Introduction to Symmetry Analysis

Chapter 16 -Backlund Transformations And Nonlocal Groups

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Burgers' Equation.

$$u_t + uu_x - \nu u_{xx} = 0, \qquad (16.27)$$

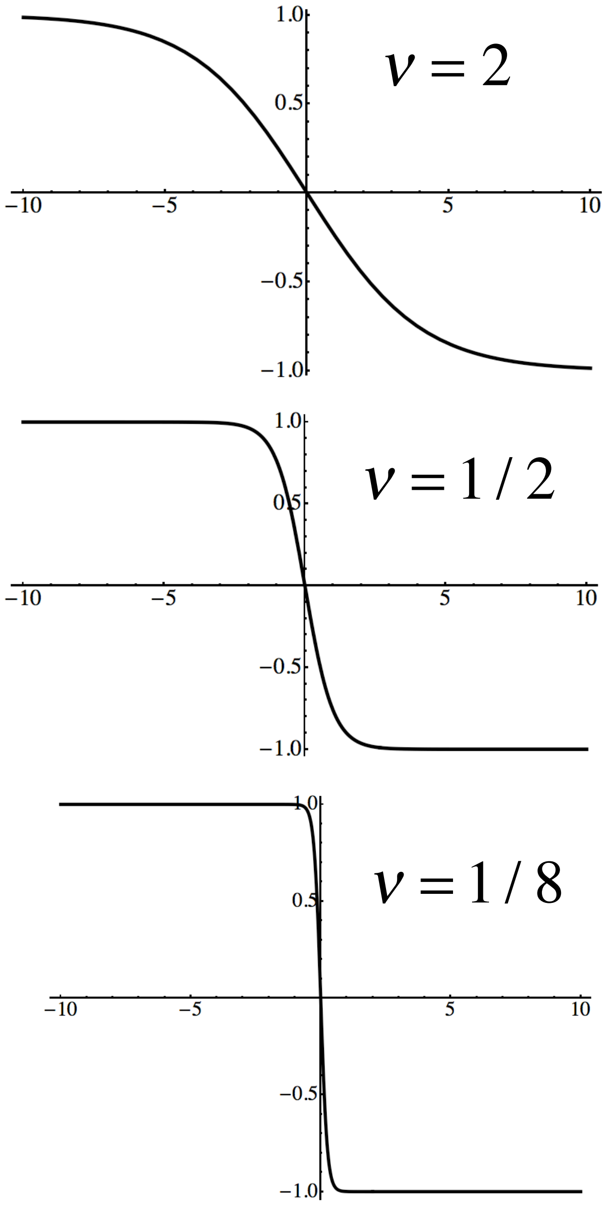
Singular behavior of Burgers' Equation. Work out the steady state solution - invariant under translation in time.

$$\frac{d}{dx}\left(\frac{u^2}{2} - \nu\frac{du}{dx}\right) = 0$$

$$\frac{u^2}{2} - \nu\frac{du}{dx} = C_1$$

$$\frac{du}{dx} = \frac{u^2}{2\nu} - \frac{C_1}{\nu}$$

$$u(x) = \sqrt{2C_1}\,Tanh\left(\sqrt{2C_1}\left(C_2 - \frac{x}{2\nu}\right)\right)$$



$$C_1 = 1/2$$
$$C_2 = 0$$

Exact solution of Burgers' Equation

$$u_t + uu_x - \nu u_{xx} = 0, \qquad (16.27)$$

Integrate the equation in space

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\,dx + \left(\frac{1}{2}u^2 - \nu\frac{\partial u}{\partial x}\right)_{-\infty}^{\infty} = 0. \qquad (16.28)$$

Conserved integral

$$A = \int_{-\infty}^{\infty} u\,dx, \qquad (16.29)$$

Initial velocity distribution

$$u[x,0] = u_0 g[x], \qquad (16.30)$$

Non-dimensionalize the equation

$$U = \frac{u}{u_0}, \qquad \chi = \frac{x u_0}{\nu}, \qquad \tau = \frac{u_0^2 t}{\nu}.$$

$$U_\tau + U U_\chi - U_{\chi\chi} = 0$$

The conserved integral becomes

$$Re = \int_{-\infty}^{\infty} U[\chi]\, d\chi,$$

where the Reynolds number is

$$R_e = A / \nu$$

Symmetries of the Burgers potential equation

$$\phi_\tau + \frac{1}{2}(\phi_\chi)^2 - \phi_{\chi\chi} = 0$$

$$\begin{aligned}
\tilde{\chi} &= \chi + s\xi[\chi, \tau, \phi], \\
\tilde{\tau} &= \tau + s\zeta[\chi, \tau, \phi], \\
\tilde{\phi} &= \phi + s\eta[\chi, \tau, \phi]
\end{aligned}$$

Invariance condition

$$\eta_{\{\tau\}} + \phi_\chi \eta_{\{\chi\}} - \eta_{\{\chi\chi\}} = 0.$$

Group operators

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial\chi}, \qquad X_2 = \frac{\partial}{\partial\tau}, \qquad X_3 = \frac{\partial}{\partial\phi}, \\
&X_4 = \tau\frac{\partial}{\partial\chi} + \chi\frac{\partial}{\partial\phi}, \qquad X_5 = \frac{1}{2}\chi\frac{\partial}{\partial\chi} + \tau\frac{\partial}{\partial\tau}, \\
&X_6 = 2\chi\tau\frac{\partial}{\partial\chi} + 2\tau^2\frac{\partial}{\partial\tau} + (\chi^2 + 2\tau)\frac{\partial}{\partial\phi}.
\end{aligned}$$

There is another solution of the invariance condition !!

$$\tilde{\chi} = \chi,$$
$$\tilde{\tau} = \tau,$$
$$\tilde{\phi} = \phi + s\eta[\chi, \tau, \phi].$$

With the independent variables not transformed,
the invariance condition takes the following form

$$D_\tau \eta + \phi_\chi D_\chi \eta - D_{\chi\chi} \eta = 0.$$

The invariance condition is satisfied by the infinite dimensional group

$$\eta = f[\chi, \tau] e^{\phi/2},$$

Where f is any solution of the heat equation

In this example we use the package IntroToSymmetry.m to work out point and Lie-Backlund groups of the Burgers potential equation,

Uxx-(1/2)Ux^2-Ut=0.

This next command turns off spurious spelling error warnings.

```
Off[General::spell]
```

Clear all symbols in the current context.

```
ClearAll[Evaluate[Context[] <> "*"]]
```

First read in the package which is located in User Home Folder/Library/*Mathematica*/Applications/SymmetryAnalysis.

```
Needs["SymmetryAnalysis`IntroToSymmetry`"]
```

Enter the input equation as a string. Don't include the == 0 at the end.

```
inputequation =
"D[U[x,t],t]+(1/2)*D[U[x,t],x]^2-D[U[x,t],x,x]";
```

The function U[x,t] is a solution of the equation and this constraint must be applied in the form of a rule to the invariance condition. Be careful to check signs.

```
rulesarray =
  {"D[U[x,t],x,x]→D[U[x,t],t]+(1/2)*D[U[x,t],x]^2"};
```

Enter the list of independent variables.

```
independentvariables = {"x", "t"};
```

Enter the list of dependent variables.

```
dependentvariables = {"U"};
```

Enter the list of function and/or constant names that need to be preserved when the equation is expressed in terms of generic variables.

```
frozennames = {};
```

Enter the maximum derivative order of the input equation(s).

```
porder = 2;
```

The maximum derivative order that the infinitesimals are assumed to depend on is specified by the input parameter rorder. This parameter is only nonzero when the user is looking for Lie contact groups or Lie-Backlund groups. For the usual case where one is searching for point groups set rorder=0.

```
rorder = 0;
```

When searching for Lie-Backlund groups (rorder=1 or greater) one can, without loss of generality, leave the independent variables untransformed. The corresponding infinitesimals (the xse's) are set to zero by setting xseon=0. If one is searching for point groups then set xseon=1. The choice xseon=1 is also an option when looking for Lie-Backlund groups and this can be useful when looking for contact symmetries.

```
xseon = 1;
```

When searching for Lie-Backlund groups it is necessary to differentiate the input equation to produce derivatives of order p+r and append these higher order differential consequences to the set of rules applied to the invariance condition. This process is carried out automatically when internalrules=1. For point groups the equation or equation system is the only rule or set of rules needed and one sets internalrules=0.

```
internalrules = 0;
```

Now work out the determining equations of the Lie point group that leaves the equation invariant. The output is available as a table of strings called zdeterminingequations.

