \\ & = V_y + (V_v - Y_y) v_y - (v_y)^2 Y_v + (V_u - v_t T_u) u_y \\ & - (u_y X_u + v_y X_v + X_y) v_x - (u_y Y_u + v_t T_v) v_y - v_t T_y \end{aligned} \tag{2.21}$$

and

$$
\begin{aligned}
U^{[xx]} &= D_x(U^{[x]}) - u_{xx}D_x(X) - u_{xy}D_x(Y)\\
&= \Big(\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xy}\frac{\partial}{\partial u_y} + v_x\frac{\partial}{\partial v} + v_{xx}\frac{\partial}{\partial v_x} + v_{xy}\frac{\partial}{\partial v_y}\Big)U^{[x]}\\
&- u_{xx}\Big(\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v}\Big)X - u_{xy}\Big(\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v}\Big)Y\\
&= U_{xx} + (2U_{xu} - X_{xx})u_x + (U_{uu} - 2X_{xu})(u_x)^2 + (U_u - 2X_x)u_{xx}\\
&\quad - X_{uu}(u_x)^3 - 2(u_xT_u + v_xT_v + T_x)u_{xt} - (u_{xx}T_u + v_{xx}T_v + T_{xx})u_t\\
&\quad - 2(u_xv_xT_{uv} + u_xT_{xu} + v_xT_{xv})u_t - [(u_x)^2T_{uu} + (v_x)^2T_{vv}]u_t\\
&\quad - (2v_xX_v + 3u_xX_u + u_yY_u)u_{xx} - 2(u_xY_u + v_xY_v + Y_x)u_{xy}\\
&\quad + (U_v - u_yX_v - u_xX_v)v_{xx} - (2v_xX_{uv} + u_yY_{uu})(u_x)^2\\
&\quad + (U_{vv} - u_xX_{vv} - u_yY_{vv})(v_x)^2 + 2(U_{uv} - X_{xv})u_xv_x\\
&\quad - 2(v_xY_{uv} + Y_{xu})u_xu_y - (2v_xY_{xv} + Y_{xx})u_y + 2U_{xv}v_x
\end{aligned}
\tag{2.22}
$$

$$
\begin{aligned}
U^{[yy]} &= D_y(U^{[y]}) - u_{xy}D_y(X) - u_{yy}D_y(Y)\\
&= \Big(\frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{xy}\frac{\partial}{\partial u_x} + u_{yy}\frac{\partial}{\partial u_y} + v_y\frac{\partial}{\partial v} + v_{xy}\frac{\partial}{\partial v_x} + v_{yy}\frac{\partial}{\partial v_y}\Big)U^{[y]}\\
&- u_{xy}\Big(\frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + v_y\frac{\partial}{\partial v}\Big)X - u_{yy}\Big(\frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + v_y\frac{\partial}{\partial v}\Big)Y\\
&= U_{yy} + (2U_{yu} - Y_{yy})u_y + (U_u - 2Y_y)u_{yy} + (U_{uu} - 2Y_{yu})(u_y)^2\\
&\quad - Y_{uu}(u_y)^3 - 2(u_yT_u + v_yT_v + T_y)u_{yt} - (u_{yy}T_u + v_{yy}T_v + T_{yy})u_t\\
&\quad - 2(u_yv_yT_{uv} + u_yT_{yu} + v_yT_{yv})u_t - [(u_y)^2T_{uu} + (v_y)^2T_{vv}]u_t\\
&\quad - (2v_yY_v + 3u_yY_u + u_xX_u)u_{yy} - 2(u_yX_u + v_yX_v + X_y)u_{xy}\\
&\quad + (U_v + u_xX_v + u_yY_v)v_{yy} - (2v_yY_{uv} + u_xX_{uu})(u_y)^2\\
&\quad + (U_{vv} - u_xX_{vv} - u_yY_{vv})(v_y)^2 + 2(U_{uv} - Y_{yv})u_yv_y\\
&\quad - 2(v_yX_{uv} + X_{yu})u_xu_y - (2v_yX_{yv} + X_{yy})u_x + 2U_{yv}v_y
\end{aligned}
\tag{2.23}
$$

