Introduction to Symmetry Analysis

Chapter 15 - Variational Symmetries and Conservation Laws

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Results from Chapter 4 - Classical Dynamics

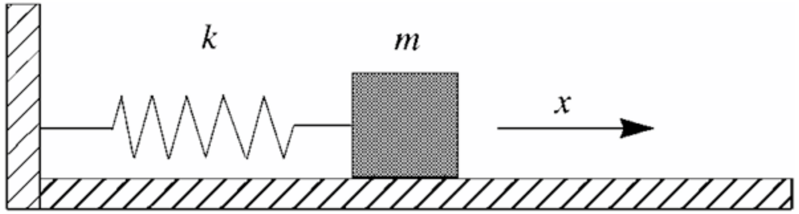

Fig. 4.1. Spring–mass system.

Equation of motion

$$m\frac{d^2x}{dt^2} + kx = 0.$$

Break into an autonomous system

$$\frac{dx}{dt} = \frac{p}{m}, \qquad \frac{dp}{dt} = -kx$$

Corresponding first order PDE

$$\frac{p}{m}\frac{\partial H}{\partial x} - kx\frac{\partial H}{\partial p} = 0.$$

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Characteristic equation

$$m\frac{dx}{p} = \frac{dp}{-kx}$$

Integrate

$$dH = \frac{p}{m}dp + kx\,dx$$

Energy is conserved.

$$H[x, p] = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kx^2$$

is called the Hamiltonian

$$H = \frac{1}{2m}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 = T + V$$

There is a very general approach to problems of this type called Lagrangian dynamics.

Dynamical systems that conserve energy follow a path in phase space that corresponds to an extremum in a certain integral of the coordinates and velocities called the action integral.

$$S = \int_{t_1}^{t_2} L[q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n, t]\, dt$$

The kernel of the integral is called the Lagrangian. Typically,

$$L = T - V$$

Usually the extremum is a minimum and this theory is often called the *principle of least action*.

Consider

$$\tilde{S} = \int_{t_1}^{t_2} L[\tilde{q}^1, \ldots, \tilde{q}^n, \dot{\tilde{q}}^1, \ldots, \dot{\tilde{q}}^n, t]\, dt$$

Apply a small variation in the coodinates and velocities.

$$\tilde{q}^i[t] = q^i[t] + \varepsilon \eta^i[t]$$

$$S + \delta S = \int_{t_1}^{t_2} L[q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n, t]\, dt + \varepsilon \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^i} \eta^i + \frac{\partial L}{\partial \dot{q}^i} \dot{\eta}^i \right) dt + O(\varepsilon^2)$$

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At an extremum in S the first variation vanishes.

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^i} \eta^i + \frac{\partial L}{\partial \dot{q}^i} \dot{\eta}^i \right) dt = 0$$

Using

$$\frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{q}^i} \eta^i \right) = \frac{\partial L}{\partial \dot{q}^i} \dot{\eta}^i + \frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{q}^i} \right) \eta^i$$

Integrate by parts.

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^i} - \frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{q}^i} \right) \right) \eta^i \, dt + \left(\frac{\partial L}{\partial \dot{q}^i} \eta^i \right)_{t_1}^{t_2} = 0$$

At the end points the variation is zero and in between the variation is arbitrary.

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The Lagrangian satisfies the Euler-Lagrange equations.

$$\frac{\partial L}{\partial q^i} - \frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = 0$$

Spring mass system

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

The Euler-Lagrange equations generate

$$m\frac{d^2x}{dt^2} + kx = 0$$

The two body problem

Example 4.3 (The two-body problem). Consider the motion of two particles moving under the action of a force field that acts between them (Figure 4.3).

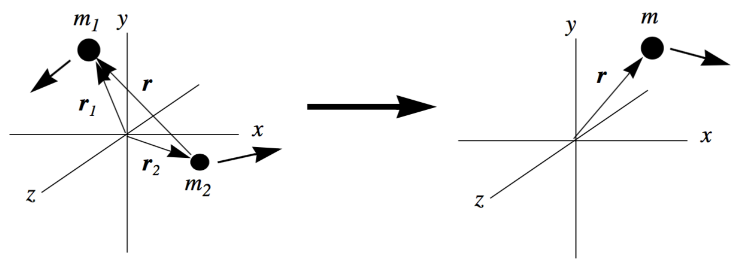

Fig. 4.3. Mapping of the two-body problem to an equivalent one-body problem in center-of-mass coordinates.

The Lagrangian is

$$L = \frac{1}{2} m_1 \dot{\boldsymbol{r}}_1^2 + \frac{1}{2} m_2 \dot{\boldsymbol{r}}_2^2 - V[|\boldsymbol{r}_1 - \boldsymbol{r}_2|], \tag{4.79}$$

where $\boldsymbol{r}_1 = (x_1, y_1, z_1)$ and $\boldsymbol{r}_2 = (x_2, y_2, z_2)$ are the radius vectors to the mass particles, $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ is the vector joining the particles, and $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$ is the distance between them. To simplify the problem let's set the origin of coordinates at the center of mass of the two points, so that $m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2 = 0$. In this system, the two position vectors can be expressed in terms of $\boldsymbol{r}$:

$$\boldsymbol{r}_1 = \frac{m_2 \boldsymbol{r}}{m_1 + m_2}, \qquad \boldsymbol{r}_2 = \frac{m_1 \boldsymbol{r}}{m_1 + m_2}. \tag{4.80}$$

If we insert these expressions into (4.79), the result is

$$L = \frac{1}{2} m \dot{\boldsymbol{r}}^2 - V[r], \tag{4.81}$$

or, in terms of the coordinates,

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V[\sqrt{x^2 + y^2 + z^2}], \tag{4.82}$$

where m is the reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}, \tag{4.83}$$

and the scalar distance r is measured from the center-of-mass origin. By using the reduced mass, the two-body problem is reduced to the motion of a single particle in a spherically symmetric force field. Once the path $\boldsymbol{r}[t]$ has been determined, the motions of the individual particles are obtained by means of (4.80). The equations of motion generated by the Euler–Lagrange equations are

$$
\begin{aligned}
m\ddot{x} + \frac{x}{r}\left(\frac{\partial V}{\partial r}\right) &= 0, \\
m\ddot{y} + \frac{y}{r}\left(\frac{\partial V}{\partial r}\right) &= 0, \\
m\ddot{z} + \frac{z}{r}\left(\frac{\partial V}{\partial r}\right) &= 0.
\end{aligned} \tag{4.84}
$$