100%

```
Timing[FindDeterminingEquations[
  independentvariables, dependentvariables, frozennames, porder, rorder, xseon,
  inputequation, rulesarray, internalrules]]
```

```
The function FindDetermining Equations has begun,
  the memory in use = 55560320, the time used = 2.1674719999999996`

The function FindDeterminingEquations is nearly complete. The invariance condition has been created with
  all rules applied. The final step in the generation of the determining equations is to sum
  together terms in the table of invariance condition terms (called infinitesimaltable) that are
  multiplied by the same combination of products of free y derivatives. The result is the table
  infinitesimaltablesums corresponding to matching y-derivative expressions. If the invariance
  condition is long as it often is this process could take a long time since it requires sorting
  through the table infinitesimaltable once for each possible combination of y derivative products.
  This is the rate limiting step in the function FindDeterminingEquations.Virtually all other
  steps are quite fast including the generation of the extended derivatives of the infinitesimals.

The determining equations have been expressed in terms of z-variables,
  the length of zdeterminingequations = 9, the byte count of zdeterminingequations
  = 1320, the memory in use = 55491648, the time used = 2.2272019999999997`

FindDeterminingEquations is done. The memory in use = 55493632, the time used = 2.227389`

FindDeterminingEquations has finished executing. You can look at the output in the table
  zdeterminingequations. Each entry in this table is a determining equation in string format expressed
  in terms of z-variables. Rules for converting between z-variables and conventional variables are
  contained in the table ztableofrules. To view the determining equations in terms of conventional
  variables use the command ToExpression[zdeterminingequations]/.ztableofrules. There are two
  other items the user may wish to look at; the equation converted to generic (x1,x2,...,y1,y2,...)
  variables is designated equationgenericvariables and the various derivatives of the equation
  that appear in the invariance condition can be viewed in the table invarconditiontable. Rules for
  converting between z-variables and generic variables are contained in the table ztableofrulesxy.

{0.067045, Null}
```

Let's look at the partial derivatives of Burgers equation thatappear in the invariance condition.

```
invarconditiontable

{0, 0, 0, y1^(1,0)[x1, x2], 1, -1, 0, 0}
```

Here are the determining equations in terms of z-variables.

```
zdeterminingequations

{2*Derivative[0, 0, 1][xse2][z1, z2, z3] == 0,
 Derivative[0, 0, 1][xse1][z1, z2, z3]/2 + Derivative[0, 0, 2][xse1][z1, z2, z3] == 0,
 -Derivative[0, 0, 1][xse2][z1, z2, z3]/2 + Derivative[0, 0, 2][xse2][z1, z2, z3] == 0,
 2*Derivative[1, 0, 0][xse2][z1, z2, z3] == 0,
 Derivative[0, 0, 1][eta1][z1, z2, z3]/2 - Derivative[0, 0,
   2][eta1][z1, z2, z3] + 2*Derivative[1, 0, 1][xse1][z1, z2, z3] == 0,
 2*Derivative[0, 0, 1][xse1][z1, z2, z3] - Derivative[1, 0, 0][xse2][z1,
   z2, z3] + 2*Derivative[1, 0, 1][xse2][z1, z2, z3] == 0,
 Derivative[0, 1, 0][eta1][z1, z2, z3] - Derivative[2, 0, 0][eta1][z1, z2, z3] == 0,
 -Derivative[0, 1, 0][xse1][z1, z2, z3] + Derivative[1, 0, 0][eta1][z1, z2, z3] -
   2*Derivative[1, 0, 1][eta1][z1, z2, z3] + Derivative[2, 0, 0][xse1][z1, z2, z3] == 0,
 -Derivative[0, 1, 0][xse2][z1, z2, z3] + 2*Derivative[1, 0, 0][xse1][z1,
   z2, z3] + Derivative[2, 0, 0][xse2][z1, z2, z3] == 0}
```

```
zdeterminingequationscolumn =
 TraditionalForm[Column[ToExpression[zdeterminingequations], {Left, Below}]] /. ztableofrules
```

$$2\,\mathrm{xse2}^{(0,0,1)}(x,t,U(x,t))=0$$

$$\tfrac{1}{2}\,\mathrm{xse1}^{(0,0,1)}(x,t,U(x,t))+\mathrm{xse1}^{(0,0,2)}(x,t,U(x,t))=0$$

$$\mathrm{xse2}^{(0,0,2)}(x,t,U(x,t))-\tfrac{1}{2}\,\mathrm{xse2}^{(0,0,1)}(x,t,U(x,t))=0$$

$$2\,\mathrm{xse2}^{(1,0,0)}(x,t,U(x,t))=0$$

$$\tfrac{1}{2}\,\mathrm{eta1}^{(0,0,1)}(x,t,U(x,t))-\mathrm{eta1}^{(0,0,2)}(x,t,U(x,t))+2\,\mathrm{xse1}^{(1,0,1)}(x,t,U(x,t))=0$$

$$2\,\mathrm{xse1}^{(0,0,1)}(x,t,U(x,t))-\mathrm{xse2}^{(1,0,0)}(x,t,U(x,t))+2\,\mathrm{xse2}^{(1,0,1)}(x,t,U(x,t))=0$$

$$\mathrm{eta1}^{(0,1,0)}(x,t,U(x,t))-\mathrm{eta1}^{(2,0,0)}(x,t,U(x,t))=0$$

$$\mathrm{eta1}^{(1,0,0)}(x,t,U(x,t))-2\,\mathrm{eta1}^{(1,0,1)}(x,t,U(x,t))-\mathrm{xse1}^{(0,1,0)}(x,t,U(x,t))+\mathrm{xse1}^{(2,0,0)}(x,t,U(x,t))=0$$

$$2\,\mathrm{xse1}^{(1,0,0)}(x,t,U(x,t))-\mathrm{xse2}^{(0,1,0)}(x,t,U(x,t))+\mathrm{xse2}^{(2,0,0)}(x,t,U(x,t))=0$$

```
Length[zdeterminingequations]

9
```

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Here is the correspondence between z-variables and conventional variables.

```
ztableofrules
{z1 → x, z2 → t, z3 → U[x, t]}
```

Now solve the determining equations in terms of multivariable polynomials. The Mathematica function Solve uses Gaussian elimination to solve a large number of linear equations for the polynomial coefficients. The time roughly follows

time/timeref=((number of equations)/(number of equationsref))^n

where the exponent is between 2.4 and 2.7. The Mathematica function Timing outputs the time required for the SolveDeterminingEquations function to execute.

```
Timing[SolveDeterminingEquations[
  independentvariables, dependentvariables, rorder, xseon, zdeterminingequations, polyorder = 2]]
The variable powertablelength is the number of terms required for each
  multivariate polynomial used for the infinitesimals. This number is determined by
  the choice of polynomial order and the number of zvariables. The time needed to solve
  the determining equations increases as powertable increases. powertablelength = 10
The polynomial expansions have been substituted into the determining equations. It is now time
  to collect the coefficients of various powers of zvariables into a table called table of
  coefficientsall. This step uses the function CoefficientList and is a fairly time consuming procedure.
The memory in use = 56032152, The time = 2.2762219999999997`
The number of unknown polynomial coefficients = 30
The number of equations for the polynomial coefficients = 35
Now it we are ready to use the function Solve to find the nonzero polynomial coefficients
  corresponding to the symmetries of the input equation(s). This can take a while.
The memory in use = 55956776, The time = 2.2775449999999995`
Solve has finished.
The function SolveDeterminingEquations is finished executing.
The memory in use = 55961664, The time = 2.2844919999999997`
You can look at the output in the tables xsefunctions and etafunctions. Each entry in these
  tables is an infinitesimal function in string format expressed in terms of z-variables
  and the group parameters. The output can also be viewed with the group parameters
  stripped away by looking at the table infinitesimalgroups. In either case you may wish
  to convert the z-variables to conventional variables using the table ztableofrules.

Keep in mind that this function only finds solutions of the determining equations that are of
  polynomial form. The determining equations may admit solutions that involve transcendental
  functions and/or integrals. Note that arbitrary functions may appear in the infinitesimals and
  that these can be detected by running the package function SolveDeterminingEquations for several
  polynomial orders. If terms of ever increasing order appear, then an arbitrary function is indicated.
{0.013889, Null}
```