3 A second order system of PDEs from Boundary Layer Theory

We consider the following system of PDEs

$$
u_x + v_y = 0, \quad uu_x + vu_y = u_{yy} \tag{3.1}
$$

which appears in Boundary Layer Theory of the Mechanics of Fluids.
2.1. Lie Symmetry Analysis of the system.
We denote by Δ_1 and Δ_2 the quantities

$$\Delta_1 = u_x + v_y \quad \text{and} \quad \Delta_2 = uu_x + vu_y - u_{yy} \tag{3.2}$$

Invariance of the first equation is being expressed by

$$\Gamma^{(2)}(\Delta_1) = 0 \Leftrightarrow U^{[x]} + V^{[y]} = 0 \tag{3.3}$$

where $\Gamma^{(2)}$ is given by (21.6). Substituting the coefficients $U^{[x]}$ and $V^{[y]}$ given by (21.20) and (21.22) into the above equation, we obtain the equation

$$\begin{aligned} &U_x + (U_u - X_x)u_x - (u_x)^2 X_u - u_x v_x X_v - u_x u_y Y_u \\ &- u_y v_x Y_v - u_y Y_x + v_x U_v \\ &+ V_y + (V_v - Y_y)v_y - (v_y)^2 Y_v + V_u u_y \\ &- (u_y X_u + v_y X_v + X_y)v_x - u_y v_y Y_u = 0 \end{aligned} \tag{3.4}$$

In the above equation, we substitute v_y by $-u_x$ and u_{yy} by $uu_x + vu_y$, to account for the conditions $\Delta_1 = 0$ and $\Delta_2 = 0$. We thus see that the condition $\Gamma^{(2)}(\Delta_1)|_{\Delta_1=0,\,\Delta_2=0} = 0$ is equivalent to the equation

$$\begin{aligned} &U_x + (U_u - X_x)u_x - (u_x)^2 X_u - u_x v_x X_v - u_x u_y Y_u \\ &- u_y v_x Y_v - u_y Y_x + v_x U_v \\ &+ V_y + (V_v - Y_y)(-u_x) - (-u_x)^2 Y_v + V_u u_y \\ &- (u_y X_u + (-u_x)X_v + X_y)v_x - u_y(-u_x)Y_u = 0 \end{aligned} \tag{3.5}$$

We now have to set to zero all the coefficients of the partial derivatives in order to determine the quantities X, Y, U and V. Before that, the above equation can be written as

$$\begin{aligned} &U_x + V_y + (U_u - X_x)u_x - (V_v - Y_y)u_x \\ &- u_y v_x Y_v + v_x U_v - (u_y X_u + X_y)v_x \\ &- (u_x)^2 X_u - (u_x)^2 Y_v + V_u u_y - Y_x u_y = 0 \end{aligned} \tag{3.6}$$

We are now in a position to equate to zero the various coefficients of the partial derivatives. We have:
Coefficient of u_x

$$U_u - X_x - V_v + Y_y = 0 \tag{3.7}$$

Coefficient of u_y

$$V_u - Y_x = 0 \tag{3.8}$$

Coefficient of $(u_x)^2$

$$X_u + Y_v = 0 \tag{3.9}$$

Coefficient of v_x

$$U_v - X_y = 0 \tag{3.10}$$

Coefficient of $u_y v_x$

$$X_u + Y_v = 0 \tag{3.11}$$

The constant term

$$U_x + V_y = 0 \tag{3.12}$$

Invariance of the second equation is being expressed by

$$\Gamma^{(2)}(\Delta_2) = 0 \Leftrightarrow U u_x + u U^{[x]} + V u_y + v U^{[y]} - U^{[yy]} = 0 \tag{3.13}$$

where $\Gamma^{(2)}$ is given by (21.6). Substituting the coefficients $U^{[x]}$, $U^{[y]}$ and $U^{[yy]}$ given by (21.20), (21.21) and (21.24) respectively, we obtain the equation