The Hamiltonian with $\boldsymbol{p} = m\dot{\boldsymbol{r}}$ is

$$
H = \frac{1}{2}\frac{\boldsymbol{p}^2}{m} + V[r]. \tag{4.85}
$$

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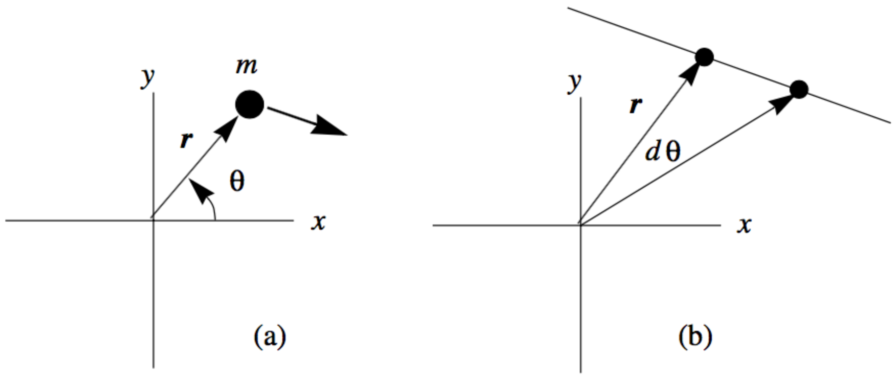

Fig. 4.4. Motion of the reduced-mass particle in cylindrical coordinates.

The motion of the particle actually takes place in a plane, and it is convenient to express the position of the particle in terms of the distance from the center of mass and the angle with respect to some reference axis, as shown in Figure 4.4. In these coordinates the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V[r], \tag{4.86}$$

and the Hamiltonian is the total energy

$$H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V[r], \tag{4.87}$$

which is conserved. The equations of motion in cylindrical coordinates simplify to

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} &= 0, \\ \frac{d}{dt}(mr^2\dot{\theta}) &= 0. \end{aligned} \tag{4.88}$$

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The second of the equations of motion expresses conservation of angular momentum in the center-of-mass system:

$$\Gamma = mr^2\dot{\theta} = \text{constant}. \tag{4.89}$$

Equation (4.89) can be interpreted using the sketch in Figure 4.4(b), which shows the sector swept out by the particle in a small period of time. The area of the sector is $dA = \boldsymbol{r} \cdot \boldsymbol{r}\ d\theta/2$, and $dA/dt = r^2\dot{\theta}/2 = \Gamma/2m$. This result is known as Kepler's second law: the particle sweeps out equal areas in equal times. Note that Kepler's second law applies for any central force field and does not assume anything about the radial dependence of the field.

The easiest way to reach the complete solution of the motion of the particle is to use the two conserved quantities. Use (4.89) to eliminate $\dot{\theta}$ from the Hamiltonian (4.87) and solve for the radial velocity

$$\frac{dr}{dt} = \pm\left(\frac{2}{m}(H - V[r]) - \frac{\Gamma^2}{m^2 r^2}\right)^{1/2}. \tag{4.90}$$

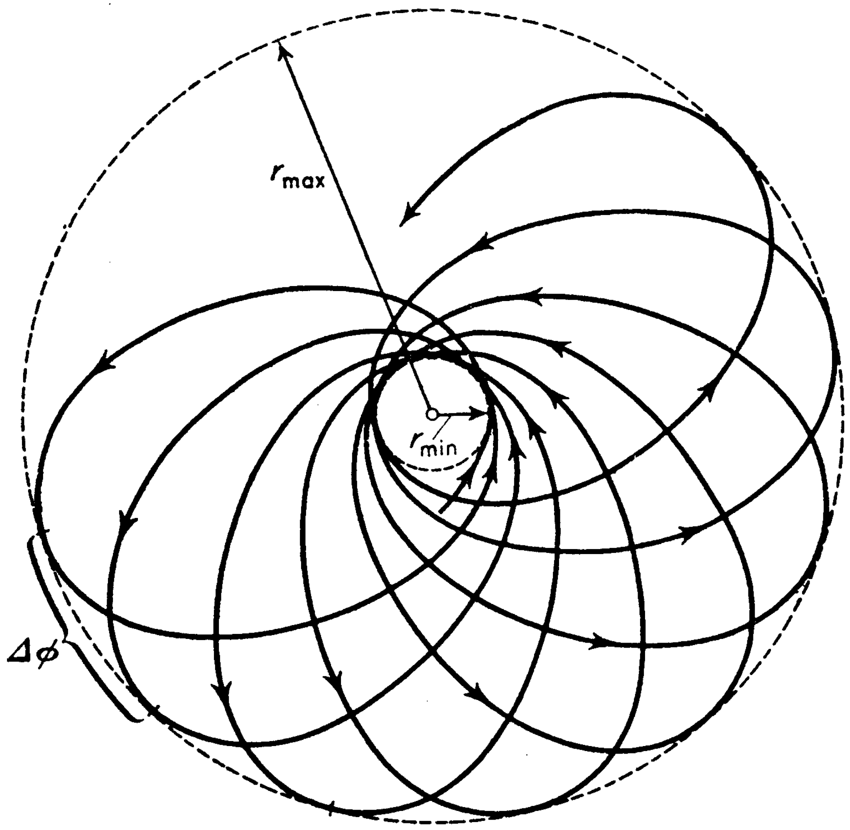
The solution for the radius is expressed implicitly in terms of the time,

$$t = \int_{r_0}^{r} \frac{dr}{\left(\frac{2}{m}(H - V[r]) - \frac{\Gamma^2}{m^2 r^2}\right)^{1/2}}, \tag{4.91}$$

and the angle is determined from conservation of angular momentum,

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{\Gamma\, dr}{r^2\left(2m(H - V[r]) - \frac{\Gamma^2}{r^2}\right)^{1/2}}. \tag{4.92}$$

As the particle moves under the influence of the central field, it is constrained to move in an annular disk between two radii, r_{min} and r_{max}. The condition for the orbit to be closed is that the angle defined by (4.92) must be an integer multiple of 2π when the radius, starting at say r_{min}, returns to r_{min}. This only occurs for the case when the potential energy varies as r^2 or $1/r$. We shall return to this issue in Chapter 15 in the context of Problem 15.3.