Here are the infinitesimal transformation functions for the independent variables.

```
xsefunctions
{xse1[z1_, z2_, z3_]=a10 + a11*z1 + a13*z2 + a14*z1*z2,
 xse2[z1_, z2_, z3_]=a20 + 2*a11*z2 + a14*z2^2}
```

and for the dependent variables.

```
etafunctions
{eta1[z1_, z2_, z3_]=b10 + a13*z1 + (a14*z1^2)/2 + a14*z2}
```

Usually Burger's equation is expressed in terms of x, t and U.Express the xse functions in terms of these variables.

```
ToExpression[xsefunctions] /. {z1 → x, z2 → t, z3 → U}
```

$\{a10 + a13\,t + a11\,x + a14\,t\,x,\ a20 + 2\,a11\,t + a14\,t^2\}$

Express the eta function in terms of these variables.

```
ToExpression[etafunctions] /. {z1 → x, z2 → t, z3 → U}
```

$\{b10 + a14\,t + a13\,x + \frac{a14\,x^2}{2}\}$

100%

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```
infinitesimalgroups1 = infinitesimalgroups /. {z1 → x, z2 → t, z3 → U};
```

Check the groups.xse's are on the left,eta's are on the right.

```
Column[infinitesimalgroups1]
```

```
{{1, 0}, {0}}
{{x, 2 t}, {0}}
{{t, 0}, {x}}
```

$\left\{\left\{t\,x,\ t^2\right\},\ \left\{t + \frac{x^2}{2}\right\}\right\}$

```
{{0, 1}, {0}}
{{0, 0}, {1}}
```

Check the infinitesimals in the determining equations.

```
Simplify[ToExpression[zdeterminingequations]]
```

```
{True, True, True, True, True, True, True, True, True}
```

This is the six-parameter point group of the Burger's potential equation. Note that the package does not capture the infinite parameter group involving an arbitrary solution of the heat equation. This is discussed extensively in Chapter 14 and Chapter 16. Let's construct the commutator table.

```
MakeCommutatorTable[
 independentvariables, dependentvariables, infinitesimalgroups1]
```

```
MakeCommutatorTable has finished executing. You can look at the output in the table commutatortable.
  To present the output in the most readable form you may want view it as a matrix using
  MatrixForm[commutatortable]. Occasionally the entries in the commutatortable will have terms
  that cancel. To get rid of these terms use the function Simplify before viewing the table.
```

```
MatrixForm[commutatortable]
```

{0, 0} {0}	{1, 0} {0}	{0, 0} {1}	{t, 0} {x}	{0, 0} {0}	{0, 0} {0}
{-1, 0} {0}	{0, 0} {0}	{t, 0} {x}	$\{2\,t\,x,\ 2\,t^2\}$ $\{2\,t + x^2\}$	{0, -2} {0}	{0, 0} {0}
{0, 0} {-1}	{-t, 0} {-x}	{0, 0} {0}	{0, 0} {0}	{-1, 0} {0}	{0, 0} {0}
{-t, 0} {-x}	$\{-2\,t\,x,\ -2\,t^2\}$ $\{-2\,t - x^2\}$	{0, 0} {0}	{0, 0} {0}	{-x, -2 t} {-1}	{0, 0} {0}
{0, 0} {0}	{0, 2} {0}	{1, 0} {0}	{x, 2 t} {1}	{0, 0} {0}	{0, 0} {0}
{0, 0} {0}	{0, 0} {0}	{0, 0} {0}	{0, 0} {0}	{0, 0} {0}	{0, 0} {0}

100%

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Note that the package does not capture the infinite parameter group involving an arbitrary solution of the heat equation. This is discussed extensively in Chapter 14 and Chapter 16. Let's see if we can find this group using a slightly different approach to the invariance condition.

```
rorder = 0
0
```

Turn off the transformation of independent variables.

```
xseon = 0
0
```

```
Timing[FindDeterminingEquations[independentvariables, dependentvariables,
  frozennames, porder, rorder, xseon, inputequation, rulesarray, internalrules]]
```

```
The function FindDetermining Equations has
  begun, the memory in use = 55956976, the time used = 2.304852`
```

```
The function FindDeterminingEquations is nearly complete. The invariance condition has been
  created with all rules applied. The final step in the generation of the determining
  equations is to sum together terms in the table of invariance condition terms (called
  infinitesimaltable) that are multiplied by the same combination of products of free y
  derivatives. The result is the table infinitesimaltablesums corresponding to matching
  y-derivative expressions. If the invariance condition is long as it often is this process
  could take a long time since it requires sorting through the table infinitesimaltable
  once for each possible combination of y derivative products. This is the rate limiting
  step in the function FindDeterminingEquations.Virtually all other steps are quite
  fast including the generation of the extended derivatives of the infinitesimals.
```

```
The determining equations have been expressed in terms of z-variables, the
  length of zdeterminingequations = 3, the byte count of zdeterminingequations
  = 424, the memory in use = 55892384, the time used = 2.3386299999999998`
```

```
FindDeterminingEquations is done. The
  memory in use = 55894296, the time used = 2.3387949999999997`
```

```
FindDeterminingEquations has finished executing. You can look at the output in the
  table zdeterminingequations. Each entry in this table is a determining equation
  in string format expressed in terms of z-variables. Rules for converting between
  z-variables and conventional variables are contained in the table ztableofrules. To
  view the determining equations in terms of conventional variables use the command
  ToExpression[zdeterminingequations]/.ztableofrules. There are two other items the user
  may wish to look at; the equation converted to generic (x1,x2,...,y1,y2,...) variables is
  designated equationgenericvariables and the various derivatives of the equation that appear in
  the invariance condition can be viewed in the table invarconditiontable. Rules for converting
  between z-variables and generic variables are contained in the table ztableofrulesxy.
```

```
{0.041013, Null}
```

```
zdeterminingequations1 = ToExpression[zdeterminingequations] /. {z1 → x, z2 → t, z3 → U}
```

$$\left\{\frac{1}{2}\,\text{eta1}^{(0,0,1)}[x,t,U]-\text{eta1}^{(0,0,2)}[x,t,U]=0,\right.$$
$$\left.\text{eta1}^{(1,0,0)}[x,t,U]-2\,\text{eta1}^{(1,0,1)}[x,t,U]=0,\ \text{eta1}^{(0,1,0)}[x,t,U]-\text{eta1}^{(2,0,0)}[x,t,U]=0\right\}$$

100%

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Note that we are now just looking for a single unknown eta1 and it is a point group. This is an example where we can turn off the transformation of independent variables and still look
for a point group rather than a Lie Backlund Group. The frst determining equation is

(1/2)eta1u-eta1uu==0.

This is clearly satisfied by a function of the form

eta1=f[x,t]Exp[u/2].

The third equation is also satisfied by the same function. Let's check

```
eta1[x_, t_, U_] := f[x, t] * Exp[U / 2]