$$\begin{aligned}
&U u_x + u\Big[U_x + (U_u - X_x)u_x - (u_x)^2 X_u - u_x v_x X_v - u_x u_y Y_u \\
&\qquad - u_y v_x Y_v - u_y Y_x + v_x U_v\Big] + V u_y \\
&+ v\Big[U_y + (U_u - Y_y)u_y - (u_y)^2 Y_u + U_v v_y \\
&\quad - (u_y X_u + v_y X_v + X_y)u_x - u_y v_y Y_v\Big] \\
&- \Big[U_{yy} + (2U_{yu} - Y_{yy})u_y + (U_u - 2Y_y)u_{yy} + (U_{uu} - 2Y_{yu})(u_y)^2 \\
&\quad - Y_{uu}(u_y)^3 \\
&\quad - (2v_y Y_v + 3u_y Y_u + u_x X_u)u_{yy} - 2(u_y X_u + v_y X_v + X_y)u_{xy} \\
&\quad + (U_v + u_x X_v + u_y Y_v)v_{yy} - (2v_y Y_{uv} + u_x X_{uu})(u_y)^2 \\
&\quad + (U_{vv} - u_x X_{vv} - u_y Y_{vv})(v_y)^2 + 2(U_{uv} - Y_{yv})u_y v_y \\
&\quad - 2(v_y X_{uv} + X_{yu})u_x u_y - (2v_y X_{yv} + X_{yy})u_x + 2U_{yv} v_y\Big] = 0
\end{aligned} \tag{3.14}$$

In the above equation, we substitute v_y by $-u_x$ and u_{yy} by $uu_x + vu_y$, to account for the conditions $\Delta_1 = 0$ and $\Delta_2 = 0$. We thus see that the condition $\Gamma^{(2)}(\Delta_2)|_{\Delta_1=0,\, \Delta_2=0} = 0$ is equivalent to the equation

$$
\begin{aligned}
&Uu_x + u\Big[U_x + (U_u - X_x)u_x - (u_x)^2X_u - u_xv_xX_v - u_xu_yY_u \\
&\qquad - u_yv_xY_v - u_yY_x + v_xU_v\Big] + Vu_y \\
&+ v\Big[U_y + (U_u - Y_y)u_y - (u_y)^2Y_u - U_vu_x \\
&\qquad - (u_yX_u - X_vu_x + X_y)u_x + u_xu_yY_v\Big] \\
&- \Big[U_{yy} + (2U_{yu} - Y_{yy})u_y + (U_u - 2Y_y)(uu_x + vu_y) \\
&\quad + (U_{uu} - 2Y_{yu})(u_y)^2 - Y_{uu}(u_y)^3 \\
&\quad - (-2Y_vu_x + 3u_yY_u + u_xX_u)(uu_x + vu_y) \\
&\quad - 2(u_yX_u - X_vu_x + X_y)u_{xy} \\
&\quad + (U_v + u_xX_v + u_yY_v)v_{yy} - (2v_yY_{uv} + u_xX_{uu})(u_y)^2 \\
&\quad + (U_{vv} - u_xX_{vv} - u_yY_{vv})(u_x)^2 - 2(U_{uv} - Y_{yv})u_xu_y \\
&\quad - 2(-X_{uv}u_x + X_{yu})u_xu_y - (-2X_{yv}u_x + X_{yy})u_x - 2U_{yv}u_x\Big] = 0
\end{aligned}
\tag{3.15}
$$

We now have to set to zero the coefficients of the partial derivatives of the functions u and v. Before that, we see that some coefficients can be identified more easily compared to some others. For example the coefficient of $(u_y)^3$ is Y_{uu}, the coefficient of u_{xy} is $2X_y$, the coefficient of u_xu_{xy} is $-2X_v$, the coefficient of u_yu_{xy} is $2X_u$, the coefficient of v_{yy} is $-U_v$, the coefficient of u_xv_{yy} is $-X_v$, the coefficient of u_yv_{yy} is $-Y_v$, the coefficient of $(u_y)^2$ is $2vY_u - (U_{uu} - 2Y_{yu})$, the coefficient of $u_x(u_y)^2$ is X_{uu}, and the coefficient of $v_y(u_y)^2$ is $2Y_{uv}$. Therefore we have the following equations

$$Y_{uu} = 0 \tag{3.16}$$

$$X_y = 0, \quad X_u = 0, \quad X_v = 0 \tag{3.17}$$

$$U_v = 0, \quad Y_v = 0 \tag{3.18}$$

$$Y_{uv} = 0, \quad X_{uu} = 0, \quad 2vY_u - (U_{uu} - 2Y_{yu}) = 0 \tag{3.19}$$