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Example 4.4 (Kepler's problem – the motion of celestial bodies). A very important class of two-body problems is defined by a potential field of the form

$$V = -\frac{\gamma}{r}, \tag{4.93}$$

for which the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\gamma}{r} \tag{4.94}$$

and the generalized momenta are

$$\begin{aligned} p_1 &= m\dot{r}, \\ p_2 &= mr^2\dot{\theta} = \Gamma. \end{aligned} \tag{4.95}$$

We are considering the interaction between two gravitating bodies, where the constant γ is

$$\gamma = Gm_1m_2 \tag{4.96}$$

and the force of attraction varies inversely as the square of the radius. This is the same problem we analyzed in Chapter 2 using dimensional analysis, except that there the final result was reached under an assumption, appropriate to the solar system, that one mass was much larger than the other. The equations of motion in Cartesian coordinates are

$$\begin{aligned} m\ddot{x} + \gamma\frac{x}{r^3} &= 0, \\ m\ddot{y} + \gamma\frac{y}{r^3} &= 0, \\ m\ddot{z} + \gamma\frac{z}{r^3} &= 0. \end{aligned} \tag{4.97}$$

These equations can be cast in terms of cylindrical coordinates in the plane of the motion:

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 + \frac{\gamma}{r^2} &= 0, \\ \frac{d}{dt}(mr^2\dot{\theta}) &= 0. \end{aligned} \tag{4.98}$$

Dilation group

$$\tilde{x} = e^{2a}x$$
$$\tilde{y} = e^{2a}y$$
$$\tilde{z} = e^{2a}z$$
$$\tilde{t} = e^{3a}t$$
$$\tilde{m} = m$$

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The two-body solution is

$$t - t_0 = \int_{r_0}^{r} \frac{dr}{\left(\frac{2}{m}\left(H + \frac{\gamma}{r}\right) - \frac{\Gamma^2}{m^2 r^2}\right)^{1/2}},$$
$$\theta - \theta_0 = \int_{r_0}^{r} \frac{\Gamma\, dr}{r^2\left(2m\left(H + \frac{\gamma}{r}\right) - \frac{\Gamma^2}{r^2}\right)^{1/2}}. \tag{4.99}$$

The integral relating the angle to the radius can be carried out, leading to

$$-\left(\frac{2Hm}{\Gamma^2} r^2 + \frac{2\gamma m}{\Gamma^2} r - 1\right)^{1/2} = \left(\left(\frac{\gamma m}{\Gamma^2}\right) r - 1\right) \tan[\theta - \theta_0] \tag{4.100}$$

Note that the initial radius r_0 does not appear in (4.100). The usual convention that is adopted is that r_0 corresponds to $\theta = \theta_0$. That is, r_0 satisfies $\frac{2Hm}{\Gamma^2} r_0^2 + \frac{2\gamma m}{\Gamma^2} r_0 - 1 = 0$ where the positive root is selected. This aligns the major axis of the orbit along the horizontal axis of coordinates. After some manipulation, the trajectory of the particle can be written as

$$\left(\frac{2H\Gamma^2}{m\gamma^2} + 1\right)^{1/2} r \cos[\theta - \theta_0] - r + \frac{\Gamma^2}{\gamma m} = 0. \tag{4.101}$$

This is the equation of a conic section with one focus at the origin and eccentricity

$$e = \left(\frac{2H\Gamma^2}{m\gamma^2} + 1\right)^{1/2}. \tag{4.102}$$

The quantity

$$h = \frac{\Gamma^2}{\gamma m} \tag{4.103}$$

is one-half the so-called *latus rectum* of the trajectory of the particle. The equation for the path of the particle in cylindrical coordinates, (4.101), can be cast into Cartesian coordinates, and the result is as follows:

$$\frac{\left(x - \frac{eh}{1-e^2}\right)^2}{\left(\frac{h}{1-e^2}\right)^2} + \frac{y^2}{\left(\frac{h^2}{1-e^2}\right)} = 1. \tag{4.104}$$

$$H = \frac{1}{2} m\left(\dot{r}^2 + \left(r\dot{\theta}\right)^2\right) - \frac{\gamma}{r}$$

If the energy $H < 0$, so that $e < 1$, then the trajectory is an elliptical orbit as shown in Figure 4.5a. The semimajor and semiminor axes of the ellipse are

$$a = \frac{h}{1-e^2} = \frac{\gamma}{-2H},$$
$$b = \frac{h}{(1-e^2)^{1/2}} = \frac{\Gamma}{(-2Hm)^{1/2}}, \quad (4.105)$$

while the apogee and perigee of the orbit are

$$r_{\min} = \frac{h}{1+e}, \qquad r_{\max} = \frac{h}{1-e}. \quad (4.106)$$

The elliptical nature of the orbit is the first of Kepler's laws. Note that $r_0 = r_{\max}$.