Column[Simplify[zdeterminingequations1]]
```

True
True
$e^{U/2}\left(f^{(0,1)}[x,t]-f^{(2,0)}[x,t]\right)=0$

The invariance condition is satisfied if f[x,t] is a solution of the heat equation.

100%

Burgers potential equation $\phi_\tau + \frac{1}{2}(\phi_\chi)^2 - \phi_{\chi\chi} = 0$

$$\tilde{\chi} = \chi,$$
$$\tilde{\tau} = \tau,$$
$$\tilde{\phi} = \phi + s\eta[\chi, \tau, \phi].$$

$$\eta = f[\chi, \tau]e^{\phi/2}$$

What finite transformation does this correspond to ?
To find out we have to sum the Lie series.

$$\tilde{\phi} = \phi + sX\phi + \frac{s^2}{2!}X(X\phi) + \frac{s^3}{3!}X(XX(\phi)) + \cdots,$$

Where

$$X = f[\chi, \tau]e^{\phi/2}\frac{\partial}{\partial\phi}.$$

First few terms

$$X\phi = 2\left(\frac{f}{2}e^{\phi/2}\right),$$
$$X^2\phi = 2(1)\left(\frac{f}{2}e^{\phi/2}\right)^2,$$
$$X^3\phi = 2(1\times 2)\left(\frac{f}{2}e^{\phi/2}\right)^3,$$
$$X^4\phi = 2(1\times 2\times 3)\left(\frac{f}{2}e^{\phi/2}\right)^4,$$
$$X^5\phi = 2(1\times 2\times 3\times 4)\left(\frac{f}{2}e^{\phi/2}\right)^5,$$
$$\vdots$$

Let

$$a = s\frac{f}{2}e^{\phi/2}.$$

$$\tilde{\phi} = \phi + 2\left\{a + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + \frac{a^5}{5} + \cdots\right\}.$$

$$g(a) = a + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + \frac{a^5}{5} + \cdots,$$

$$\frac{dg}{da} = a + a^2 + a^3 + a^4 + a^5 + \cdots = \frac{1}{1-a}.$$

$$g(a) = -\ln[1-a].$$

The finite transformation of the Burgers potential equation is

$$\tilde{\phi} = \phi - 2\ln\left[1 - \frac{s}{2} f[\chi, \tau] e^{\phi/2}\right], \qquad \tilde{\chi} = \chi, \qquad \tilde{\tau} = \tau. \tag{16.53}$$

This group can be used to generate a corresponding transformation of the Burgers equation. Let

$$\phi = \int_{-\infty}^{\chi} U \, d\hat{\chi} = D_\chi^{-1} U.$$

The transformation of Burgers equation is

$$\begin{aligned}
\tilde{\chi} &= \chi, \\
\tilde{\tau} &= \tau, \\
\tilde{U} &= U + D_\chi \ln\left[1 - \frac{s}{2} f[\chi, \tau] e^{D_\chi^{-1} U/2}\right]^{-2} \\
&= U + \frac{s\left(f_\chi + \frac{f}{2} U\right) e^{D_\chi^{-1} U/2}}{1 - \frac{s}{2} f e^{D_\chi^{-1} U/2}},
\end{aligned}$$

This is an example of a nonlocal group

$$\begin{aligned}
\tilde{\chi} &= F_1[\chi, \tau, U, s], \\
\tilde{\tau} &= F_2[\chi, \tau, U, s], \\
\tilde{U} &= G[\chi, \tau, U, D_\chi^{-1} U, s]
\end{aligned}$$

The Cole-Hopf transformation. Let $U = 0$

$$\tilde{U} = \frac{s f_\chi}{1 - \frac{s}{2} f}.$$

Let

$$\theta[\chi, \tau] = 1 - \frac{s}{2} f[\chi, \tau].$$

$$\tilde{U} = -2\frac{\theta_\chi}{\theta}.$$

The classical single hump solution of Burgers equation. Let

$$\theta = 1 + \frac{e^{Re/2} - 1}{\sqrt{\pi}} \int_{\chi/\sqrt{4\tau}}^{\infty} e^{-\zeta^2} d\zeta.$$

The Cole-Hopf transformation gives

$$U = \frac{(e^{Re/2} - 1)e^{-\chi^2/4\tau}}{\sqrt{\tau}\left(\sqrt{\pi} + (e^{Re/2} - 1) \int_{\chi/\sqrt{4\tau}}^{\infty} e^{-\zeta^2} d\zeta\right)}.$$

$$V = \frac{U\sqrt{\pi\tau}}{\sqrt{Re}} = \frac{(e^{Re/2} - 1)e^{-\lambda^2 Re}}{\sqrt{Re}\left(1 + \frac{e^{Re/2}-1}{\sqrt{\pi}} \int_{\lambda\sqrt{Re}}^{\infty} e^{-\zeta^2} d\zeta\right)}.$$

where

$$R_e = A / \nu$$

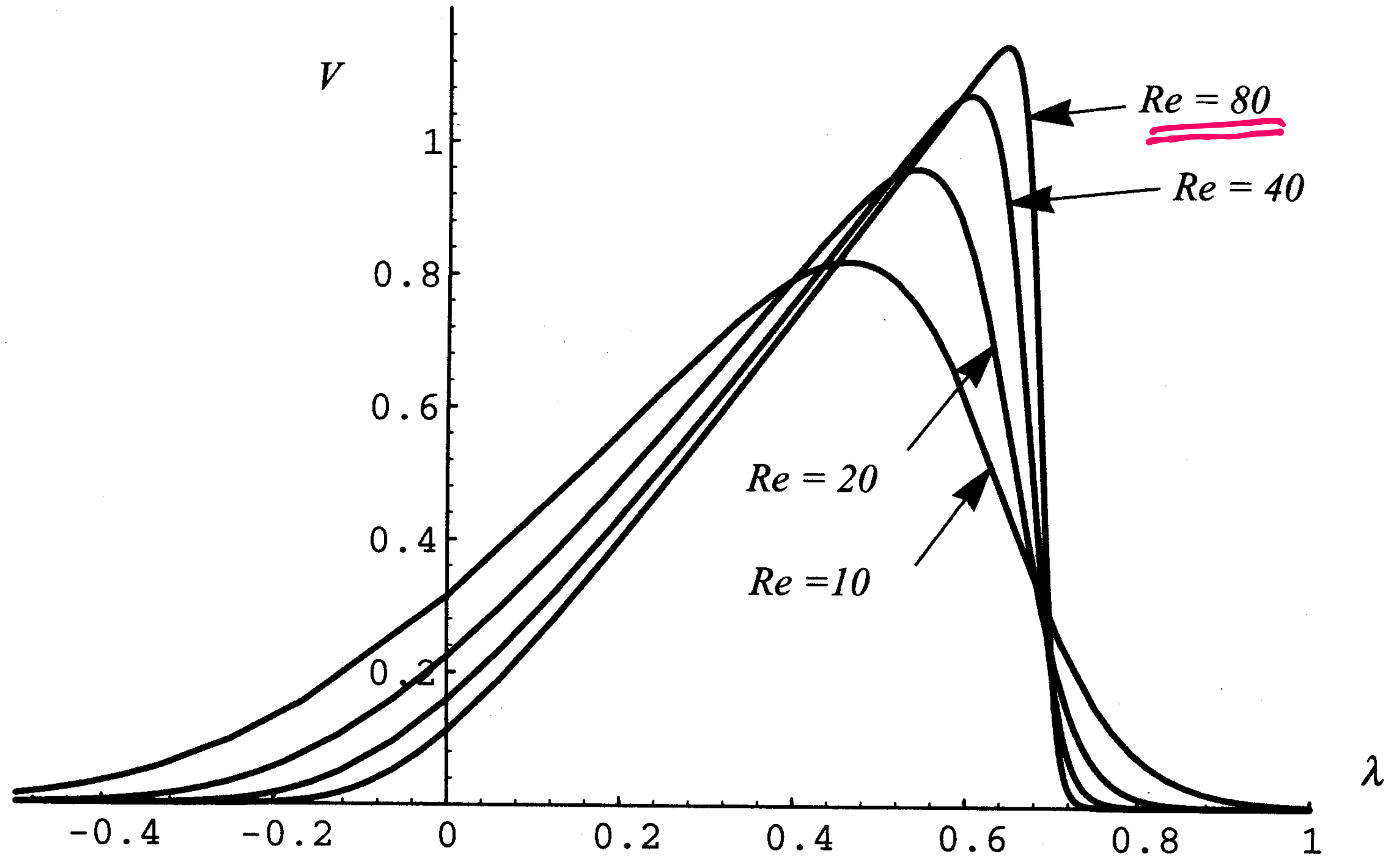


Fig. 16.3. The single-hump solution of the Burgers equation at several Reynolds numbers.