Because of the above four systems, equation (21.39) becomes

$$\begin{aligned}
&Uu_x + u\Big[U_x + (U_u - X_x)u_x - (u_x)^2 X_u - u_x v_x X_v - u_x u_y Y_u \\
&\qquad\qquad - u_y v_x Y_v - u_y Y_x + v_x U_v\Big] + V u_y \\
&+ v\Big[U_y + (U_u - Y_y)u_y - U_v u_x \\
&\qquad - (u_y X_u - X_v u_x + X_y)u_x + u_x u_y Y_v\Big] \\
&- \Big[U_{yy} + (2U_{yu} - Y_{yy})u_y + (U_u - 2Y_y)(uu_x + vu_y) \\
&\quad - (-2Y_v u_x + 3u_y Y_u + u_x X_u)(uu_x) - (-2Y_v u_x + u_x X_u)(vu_y) \\
&\quad + (U_{vv} - u_x X_{vv} - u_y Y_{vv})(u_x)^2 - 2(U_{uv} - Y_{yv})u_x u_y \\
&\quad - 2(-X_{uv}u_x + X_{yu})u_x u_y - (-2X_{yv}u_x + X_{yy})u_x - 2U_{yv}u_x\Big] = 0
\end{aligned} \tag{3.20}$$

From the above equation we obtain:
Coefficient of $(u_x)^3$ is equal to X_{vv}, coefficient of $u_x v_x$ is equal to $-uX_v$, coefficient of v_x is equal to uU_v, coefficient of $u_y v_x$ is equal to $-uY_v$. We thus obtain the system

$$X_v = 0, \quad U_v = 0 \quad \text{and} \quad Y_v = 0 \tag{3.21}$$

We further equate to zero the coefficients of $(u_x)^2$ and u_x and obtain the two equations

$$vX_v - 2uY_v - U_{vv} - 2X_{yv} = 0 \tag{3.22}$$

and

$$U - uX_x - vU_v - vX_y + 2uY_y + X_{yy} + 2U_{yv} = 0 \tag{3.23}$$

respectively. We then equate to zero the terms which are independent of the partial derivatives of the two functions. We obtain the equation

$$uU_x + vU_y - U_{yy} = 0 \tag{3.24}$$

Equation (21.44), because of the previous four systems, gets simplified into

$$
\begin{aligned}
&- u(u_x u_y Y_u + u_y Y_x) + V u_y \\
&+ v\Big[(U_u - Y_y)u_y - X_u u_x u_y + u_x u_y Y_v\Big] \\
&- \Big[(2U_{yu} - Y_{yy})u_y + v(U_u - 2Y_y)u_y \\
&\quad - (3u_y Y_u)(uu_x) - (-2Y_v u_x + u_x X_u)(vu_y) \\
&\quad - 2(U_{uv} - Y_{yv})u_x u_y - 2(X_{yu})u_x u_y\Big] = 0
\end{aligned}
\tag{3.25}
$$

Equating to zero the coefficient of u_y, we obtain the equation

$$-uY_x + V - 2U_{yu} + Y_{yy} + vY_y = 0 \tag{3.26}$$

We finally equate to zero the coefficient of $u_x u_y$ and obtain the equation

$$2uY_u - vY_v + 2U_{uv} - 2Y_{yv} + 2X_{yu} = 0 \tag{3.27}$$

We now have to solve the systems of equations (21.40)-(21.43), (21.45)-(21.48) and (21.50)-(21.51).
Since $X_u = 0$ and $X_y = 0$ from (21.41) and $Y_v = U_v = 0$ from (21.42), we obtain from (21.51) that $Y_u = 0$. Because of

$$X_u = 0, \quad X_y = 0, \quad Y_u = 0 \tag{3.28}$$

equation (21.47) takes the form

$$U - uX_x + 2uY_y = 0 \tag{3.29}$$

From (21.34), (21.36) and (21.48) we obtain

$$X_{xx} = 0, \quad Y_{xy} = 0, \quad Y_{yy} = 0 \tag{3.30}$$

Therefore

$$X = a_1 x + a_2 \quad \text{and} \quad Y = a_3 y + a(x) \tag{3.31}$$

where $a(x)$ is an arbitrary function. Finally solving (21.53) with respect to U and (21.50) with respect to V and substituting the known functions, we find