The minimum-energy case $H = -\frac{m\gamma^2}{2\Gamma^2}$ corresponds to $e = 0$, for which the orbit is a circle. If the total energy is positive ($H > 0$), then $e > 1$, the sign of the second term in (4.104) becomes negative, and the trajectory is a hyperbola

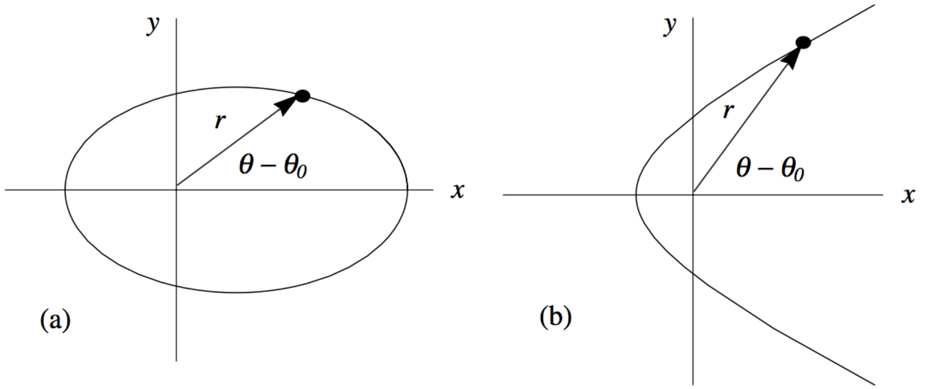

Fig. 4.5. Solutions of the Kepler problem for (a) negative total energy and (b) positive total energy.

as shown in Figure 4.5b. The radius of closest approach (the perihelion) is $r_{\min} = h/(1+e)$. Finally, if $H = 0$, the trajectory is a parabola.

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We can use the results of our analysis to develop the complete theory for the relationship between the radius of the orbit and the orbital period, which was treated using dimensional analysis in Chapter 2. For $H < 0$ the period T of the orbit can be derived from the second of Kepler's laws,

$$dA/dt = r^2\dot{\theta} = \text{constant}. \tag{4.107}$$

The area of the orbit is

$$A = r^2\dot{\theta}T = \frac{\Gamma}{2m}T. \tag{4.108}$$

The area of an ellipse is $A = \pi ab$, and so the area of the orbit is

$$A = \frac{\pi\gamma\Gamma}{m^{1/2}(-2H)^{3/2}}. \tag{4.109}$$

Equating (4.108) and (4.109) the orbital period is

$$T = \frac{2\pi\gamma m^{1/2}}{(-2H)^{3/2}}. \tag{4.110}$$

Using $-2H = \gamma/a$, this result can be cast in terms of $r_{\text{mean}} = \sqrt{ab}$ and the gravitational constant $G = 6.670 \times 10^{-11}$ m^3/(kg-s^2). The result is Kepler's third law,

$$\frac{G(m_1 + m_2)T^2}{(r_{\text{mean}})^3} = \frac{4\pi^2}{(1 - e^2)^{3/4}}. \tag{4.111}$$

Group $\quad \tilde{x} = e^{2a}x, \quad \tilde{y} = e^{2a}y, \quad \tilde{z} = e^{2a}z, \quad \tilde{t} = e^{3a}t, \quad \tilde{m} = m, \quad (4.112)$

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15.1.1 Transformation of Integrals

Let the Lagrangian of a differential function be

$$L = L[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \ldots, \mathbf{y}_p]. \tag{15.1}$$

We want to establish the conditions under which the action integral

$$S = \int_V L[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \ldots, \mathbf{y}_p]\, dx_1\, dx_2 \cdots dx_n \tag{15.2}$$

is invariant under the extended group

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \ldots, \mathbf{y}_r], \\
\tilde{y}^i &= y^i + s\eta^i[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \ldots, \mathbf{y}_r], \\
\tilde{y}^i_j &= y^i_j + s\eta^i_{\{j\}}[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \ldots, \mathbf{y}_r, \mathbf{y}_{r+1}], \\
\tilde{y}^i_{j_1 j_2} &= y^i_{j_1 j_2} + s\eta^i_{\{j_1 j_2\}}[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \ldots, \mathbf{y}_r, \mathbf{y}_{r+1}, \mathbf{y}_{r+2}], \\
&\vdots \\
\tilde{y}^i_{j_1 j_2 \cdots j_p} &= y^i_{j_1 j_2 \cdots j_p} + s\eta^i_{\{j_1 j_2 \cdots j_p\}}[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \ldots, \mathbf{y}_r, \ldots, \mathbf{y}_{r+p}],
\end{aligned} \tag{15.3}$$

where for any order p

$$\eta^i_{\{j_1 j_2 \cdots j_p\}} = D_{j_p}\eta^i_{\{j_1 j_2 \cdots j_{p-1}\}} - y^i_{j_1 j_2 \cdots j_{p-1}\alpha} D_{j_p}\xi^\alpha. \tag{15.4}$$

15.1.2 Transformation of the differential volume

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$$dV = dx^1 dx^2 \cdots dx^n. \tag{15.5}$$

$$d\tilde{x}^j = dx^j + s\left(\frac{\partial \xi^j}{\partial x_\alpha} + \frac{\partial \xi^j}{\partial y_\beta}\frac{dy_\beta}{dx_\alpha} + \frac{\partial \xi^j}{\partial y^\gamma_\beta}\frac{\partial y^\gamma_\beta}{\partial x^\alpha} + \cdots\right) dx^\alpha. \tag{15.6}$$

$$d\tilde{x}^j = dx^j + s\left(\frac{\partial \xi^j}{\partial x^j} + \frac{\partial \xi^j}{\partial y_\beta}\frac{dy_\beta}{dx^j} + \frac{\partial \xi^j}{\partial y^\gamma_\beta}\frac{\partial y^\gamma_\beta}{\partial x^j} + \cdots\right) dx^j \quad \text{(no sum over } j\text{)}, \tag{15.7}$$

$$d\tilde{x}^j = (1 + sD_j\xi^j)\, dx^j \qquad \text{(no sum over } j\text{)}. \tag{15.8}$$

Now the transformation of the differential volume becomes

$$d\tilde{x}^1 d\tilde{x}^2 \cdots d\tilde{x}^n = (1 + sD_1\xi^1)(1 + sD_2\xi^2)\cdots(1 + sD_n\xi^n)dx^1 dx^2 \cdots dx^n \tag{15.9}$$