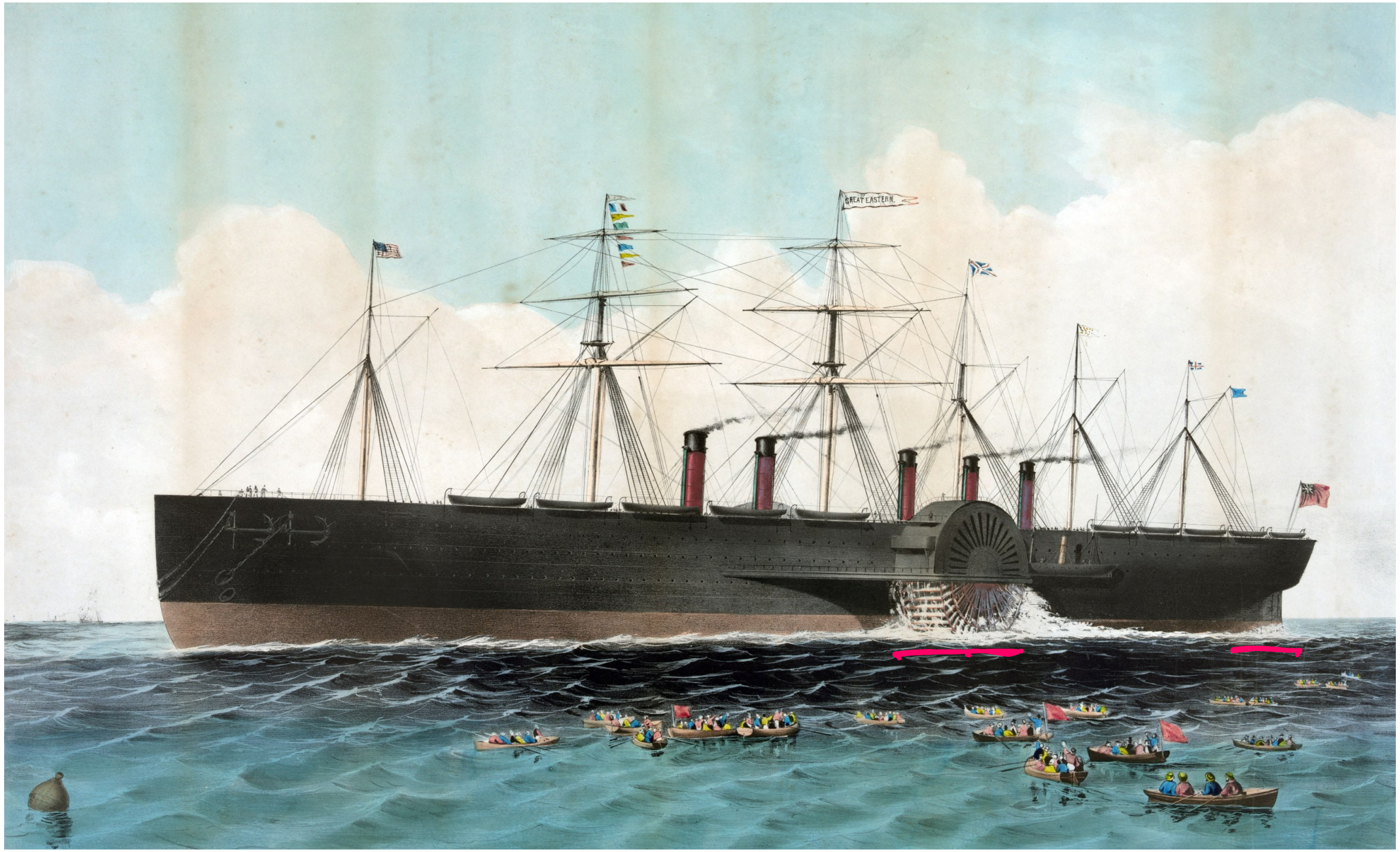
Solitary Waves

John Scott Russell

John Scott Russell

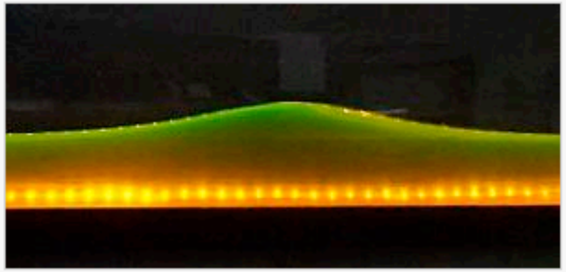
I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour [14 km/h], preserving its original figure some thirty feet [9 m] long and a foot to a foot and a half [300–450 mm] in height. Its height gradually diminished, and after a chase of one or two miles [2–3 km] I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

The Great Eastern - 1858



Scott Russell spent some time making practical and theoretical investigations of these waves, he built wave tanks at his home and noticed some key properties:

- The waves are stable, and can travel over very large distances (normal waves would tend to either flatten out, or steepen and topple over)
- The speed depends on the size of the wave, and its width on the depth of water.
- Unlike normal waves they will never merge—so a small wave is overtaken by a large one, rather than the two combining.
- If a wave is too big for the depth of water, it splits into two, one big and one small.



Solitary wave in a laboratory wave channel.

Scott Russell's experimental work seemed at contrast with the Isaac Newton and Daniel Bernoulli's theories of hydrodynamics. George Biddell Airy and George Gabriel Stokes had difficulty to accept Scott Russell's experimental observations because Scott Russell's observations could not be explained by the existing water-wave theories. His contemporaries spent some time attempting to extend the theory but it would take until the 1870s before an explanation was provided.

Lord Rayleigh published a paper in Philosophical Magazine in 1876 to support John Scott Russell's experimental observation with his mathematical theory.[4] In his 1876 paper, Lord Rayleigh mentioned Scott Russell's name and also admitted that the first theoretical treatment was by Joseph Valentin Boussinesq in 1871. Joseph Boussinesq mentioned Scott Russell's name in his 1871 paper.[5] Thus Scott Russell's observations on solitons were accepted as true by some prominent scientists within his own lifetime.

Korteweg and de Vries did not mention John Scott Russell's name at all in their 1895 paper but they did quote Boussinesq's paper in 1871 and Lord Rayleigh's paper in 1876. Although the paper by Korteweg and de Vries in 1895 was not the first theoretical treatment of this subject, it was a very important milestone in the history of the development of soliton theory.[6]

It was not until the 1960s and the advent of modern computers that the significance of Scott Russell's discovery in physics, electronics, biology and especially fibre optics started to become understood, leading to the modern general theory of solitons.

A book was written by George Sinclair Emmerson on Scott Russell with the title *John Scott Russell: a great Victorian engineer and naval architect*, which was published in 1977.[7] However, this book has very little discussion on the discovery of solitons by John Scott Russell. In 2005, Olivier Darrigol published a book *Worlds of Flow*, which covers the history of hydrodynamics from the years before John Scott Russell and to many years after his death.[8] Inside this book, Darrigol provided a comprehensive list of classical papers written by John Scott Russell and other scientists on hydrodynamics. The book by Darrigol has a much better discussion on the discovery of solitons.