$$U = uX_x - 2uY_y = a_1 u - 2a_3 u = (a_1 - 2a_3)u \tag{3.32}$$

and

$$V = uY_x + 2U_{yu} - Y_{yy} - vY_y = a'(x)u - a_3 v \tag{3.33}$$

respectively.
We are now in a position to obtain the explicit expression of the operator Γ, defined in (21.4):

$$\begin{aligned}
\Gamma &= X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} \\
&= (a_1 x + a_2)\frac{\partial}{\partial x} + (a_3 y + a(x))\frac{\partial}{\partial y} + (a_1 - 2a_3)u\frac{\partial}{\partial u} \\
&+ (a'(x)u - a_3 v)\frac{\partial}{\partial v} \\
&= a_1\Big(x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\Big) + a_2\frac{\partial}{\partial x} + a_3\Big(y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\Big) \\
&+ a(x)\frac{\partial}{\partial y} + a'(x)u\frac{\partial}{\partial v}
\end{aligned} \tag{3.34}$$

We thus have the following generators of the Lie Algebra of symmetries for the system (21.25)

$$\begin{aligned}
&\Gamma^{(1)} = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad \Gamma^{(2)} = \frac{\partial}{\partial x} \\
&\Gamma^{(3)} = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \quad \Gamma^{(4)} = a(x)\frac{\partial}{\partial y}, \\
&\Gamma^{(5)} = a'(x)u\frac{\partial}{\partial v}
\end{aligned} \tag{3.35}$$

2.2. Some general solutions of the system.
In order to find some general solutions of the system, we introduce a potential $w(x,y)$ such that

$$u = w_y, \quad v = -w_x \tag{3.36}$$

Then the first equation of (21.25) is being automatically satisfied whereas the second equation becomes

$$w_y w_{xy} - w_x w_{yy} = w_{yyy} \tag{3.37}$$

The invariant surface conditions

$$Xu_x + Yu_y = U, \quad Xv_x + Yv_y = V$$

taking into account

$$\begin{aligned} &X = a_1 x + a_2, \quad Y = a_3 y + a(x), \\ &U = (a_1 - 2a_3)u, \quad V = a'(x)u - a_3 v \end{aligned} \tag{3.38}$$

take the form

$$(a_1 x + a_2)u_x + (a_3 y + a(x))u_y = (a_1 - 2a_3)u \tag{3.39}$$

and

$$(a_1 x + a_2)v_x + (a_3 y + a(x))v_y = a'(x)u - a_3 v \tag{3.40}$$

respectively.
Using relations (21.60), the invariant surface conditions (21.63) and (21.64) become

$$(a_1 x + a_2)w_{xy} + (a_3 y + a(x))w_{yy} = (a_1 - 2a_3)w_y \tag{3.41}$$

and

$$-(a_1 x + a_2)w_{xx} - (a_3 y + a(x))w_{xy} = a'(x)w_y + a_3 w_x \tag{3.42}$$

Integrating (26.65) with respect to y, we obtain (the integration constant is taken equal to zero)

$$(a_1 x + a_2)w_x + (a_3 y + a(x))w_y = (a_1 - 2a_3)w \tag{3.43}$$

The above equation is to be solved in the special case $a_2 = a_3 = 0$.
In this case (26.67) becomes

$$a_1 x w_x + a(x)w_y = a_1\, w \tag{3.44}$$

The general solution of (26.68) is given by

$$w = x \cdot F(\xi), \quad \xi = y - J(x) \tag{3.45}$$

where

$$J(x) = \frac{1}{a_1} \int \frac{a(x)}{x} dx \tag{3.46}$$

Upon substitution of (26.69) into (21.61), we obtain

$$F'''(\xi) - F(\xi)F''(\xi) - (F'(\xi))^2 = 0 \tag{3.47}$$

Equation (26.71) is a nonlinear ordinary differential equation which can be simplified using Nucci's reduction algorithm.
We introduce the variables

$$w_1 = F(\xi), \quad w_2 = F'(\xi), \quad w_3 = F''(\xi) \tag{3.48}$$

We then have

$$w_1' = w_2, \quad w_2' = w_3, \quad w_3' - w_1 w_3 - (w_2)^2 = 0 \tag{3.49}$$

The above equations can also be written as

$$\frac{dw_1}{d\xi} = w_2, \quad \frac{dw_2}{d\xi} = w_3, \quad \frac{dw_3}{d\xi} = w_1 w_3 + (w_2)^2 \tag{3.50}$$