Retain only terms of order s.

$$d\tilde{V} = dV + sD_j\xi^j\, dV. \qquad \text{sum over } j = 1, 2, \ldots, n \tag{15.10}$$

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Expand the Lagrangian in a Lie series.

$$L[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \ldots, \tilde{\boldsymbol{y}}_p] = L[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p] + sX_{\{p\}}L + O(s^2) + \cdots, \tag{15.11}$$

Where the group operator is.

$$X_{\{p\}} = \xi^j \frac{\partial}{\partial x^j} + \eta^i \frac{\partial}{\partial y^i} + \eta^i_{\{j_1\}} \frac{\partial}{\partial y^i_{j_1}} + \cdots + \eta^i_{\{j_1 j_2 \cdots j_p\}} \frac{\partial}{\partial y^i_{j_1 j_2 \cdots j_p}}. \tag{15.12}$$

The action integral becomes

$$\begin{aligned} S &= \int L[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}_1, \ldots, \tilde{\boldsymbol{y}}_p]\, d\tilde{x}^1\, d\tilde{x}^2 \cdots d\tilde{x}^n \\ &\approx \int (L[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p] + sX_{\{p\}}L)(1 + sD_j\xi^j)\, dV. \end{aligned} \tag{15.13}$$

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15.1.3 Invariance condition for integrals

$$\tilde{S} = S + s\int (X_{\{p\}}L + L(D_j\xi^j))\,dV + O(s^2) \;+\cdots \tag{15.14}$$

The integral is invariant if and only if

$$X_{\{p\}}L + L(D_j\xi^j) = 0. \tag{15.15}$$

A slightly more general condition is

$$X_{\{p\}}L + L(D_j\xi^j) = D_j\beta^j \tag{15.16}$$

where

$$\tilde{S} = S + s\int_A \beta^j\,dA_j, \tag{15.17}$$

$$\int_A \beta^j\,dA_j = 0. \tag{15.18}$$

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It is convenient to recast the problem in terms of the characteristic function

$$\mu^i = \eta^i - y^i_\alpha \xi^\alpha. \tag{15.19}$$

The infinitesimals of the extended transformation become

$$\begin{aligned}
\eta^i_{\{j_1\}} &= D_{j_1}\mu^i + y^i_{j_1\alpha}\xi^\alpha, \\
\eta^i_{\{j_1 j_2\}} &= D_{j_1 j_2}\mu^i + y^i_{j_1 j_2\alpha}\xi^\alpha, \\
&\vdots \\
\eta^i_{\{j_1 j_2 \cdots j_p\}} &= D_{j_1 j_2 \cdots j_p}\mu^i + y^i_{j_1 j_2 \cdots j_p\alpha}\xi^\alpha,
\end{aligned} \tag{15.20}$$

Now

$$
\begin{aligned}
X_{\{p\}}L &+ L(D_j\xi^j) \\
&= L(D_j\xi^j) + \xi^j \frac{\partial L}{\partial x^j} + \eta^i \frac{\partial L}{\partial y^i} \\
&\quad + D_{j_1}\mu^i \frac{\partial L}{\partial y^i_{j_1}} + D_{j_1 j_2}\mu^i \frac{\partial L}{\partial y^i_{j_1 j_2}} + \cdots + D_{j_1 j_2 \cdots j_p}\mu^i \frac{\partial L}{\partial y^i_{j_1 j_2 \cdots j_p}} \\
&\quad + \frac{\partial L}{\partial y^i_{j_1}} y^i_{j_1 j}\xi^j + \frac{\partial L}{\partial y^i_{j_1 j_2}} y^i_{j_1 j_2 j}\xi^j + \cdots + \frac{\partial L}{\partial y^i_{j_1 j_2 \cdots j_p}} y^i_{j_1 j_2 \cdots j_p j}\xi^j .
\end{aligned}
\tag{15.21}
$$

or

$$
\begin{aligned}
X_{\{p\}}L + L(D_j\xi^j) &= D_j(L\xi^j) + \mu^i \frac{\partial L}{\partial y^i} \\
&\quad + D_{j_1}\mu^i \frac{\partial L}{\partial y^i_{j_1}} + D_{j_1 j_2}\mu^i \frac{\partial L}{\partial y^i_{j_1 j_2}} + \cdots + D_{j_1 j_2 \cdots j_q}\mu^i \frac{\partial L}{\partial y^i_{j_1 j_2 \cdots j_p}} .
\end{aligned}
\tag{15.22}
$$

Introduce the Euler operator

$$E_i(\;) = \frac{\partial(\;)}{\partial y^i} - D_{j_1}\left(\frac{\partial(\;)}{\partial y^i_{j_1}}\right) + D_{j_1 j_2}\left(\frac{\partial(\;)}{\partial y^i_{j_1 j_2}}\right) - \cdots$$

$$+ \cdots + (-1)^p D_{j_1 j_2 \cdots j_p}\left(\frac{\partial(\;)}{\partial y^i_{j_1 j_2 \cdots j_p}}\right), \qquad (15.23)$$

$$X_{\{p\}}L + L(D_j \xi^j)$$

$$= D_j(L\xi^j) + \mu^i(E_i L)$$

$$+ \mu^i D_{j_1}\left(\frac{\partial L}{\partial y^i_{j_1}}\right) - \mu^i D_{j_1 j_2}\left(\frac{\partial L}{\partial y^i_{j_1 j_2}}\right) + \cdots (-1)^{p-1}\mu^i D_{j_1 j_2 \cdots j_p}\left(\frac{\partial L}{\partial y^i_{j_1 j_2 \cdots j_p}}\right)$$

$$+ D_{j_1}\mu^i\left(\frac{\partial L}{\partial y^i_{j_1}}\right) + D_{j_1 j_2}\mu^i\left(\frac{\partial L}{\partial y^i_{j_1 j_2}}\right) + \cdots + D_{j_1 j_2 \cdots j_p}\mu^i\left(\frac{\partial L}{\partial y^i_{j_1 j_2 \cdots j_p}}\right). \qquad (15.24)$$