The Korteweg de Vries equation

$$u_t + 3\beta u u_x + \frac{\beta}{2} u_{xxx} = 0,$$

is often used to study the relationship between nonlinear convection and dispersion in water waves as well as other applications.

Begin with the KdV potential equation

$$\theta_t + \frac{\beta}{2}\theta_x^3 - \beta\theta_{xxx} = 0,$$

Symmetries of the KdV potential equation

$$\theta_t + \frac{\beta}{2}\theta_x^{\ 3} - \beta\theta_{xxx} = 0$$

$$\tilde{x} = x + s\xi[x, t, \theta]$$
$$\tilde{t} = t + s\tau[x, t, \theta]$$
$$\tilde{\theta} = \theta + s\eta[x, t, \theta]$$

Invariance condition

$$\eta_{\{t\}} + \frac{3}{2}\beta\theta_x^{\ 2}\eta_{\{x\}} - \beta\eta_{\{xxx\}} = 0$$

Four parameter point group

$$X^1 = \frac{\partial}{\partial x}, \quad X^2 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t}, \quad X^3 = \frac{\partial}{\partial t}, \quad X^4 = \frac{\partial}{\partial \theta}$$

Invariance condition for the KdV potential equation

$$\eta_{\{t\}} + \frac{3\beta}{2}\theta_x^2 \eta_{\{x\}} - \beta \eta_{\{xxx\}} = 0.$$

Assume an infinitesimal transformation of the form

$$\tilde{x} = x,$$
$$\tilde{t} = t,$$
$$\tilde{\theta} = \theta + s\eta[x, t, \theta].$$

The invariance condition becomes

$$D_t \eta + \frac{3\beta}{2}\theta_x^2 D_x \eta - \beta D_{xxx} \eta = 0.$$

The KdV potential equation admits the group with infinitesimal

$$\eta = \int e^{-\theta}\,dx = D_x^{-1}(e^{-\theta}).$$

The Lie series is

$$\tilde{\theta} = \theta + sX\theta + \frac{s^2}{2!}X^2\theta + \frac{s^3}{3!}X^3\theta + \frac{s^4}{4!}X^4\theta + \cdots,$$

where

$$X = D_x^{-1}(e^{-\theta})\frac{\partial}{\partial\theta}.$$

Summing the Lie series leads to the non-local finite transformation

θ_0

$$\boxed{\tilde{\theta} = \theta + 2\ln\left[1 - sD_x^{-1}(e^{-\theta})\right], \qquad \tilde{x} = x, \qquad \tilde{t} = t}$$

The solution of the KdV potential equation is related to the KdV equation by

$$w[x,t] = \frac{\beta}{4}\theta_x[x,t]^2 - \frac{\beta}{2}\theta_{xx}[x,t]$$

w satisfies

$$w_t + 6ww_x - \beta w_{xxx} = 0$$

The simplest propagating solution of the KdV potential equation is

$$\theta_0 = \kappa(x - x_a) - \frac{\beta}{2}\kappa^3(t - t_a) + c$$

which generates the solution

$$\theta_1 = \theta_0 + 2\ln\left[1 + \frac{s}{\kappa}e^{-\theta_0}\right]$$

The solution of the KdV equation generated by this solution is

$$u[x,t] = \frac{\kappa^2}{4} - \frac{1}{\beta}w[x,t] = 2s\kappa \frac{e^{\theta_0}}{\left(e^{\theta_0} + \frac{s}{\kappa}\right)^2}$$

u satisfies the form of the KdV equation we began with

$$u_t + 3\beta u u_x + \frac{\beta}{2}u_{xxx} = 0$$

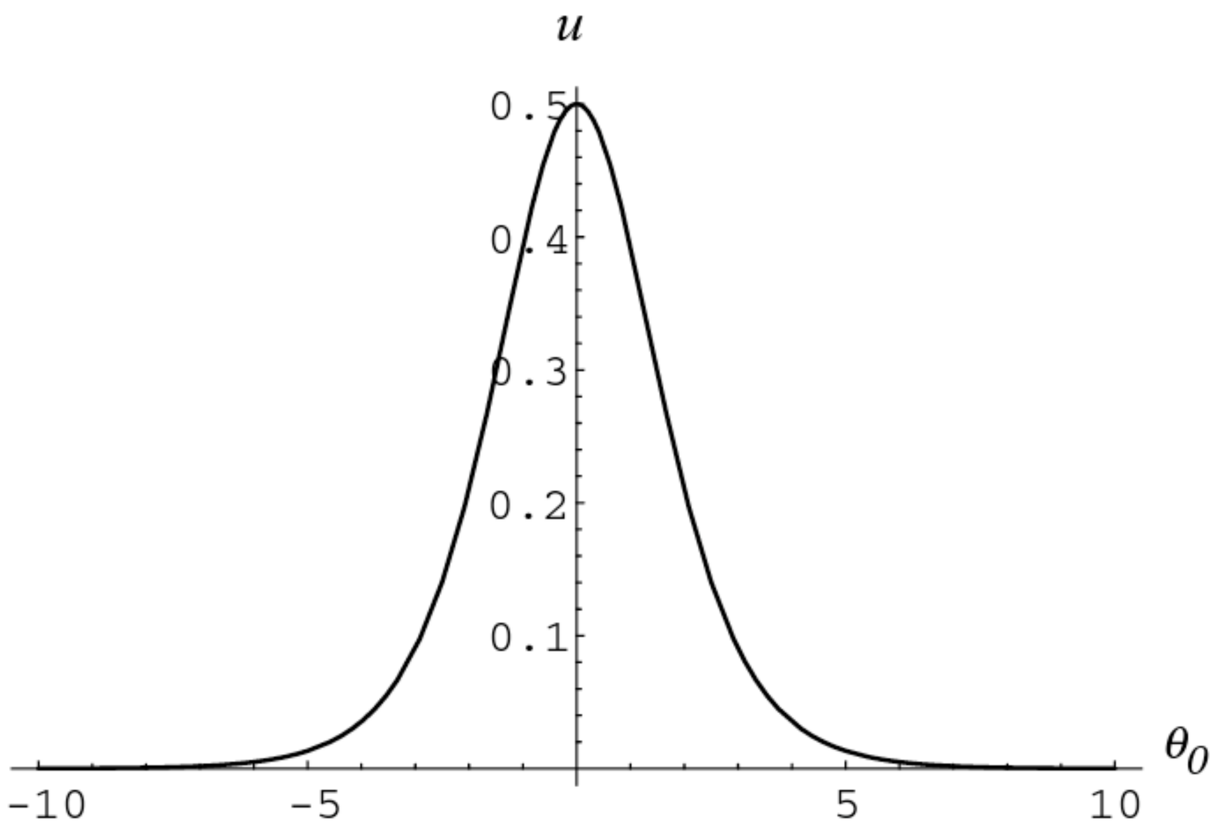


Fig. 16.6. Classical single-hump solution of the KdV equation presented in Equation (16.144). Parameter values are $k = 1$, $s = 1$.

1995 Conference on Coherent Structures in Physics and Biology

Scott Russell Aqueduct near Edinburgh UK

A group of participants attempting to produce a solitary wave

One can use the group to generate an exact solution for two colliding solitons.