Upon dividing the last two equations by the first one, we obtain the equations

$$\frac{dw_2}{dw_1} = \frac{w_3}{w_2} \quad \text{and} \quad \frac{dw_3}{dw_1} = \frac{w_1 w_3 + (w_2)^2}{w_2} \tag{3.51}$$

Solving the first of the above equations with respect to w_3, i.e.

$$w_3 = w_2 \frac{dw_2}{dw_1} \tag{3.52}$$

and substitution into the second, we get the equation

$$w_2 \frac{d}{dw_1}\left(w_2 \frac{dw_2}{dw_1}\right) = w_1 \left(w_2 \frac{dw_2}{dw_1}\right) + (w_2)^2$$

The above equation is equivalent to

$$w_2 \frac{d^2 w_2}{dw_1^2} + \left(\frac{dw_2}{dw_1}\right)^2 - w_1 \frac{dw_2}{dw_1} - w_2 = 0 \tag{3.53}$$

We thus see that we have reduced the third order ODE (26.71) into a second order ODE with unknown function $w_2(w_1)$.
Equation (26.77), written in moderate notation as

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2-x\frac{dy}{dx}-y=0,\quad y=y(x) \tag{3.54}$$

will be solved using Lie point symmetry analysis.
Equation (26.78) admits the infinitesimals

$$\{X=x,\quad Y=2y\} \tag{3.55}$$

We can go over to canonical coordinates (r,s) by solving the system

$$\begin{aligned}&\frac{\partial r}{\partial x}X+\frac{\partial r}{\partial y}Y=0\\&\frac{\partial s}{\partial x}X+\frac{\partial s}{\partial y}Y=1\end{aligned} \tag{3.56}$$

for $X=x$, $Y=2y$. We find that the general solution of the above system is given by

$$r=F\left(\frac{y}{x^2}\right),\quad s=\ln(x)+G\left(\frac{y}{x^2}\right) \tag{3.57}$$

We make the choice

$$r=\frac{y}{x^2},\quad s=\ln(x) \tag{3.58}$$

Inverting the above system, i.e. considering the substitution

$$x=e^{s(r)},\quad y=r\,e^{2s(r)} \tag{3.59}$$

equation (26.78) gets transformed into the equation

$$r\frac{d^2s}{dr^2}-(6r^2-3r)\left(\frac{ds}{dr}\right)^3-(7r-1)\left(\frac{ds}{dr}\right)^2-\frac{ds}{dr}=0 \tag{3.60}$$

The substitution

$$U=\frac{ds}{dr} \tag{3.61}$$

transforms equation (26.84) into the first order ODE

$$r\frac{dU}{dr} - (6r^2 - 3r)U^3 - (7r-1)U^2 - U = 0 \tag{3.62}$$

which is a first kind Abel's equation. This equation under the substitution

$$U = \frac{r}{Y(r)} \tag{3.63}$$

gets transformed into the equation

$$w\frac{dY}{dr} + (7r-1)w + 6r^3 - 3r^2 = 0 \tag{3.64}$$

The above equation under the substitution

$$z = -\int (7r-1)dr = -\frac{7}{2}r^2 + r = f(r) \tag{3.65}$$

is being transformed into the equation

$$Y(z)\frac{dY(z)}{dz} - Y(z) = \left.\frac{6r^3 - 3r^2}{7r-1}\right|_{r=f^{-1}(z)} \tag{3.66}$$

Since

$$f^{-1}(z) = \frac{1}{7}\left(1 \pm \sqrt{1-14z}\right) \tag{3.67}$$

equation (26.90) splits into the two equations

$$Y(z)\frac{dY(z)}{dz} - Y(z) = \frac{3}{343}\frac{\left(1+\sqrt{1-14z}\right)^2\left(-5+2\sqrt{1-14z}\right)}{\sqrt{1-14z}} \tag{3.68}$$

and

$$Y(z)\frac{dY(z)}{dz} - Y(z) = \frac{3}{343}\frac{\left(-1+\sqrt{1-14z}\right)^2\left(5+2\sqrt{1-14z}\right)}{\sqrt{1-14z}} \tag{3.69}$$