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Integrate by parts

$$X_{\{p\}}L + L(D_j\xi^j) = D_j(L\xi^j) + \mu^i(E_iL) + D_{j_1}\theta^{j_1}, \qquad (15.25)$$

where

$$\theta^{j_1} = \Bigg[\mu^i\left\{\frac{\partial L}{\partial y^i_{j_1}} - D_{j_2}\frac{\partial L}{\partial y^i_{j_1j_2}} + D_{j_2j_3}\frac{\partial L}{\partial y^i_{j_1j_2j_3}} - \cdots + (-1)^{p-1}D_{j_2\cdots j_p}\frac{\partial L}{\partial y^i_{j_1\cdots j_p}}\right\}$$

$$+ D_{j_2}\mu^i\left\{\frac{\partial L}{\partial y^i_{j_1j_2}} - D_{j_3}\frac{\partial L}{\partial y^i_{j_1j_2j_3}} + D_{j_3j_4}\frac{\partial L}{\partial y^i_{j_1\cdots j_4}}\cdots\right.$$

$$\left.(-1)^{p-2}D_{j_3\cdots j_p}\frac{\partial L}{\partial y^i_{j_1\cdots j_p}}\right\}$$

$$+ D_{j_2j_3}\mu^i\left\{\frac{\partial L}{\partial y^i_{j_1j_2j_3}} - D_{j_4}\frac{\partial L}{\partial y^i_{j_1\cdots j_4}} + \cdots(-1)^{p-3}D_{j_4\cdots j_p}\frac{\partial L}{\partial y^i_{j_1\cdots j_p}}\right\}$$

$$+\cdots$$

$$+ D_{j_2\cdots j_{p-1}}\mu^i\left\{\frac{\partial L}{\partial y^i_{j_1j_2\cdots j_{p-1}}} - D_{j_p}\frac{\partial L}{\partial y^i_{j_1\cdots j_p}}\right\}$$

$$+ D_{j_2\cdots j_p}\mu^i\left\{\frac{\partial L}{\partial y^i_{j_1j_2\cdots j_p}}\right\}\Bigg]. \qquad (15.26)$$

Finally the transformation of the action integral becomes

$$\tilde{S} = S + s \int \left(\mu^i (E_i L) + D_{j_1} (L \xi^{j_1} + \theta^{j_1})\right) dV. \tag{15.27}$$

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Noether's theorem

Theorem 15.1. *Let* $L[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p]$ *be a differential function. The action integral*

$$S = \int L[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p]\, dx^1\, dx^2 \cdots dx^n \qquad (15.28)$$

is invariant under the Lie–Bäcklund group (15.3) with infinitesimals (ξ^j, η^i) $(j = 1, \ldots, n, i = 1, \ldots, m)$ *if and only if*

$$\mu^i(E_i L) + D_{j_1}(L\xi^{j_1} + \theta^{j_1}) = D_{j_1}\beta^{j_1}, \qquad (15.29)$$

where $\mu^i = \eta^i - y^i_\alpha \xi^\alpha$ *and* $\int \beta^j\, dA_j = 0$. *The vector* θ^{j_1} *is*

$$\theta^{j_1} = \left[\mu^i \left\{ \frac{\partial L}{\partial y^i_{j_1}} + \sum_{k=2}^{p} (-1)^{k-1} D_{j_2\cdots j_k} \frac{\partial L}{\partial y^i_{j_1\cdots j_k}} \right\} \right.$$
$$+ \sum_{\lambda=2}^{p-1} \left[D_{j_2\cdots j_\lambda}\mu^i \left\{ \frac{\partial L}{\partial y^i_{j_1\cdots j_\lambda}} + \sum_{k=\lambda}^{p-1} (-1)^{k-\lambda+1} D_{j_{\lambda+1}\cdots j_{k+1}} \frac{\partial L}{\partial y^i_{j_1\cdots j_{\lambda+1}\cdots j_{k+1}}} \right\} \right]$$
$$\left. + D_{j_2\cdots j_p}\mu^i \left\{ \frac{\partial L}{\partial y^i_{j_1 j_2\cdots j_p}} \right\} \right]. \qquad j_1 = i, \ldots, n \qquad (15.30)$$

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The condition (15.29) is met if $\mathbf{y}$ *is a solution of the generalized Euler–Lagrange system*

$$E_i L = \frac{\partial L}{\partial y^i} + \sum_{k=1}^{p} (-1)^k D_{j_1 j_2 \cdots j_k} \frac{\partial L}{\partial y^i_{j_1 j_2 \cdots j_k}} = 0 \qquad (15.31)$$

and if

$$D_j(L\xi^j + \theta^j - \beta^j) = 0 \qquad (15.32)$$

holds on solutions of (15.31). The combination

$$\boxed{\Gamma^j = L\xi^j + \theta^j - \beta^j} \qquad (15.33)$$

is a conserved vector for the system (15.31), and (15.32) is a conservation law.

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$$\gamma = Gm_1 m_2$$

Example 15.2 (A particle moving under the influence of a spherically symmetric inverse-square body force). The Lagrangian for such a particle is

$$L = \frac{1}{2}m\left((x_t^1)^2 + (x_t^2)^2 + (x_t^3)^2\right) + \frac{\gamma}{((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}}. \quad (15.44)$$

If $\gamma < 0$, the force is repelling from the origin. If $\gamma > 0$, the force is attracting to the origin. The corresponding Euler-Lagrange equations are

$$\frac{\partial L}{\partial x^i} - D_t\left(\frac{\partial L}{\partial x_t^i}\right) = -\left(\frac{\gamma x^i}{r^3} + m x_{tt}^i\right) = 0, \qquad i = 1, 2, 3, \quad (15.45)$$

where

$$r = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}. \quad (15.46)$$

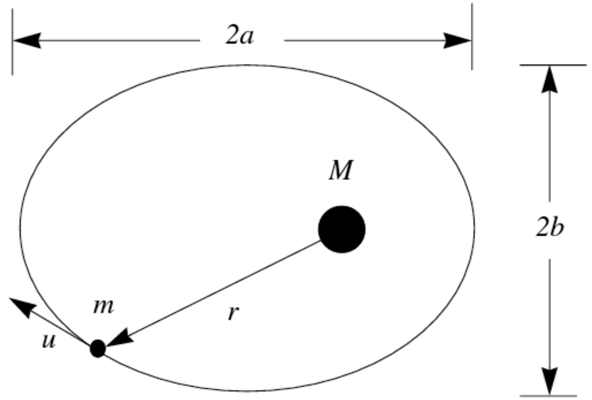

Fig. 2.1. Elliptical orbit of a planet about the sun.