$$u = 2\left\{\frac{\kappa_0^2 e^{-\theta_0} + \kappa_1^2 e^{-\theta_1} + (\kappa_1 - \kappa_0)^2 e^{-(\theta_1+\theta_0)}}{1 + e^{-\theta_0} + e^{-\theta_1} + \left(\frac{\kappa_1-\kappa_0}{\kappa_1+\kappa_0}\right)^2 e^{-(\theta_1+\theta_0)}}\right\}$$

$$- \not{+}\, 2\left\{\frac{\left(\kappa_0 e^{-\theta_0} + \kappa_1 e^{-\theta_1} + \frac{(\kappa_1-\kappa_0)^2}{\kappa_1+\kappa_0} e^{-(\theta_1+\theta_0)}\right)^2}{\left(1 + e^{-\theta_0} + e^{-\theta_1} + \left(\frac{\kappa_1-\kappa_0}{\kappa_1+\kappa_0}\right)^2 e^{-(\theta_1+\theta_0)}\right)^2}\right\}$$

where

$$\theta_0 = \kappa_0\left(x - x_0\right) - \kappa_0^{\,3}\left(t - t_0\right) + c_0$$

$$\theta_1 = \kappa_1\left(x - x_1\right) - \kappa_1^{\,3}\left(t - t_1\right) + c_1$$

$\beta = 2.0 \quad t_0 = 0 \quad c_0 = 0$

$\kappa_0 = 1.0 \quad t_1 = 0 \quad c_1 = 0$

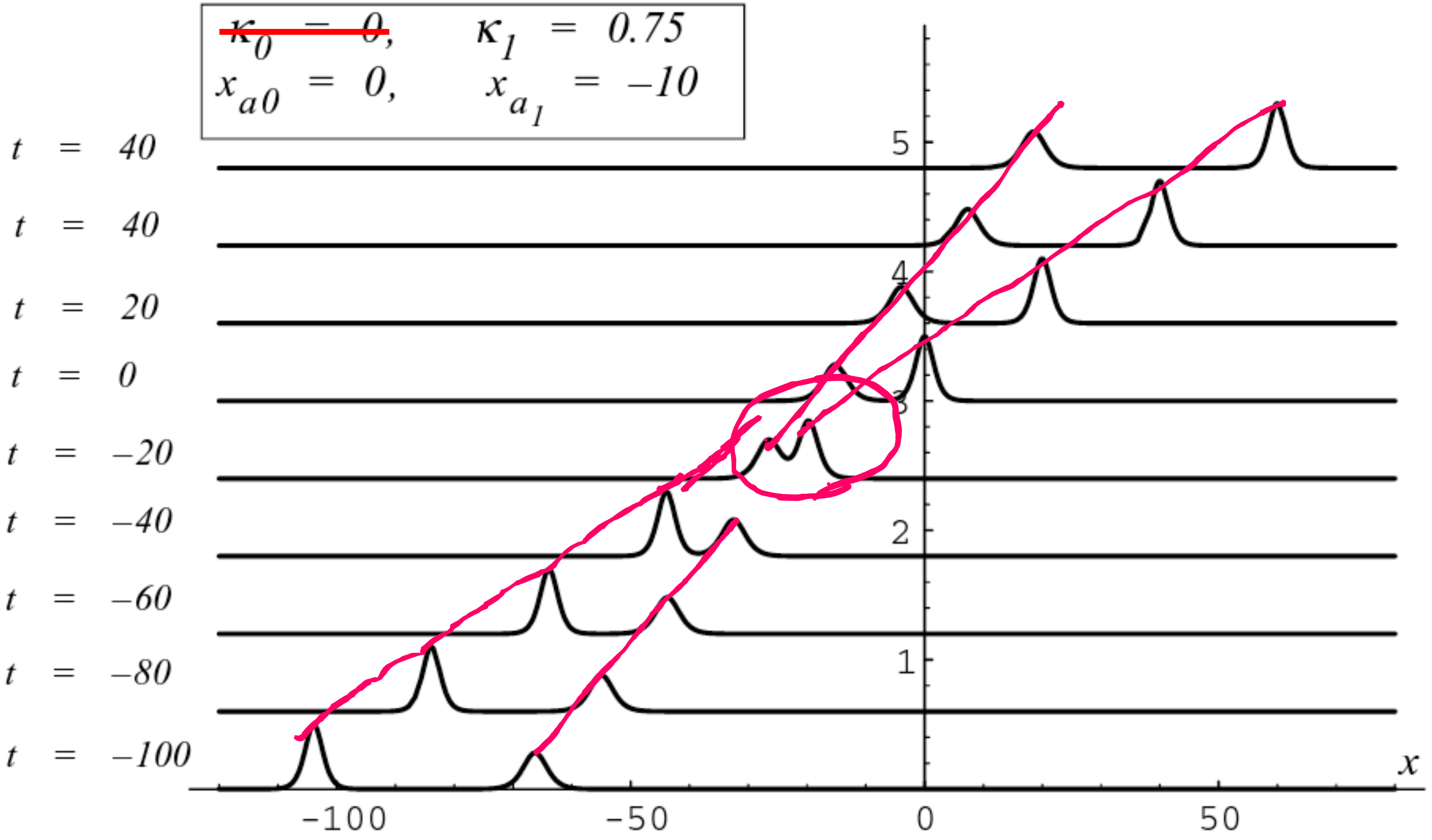
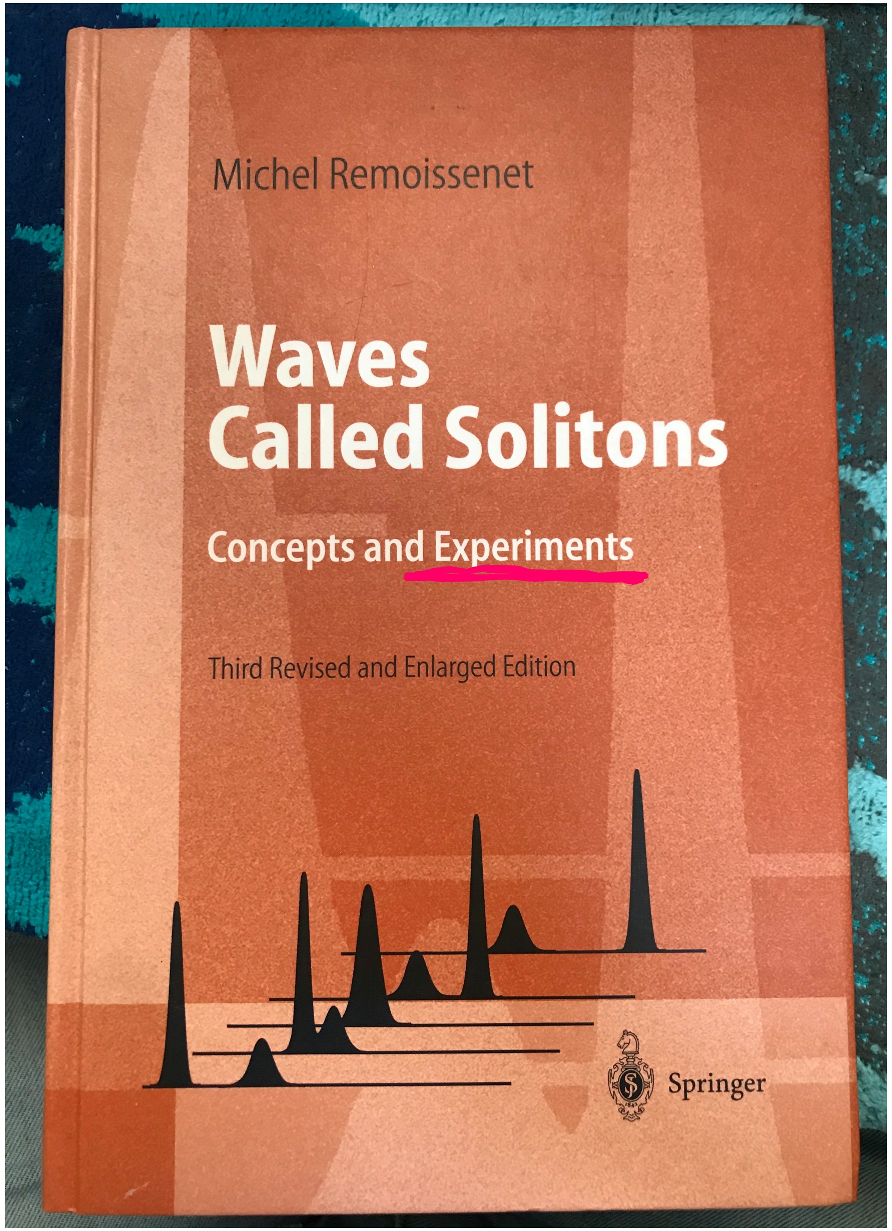


Fig. 16.7. Exact solution of the KdV equation depicting the collision of two solitons.


Michel Remoissenet
Waves
Called Solitons
Concepts and Experiments
Third Revised and Enlarged Edition
Springer

AA 218 – Virtual Exam 2
June 2020

20 points

Problem 1

1) (2 points) Show that the first order PDE

$$U_t + 6UU_x = 0$$

is invariant under a one-parameter dilation group and work out the infinitesimals of the group.

2) (2 points) Write down the invariance condition for this equation.

3) (2 points) The first derivative in t transforms infinitesimally according to

$$\tilde{U}_{\tilde{t}} = U_t + s\eta_{\{t\}}$$

Work out $\eta_{\{t\}}$ for the given group.

4) (2 points) The first derivative in x transforms infinitesimally according to

$$\tilde{U}_{\tilde{x}} = U_x + s\eta_{\{x\}}$$

Work out $\eta_{\{x\}}$ for the given group.

5) (2 points) Show that the invariance condition is satisfied by the group.

6) (3 points) Determine the invariants of the group and use these to generate similarity variables and reduce the equation to a first order ODE.

Problem 2

1) (4 points) Show that the nonlinear third order KdV equation

$$U_t + 6UU_x + U_{xxx} = 0$$

is also invariant under the group in problem 1.

2) (3 points) Use the group to generate similarity variables and reduce the PDE to a third-order ODE.

AA 218 – Virtual Exam 2
June 2020

SOLUTIONS

Problem 1

The first order PDE

$$U_t + 6UU_x = 0$$

is invariant under the group

$$\tilde{x} = e^s x \qquad \tilde{t} = e^{3s} t \qquad \tilde{U} = e^{-2s} U$$

1) (3 points) Write down the invariance condition for this equation.

SOLUTION

$$X_{\{1\}}\Psi = \eta_{\{t\}} + 6U\eta_{\{x\}} + 6U_x\eta = 0$$

2) (2 points) The first derivative in t transforms according to

$$\tilde{U}_{\tilde{t}} = U_t + s\eta_{\{t\}}.$$

Work out $\eta_{\{t\}}$ for the given group.

SOLUTION

$$\eta_{\{t\}} = \frac{D\eta}{Dt} - U_x\frac{D\xi}{Dt} - U_t\frac{D\tau}{Dt} =$$

$$\eta_t + U_t\eta_U - U_x\left(\xi_t + U_t\xi_U\right) - U_t\left(\tau_t + U_t\tau_U\right) =$$

$$0 - 2U_t - U_x\left(0 + U_t 0\right) - U_t\left(3 + U_t 0\right) = -5U_t$$

3) (2 points) The first derivative in x transforms according to

$$\tilde{U}_{\tilde{x}} = U_x + s\eta_{\{x\}}.$$

Work out $\eta_{\{x\}}$ for the given group.