The above equations (26.92) and (26.93) are Abels's quations of the second kind and thus can be solved by **Panayotounakos algorithm**.
Integrating (26.85) and using (26.87), we find

$$s = \int U(r)dr = \int \frac{r}{Y(r)}dr \tag{3.70}$$

where $Y(r)$ has been determined from either (26.92) or (26.93) using the Panayotounakos algorithm. Going back to the original variables, i.e. using (26.82), we get from (26.94) that

$$\ln(x) = \int \frac{r}{Y(r)} dr \bigg|_{r=y/x^2} \tag{3.71}$$

This is the general solution of equation (26.78).
Changing once more variables, i.e. $x \to w_1, y \to w_2$ we see that the solution of (26.77) is given by

$$\ln(w_1) = \int \frac{r}{Y(r)} dr \bigg|_{r=w_2/w_1^2} \tag{3.72}$$

We thus suppose that we have found $w_2 = w_2(w_1)$, thanks to (26.96). Using the first of (26.74), i.e.

$$\frac{dw_1}{d\xi} = w_2(w_1) \tag{3.73}$$

we obtain w_1 by solving the above differential equation and thus $F(\xi)$:

$$w_1 = F(\xi) \tag{3.74}$$

After determining $F(\xi)$, we can determine w from (26.69). The functions $u(x, y)$ and $v(x, y)$ which satisfy the original system (26.25) are determine from (26.60), i.e. by

$$u(x, y) = w_y \quad \text{and} \quad v(x, y) = -w_x \tag{3.75}$$

4 Appendix A

The **Panayotounakos algorithm** solves Abel's equation of the second kind as follows: The solution of the differential equation (Abel's equation of the second kind)

$$yy' - y = f(x), \quad y \equiv y(x), \quad y' = \frac{dy}{dx} \tag{4.1}$$

is given by

$$y(x) = \frac{1}{3}(x + 2C)\Big(r(x) + \frac{1}{3}\Big) \tag{4.2}$$

where C is a constant and $r(x)$ are the roots of the cubic equation

$$r^3(t) + p\,r(t) + q = 0, \quad t = ln|x + 2C| \tag{4.3}$$

The quantities p and q are given by

$$p = b - \frac{1}{3}a^3, \quad q = c - \frac{1}{3}ab + 2\Big(\frac{a}{3}\Big)^3 \tag{4.4}$$

$$a = -4, \quad b = 3 + 4[g(t) + f(t)]e^{-t}, \quad c = -4[g(t) + 2f(t)]e^{-t} \tag{4.5}$$

and $g(t)$ is a function defined by

$$g(t) = \frac{[(t\,sin(t) + cos(t))ci(t) + cos^2(t)][4t\,ci(t) + cos(t)]e^{-t}}{2[2t\,ci(t)]^3} - 2f(t) \tag{4.6}$$

The function $ci(t)$ is the known cosine integral function defined by

$$ci(t) = \gamma + ln(t) + \int_0^t \frac{cosu - 1}{u} du \tag{4.7}$$

where γ is the Euler-Mascheroni constant.

5 Appendix B. Transformations of Abel's equations

Abel's equation of first kind

$$\frac{dy}{dx} = A_3(x)y^3 + A_2(x)y^2 + A_1(x)y \tag{5.1}$$

under the substitution

$$y(x) = \frac{H(x)}{w(x)}, \quad H(x) = exp\Big(\int A_1(x)\,dx\Big) \tag{5.2}$$

takes on the form

$$w(x)\frac{dw(x)}{dx} = B_1(x)\,w(x) + B_0(x) \tag{5.3}$$

where $B_1(x), B_0(x)$ are functions expressed in terms of $A_3(x), A_2(x)$ and $A_0(x)$. The transformation

$$z = \int B_1(x)\, dx \equiv f(x) \tag{5.4}$$

converts equation (8.3) into the equation

$$w(z)\frac{dw(z)}{dz} - w(z) = \left.\frac{B_0(x)}{B_1(x)}\right|_{x=f^{-1}(z)} \tag{5.5}$$

which is Abel's equation of second kind and thus accessible by Panayotounakos algorithm. The reader can consult Polyanin and Zaitsev (Reference [18]).

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