Multi-parameter group

From Chapter 14 Section 14.4.3 we know that the system (15.45) is invariant under a five-parameter group of time translation, three rotations, and one dilation,

$$X^1 = \frac{\partial}{\partial t},$$

$$X^2 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad X^3 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad X^4 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{15.47}$$

$$X^5 = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} + 3t\frac{\partial}{\partial t}.$$

Conserved vector in y[x] notation

$$\Gamma^j = L\xi^j + \left(\eta^i - y^i_\alpha \xi^\alpha\right)\frac{\partial L}{\partial y^i_j}$$

In this problem $y^i \rightarrow x^i$, $x \rightarrow t$ — One independent and three dependent variables

$$\Gamma = L\xi + \left(\eta^i - x^i_t \xi\right)\frac{\partial L}{\partial x^i_t}$$

Conservation law connected to invariance under time translation

$$X^1 = \frac{\partial}{\partial t}$$

$$\xi^1 = 1 \qquad \eta^1 = 0 \qquad \eta^2 = 0 \qquad \eta^3 = 0$$

$$\Gamma = L\xi + \left(\eta^i - x_t^i \xi\right)\frac{\partial L}{\partial x_t^i} = L + \left(-x_t^i\right)\frac{\partial L}{\partial x_t^i}$$

$$= \frac{1}{2}m\left(\left(x_t^1\right)^2 + \left(x_t^2\right)^2 + \left(x_t^3\right)^2\right) + \frac{\gamma}{((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}}$$

$$- m\left(\left(x_t^1\right)^2 + \left(x_t^2\right)^2 + \left(x_t^3\right)^2\right)$$

$$= -\frac{1}{2}m\left(\left(x_t^1\right)^2 + \left(x_t^2\right)^2 + \left(x_t^3\right)^2\right) + \frac{\gamma}{((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}}, \qquad (15.48)$$

$\Gamma \to -E$

The energy is a conserved "vector"

$$E = \frac{1}{2}mv^2 - \frac{\gamma}{r}, \qquad (15.49)$$

Check $\frac{DE}{Dt} = 0$

$$E = \frac{1}{2} m v^2 - \frac{\gamma}{r}, \tag{15.49}$$

$$E = \frac{1}{2} m \left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right) - \frac{\gamma}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\frac{DE}{Dt} = \frac{\partial E}{\partial t} + \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} + \frac{\partial E}{\partial z}\frac{dz}{dt} + \frac{\partial E}{\partial \left(\frac{dx}{dt} \right)}\frac{d^2x}{dt^2} + \frac{\partial E}{\partial \left(\frac{dy}{dt} \right)}\frac{d^2y}{dt^2} + \frac{\partial E}{\partial \left(\frac{dz}{dt} \right)}\frac{d^2z}{dt^2}$$

$$\frac{DE}{Dt} = \frac{\gamma}{r^3}\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} \right) + m\left(\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2} \right)$$

$$\frac{DE}{Dt} = \frac{dx}{dt}\left(m\frac{d^2x}{dt^2} + \frac{\gamma x}{r^3} \right) + \frac{dy}{dt}\left(m\frac{d^2y}{dt^2} + \frac{\gamma y}{r^3} \right) + \frac{dz}{dt}\left(m\frac{d^2z}{dt^2} + \frac{\gamma z}{r^3} \right) = 0$$

Conservation law connected to invariance under rotation

$$\xi = 0$$
$$\eta^1 = 0$$
$$\eta^2 = -x^3$$
$$\eta^3 = x^2$$

$$X^2 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad X^3 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad X^4 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$\Gamma = L\xi + \left(\eta^i - x_t^i\xi\right)\frac{\partial L}{\partial x_t^i} = \eta^i\frac{\partial L}{\partial x_t^i}$$

$$M^1 = L\xi + (\eta^1 - x_t^1\xi)\frac{\partial L}{\partial x_t^1} + (\eta^2 - x_t^2\xi)\frac{\partial L}{\partial x_t^2} + (\eta^3 - x_t^3\xi)\frac{\partial L}{\partial x_t^3} = \eta^1\frac{\partial L}{\partial x_t^1} + \eta^2\frac{\partial L}{\partial x_t^2} + \eta^3\frac{\partial L}{\partial x_t^3} = -x^3\frac{\partial L}{\partial x_t^2} + x^2\frac{\partial L}{\partial x_t^3} = -mx^3x_t^2 + mx^2x_t^3$$

$$\Gamma \to M^i$$

$$y^i \to x^i$$

$$x \to t$$

Invariance under rotation leads to conservation of the three components of angular momentum,

$$M^1 = L\xi^j + (\eta^i - y_\alpha^i\xi^\alpha)\frac{\partial L}{\partial y_j^i} = x^3\frac{\partial L}{\partial x_t^2} - x^2\frac{\partial L}{\partial x_t^3} = m\left(x^2x_t^3 - x^3x_t^2\right),$$

$$M^2 = L\xi^j + (\eta^i - y_\alpha^i\xi^\alpha)\frac{\partial L}{\partial y_j^i} = x^1\frac{\partial L}{\partial x_t^3} - x^3\frac{\partial L}{\partial x_t^1} = m\left(x^3x_t^1 - x^1x_t^3\right),$$

$$M^3 = L\xi^j + (\eta^i - y_\alpha^i\xi^\alpha)\frac{\partial L}{\partial y_j^i} = x^2\frac{\partial L}{\partial x_t^1} - x^1\frac{\partial L}{\partial x_t^2} = m\left(x^1x_t^2 - x^2x_t^1\right),$$