SOLUTION

$$\eta_{\{x\}} = \frac{D\eta}{Dx} - U_x\frac{D\xi}{Dx} - U_t\frac{D\tau}{Dx} =$$

$$\eta_x + U_x\eta_U - U_x\left(\xi_x + U_x\xi_U\right) - U_t\left(\tau_x + U_x\tau_U\right) =$$

$$0 - 2U_x - U_x\left(1 + U_x 0\right) - U_t\left(0 + U_x 0\right) = -3U_x$$

4) (2 points) Show that the invariance condition is satisfied by the group.

SOLUTION

$$X_{\{1\}}\Psi = \eta_{\{t\}} + 6U\eta_{\{x\}} + 6U_x\eta =$$

$$-5U_t + 6U\left(-3U_x\right) + 6U_x\left(-2U\right) =$$

$$-5\left(U_t + 6UU_x\right) = 0$$

5) (4 points) Determine the invariants of the group and use these to generate similarity variables and reduce the equation to a first order ODE.

SOLUTION

The characteristic equations are:

$$\frac{dx}{x} = \frac{dt}{3t} = \frac{dU}{-2U}$$

The invariants are

$$\phi = x / t^{1/3}$$

and the solution of

$$\frac{dt}{3t} = \frac{dU}{-2U} \Rightarrow G = t^{2/3}U$$

Let

$$U = t^{-2/3}G(\phi)$$

Now the derivatives are

$$U = t^{-2/3}G(\phi)$$

$$U_t = -\frac{2}{3}t^{-5/3}G - \frac{1}{3}t^{-5/3}\phi G_\phi$$

$$U_x = t^{-3/3}G_\phi$$

$$U_t + 6UU_x = -\frac{2}{3}t^{-5/3}G - \frac{1}{3}t^{-5/3}\phi G_\phi + 6t^{-5/3}GG_\phi = 0$$

$$-\frac{2}{3}G - \frac{1}{3}\phi G_\phi + 6GG_\phi = 0$$

The problem reduces to the ODE $-\frac{2}{3}G - \frac{1}{3}\phi G_\phi + 6GG_\phi = 0$.

Problem 2

The nonlinear third order KdV equation

$$U_t + 6UU_x + U_{xxx} = 0$$

is also invariant under the group in problem 1.

1) (2 points) Write down the invariance condition for this equation.

$$X_{\{1\}}\Psi = \eta_{\{t\}} + 6U\eta_{\{x\}} + 6U_x\eta + \eta_{\{xxx\}} = 0$$

2) (2 points) Work out $\eta_{\{xxx\}}$ for the given group and show that the invariance condition is satisfied.

$$X_{\{1\}}\Psi = \eta_{\{t\}} + 6U\eta_{\{x\}} + 6U_x\eta + \eta_{\{xxx\}} =$$
$$-5U_t + 6U(-3U_x) + 6U_x(-2U) - 5U_{xxx} =$$
$$-5(U_t + 6UU_x + U_{xxx}) = 0$$

3) (4 points) Use the group to reduce the PDE to a third-order ODE.

SOLUTION

The invariants are

$$\phi = x/t^{1/3}$$

and the solution of

$$\frac{dt}{3t} = \frac{dU}{-2U} \Rightarrow G = t^{2/3}U$$

Let

$$U = t^{-2/3}G(\phi)$$

Now the derivatives are

$$U = t^{-2/3}G(\phi)$$
$$U_t = -\frac{2}{3}t^{-5/3}G - \frac{1}{3}t^{-5/3}\phi G_\phi$$
$$U_x = t^{-3/3}G_\phi$$
$$U_{xx} = t^{-4/3}G_{\phi\phi}$$
$$U_{xxx} = t^{-5/3}G_{\phi\phi\phi}$$
$$U_t + 6UU_x + U_{xxx} = -\frac{2}{3}t^{-5/3}G - \frac{1}{3}t^{-5/3}\phi G_\phi + 6t^{-5/3}GG_\phi + t^{-5/3}G_{\phi\phi\phi} = 0$$
$$-\frac{2}{3}G - \frac{1}{3}\phi G_\phi + 6GG_\phi + G_{\phi\phi\phi} = 0$$

16.5 The point-source solution of the heat equation is

$$\phi = \left(\frac{2}{\pi}\right)^{1/2} A(2\nu t)^{-1/2} e^{-\xi^2/2} + B, \qquad \xi = \frac{x}{(2\nu t)^{1/2}}. \quad (16.172)$$

(i) Use this to generate the classical N-wave solution of the Burgers equation. Discuss the nature of this solution at small and large time. This solution has been used to model the shock structure associated with the sonic boom.

16.6 Show by direct substitution that

$$\tilde{\theta} = \theta + 2\ln\left(1 - sD_x^{-1}(e^{-\theta})\right) \quad (16.173)$$

is a solution of the KdV potential equation

$$\theta_t + \frac{\beta}{2}\theta_x^3 - \beta\theta_{xxx} = 0 \quad (16.174)$$

as long as θ is a solution.