(15.50)

or

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{P}. \qquad (15.51)$$

Conservation law connected to invariance under rotation

$$X^2 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad X^3 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad X^4 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$\Gamma = L\xi + \left(\eta^i - x_t^i\xi\right)\frac{\partial L}{\partial x_t^i} = \eta^i\frac{\partial L}{\partial x_t^i}$$

Invariance under rotation leads to conservation of the three components of angular momentum,

$\Gamma \rightarrow M^i$

$y^i \rightarrow x^i$

$x \rightarrow t$

$$M^1 = L\xi^j + (\eta^i - y_\alpha^i\xi^\alpha)\frac{\partial L}{\partial y_j^i} = x^3\frac{\partial L}{\partial x_t^2} - x^2\frac{\partial L}{\partial x_t^3} = m\left(x^2x_t^3 - x^3x_t^2\right),$$

$$M^2 = L\xi^j + (\eta^i - y_\alpha^i\xi^\alpha)\frac{\partial L}{\partial y_j^i} = x^1\frac{\partial L}{\partial x_t^3} - x^3\frac{\partial L}{\partial x_t^1} = m\left(x^3x_t^1 - x^1x_t^3\right),$$

$$M^3 = L\xi^j + (\eta^i - y_\alpha^i\xi^\alpha)\frac{\partial L}{\partial y_j^i} = x^2\frac{\partial L}{\partial x_t^1} - x^1\frac{\partial L}{\partial x_t^2} = m\left(x^1x_t^2 - x^2x_t^1\right),$$

(15.50)

or

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{P}. \qquad (15.51)$$

Check $\dfrac{DM^1}{Dt} = 0$

$$M^1 = m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)$$

$$\frac{DM^1}{Dt} = \frac{\partial M^1}{\partial t} + \frac{\partial M^1}{\partial x}\frac{dx}{dt} + \frac{\partial M^1}{\partial y}\frac{dy}{dt} + \frac{\partial M^1}{\partial z}\frac{dz}{dt} + \frac{\partial M^1}{\partial\left(\frac{dx}{dt}\right)}\frac{d^2x}{dt^2} + \frac{\partial M^1}{\partial\left(\frac{dy}{dt}\right)}\frac{d^2y}{dt^2} + \frac{\partial M^1}{\partial\left(\frac{dz}{dt}\right)}\frac{d^2z}{dt^2}$$

$$\frac{DM^1}{Dt} = m\frac{dz}{dt}\frac{dy}{dt} - m\frac{dy}{dt}\frac{dz}{dt} - mz\frac{d^2y}{dt^2} + my\frac{d^2z}{dt^2}$$

$$\frac{DM^1}{Dt} = z\frac{\gamma y}{r^3} - y\frac{\gamma z}{r^3} = 0$$

Is there a conservation law connected to invariance under dilation?

$$X^5 = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + \frac{3}{2} t \frac{\partial}{\partial t}$$

$$\Gamma = L\xi + \left(\eta^i - x_t^i \xi\right)\frac{\partial L}{\partial x_t^i} = \frac{3}{2} tL + \left(x^1 - \frac{3}{2} t x_t^1\right)\frac{\partial L}{\partial x_t^1} + \left(x^2 - \frac{3}{2} t x_t^2\right)\frac{\partial L}{\partial x_t^2} + \left(x^3 - \frac{3}{2} t x_t^3\right)\frac{\partial L}{\partial x_t^3} =$$

$$\frac{3}{2} t \left(\frac{1}{2} m\left(\left(x_t^1\right)^2 + \left(x_t^2\right)^2 + \left(x_t^3\right)^2\right) + \frac{\gamma}{\left(\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2\right)^{1/2}} \right) +$$

$$\left(x^1 - \frac{3}{2} t x_t^1\right) m x_t^1 + \left(x^2 - \frac{3}{2} t x_t^2\right) m x_t^2 + \left(x^3 - \frac{3}{2} t x_t^3\right) m x_t^3 =$$

$$-\frac{3}{4} tm\left(\left(x_t^1\right)^2 + \left(x_t^2\right)^2 + \left(x_t^3\right)^2\right) + \frac{3}{2} t \frac{\gamma}{\left(\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2\right)^{1/2}} + \tag{15.52}$$

$$m\left(x^1 x_t^1 + x^2 x_t^2 + x^2 x_t^2\right) =$$

$$\Gamma \rightarrow R \qquad\qquad R = -\frac{3}{2} Et + P \cdot x \tag{15.53}$$

Is this a constant of the motion?

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Take the divergence of R

$$\frac{DR}{Dt} = \frac{\partial R}{\partial t} + \frac{\partial R}{\partial x}\frac{dx}{dt} + \frac{\partial R}{\partial y}\frac{dy}{dt} + \frac{\partial R}{\partial z}\frac{dz}{dt} + \frac{\partial R}{\partial\left(\frac{dx}{dt}\right)}\frac{d^2x}{dt^2} + \frac{\partial R}{\partial\left(\frac{dy}{dt}\right)}\frac{d^2y}{dt^2} + \frac{\partial R}{\partial\left(\frac{dz}{dt}\right)}\frac{d^2z}{dt^2}$$

$$R = -\frac{3}{2}Et + P\cdot x = -\frac{3}{2}Et + m\left(x^i x_t^i\right)$$

$$\frac{DR}{Dt} = -\frac{3}{2}E + m\left(x_t^i\right)^2 + m\left(x^i x_{tt}^i\right) = -\frac{3}{2}E + m\left(x_t^i\right)^2 + x^i\left(-\gamma\frac{x^i}{r^3}\right) =$$

$$-\frac{3}{2}E + mv^2 - \frac{\gamma}{r} = -\frac{3}{2}\left(\frac{1}{2}mv^2 - \frac{\gamma}{r}\right) + mv^2 - \frac{\gamma}{r} =$$

$$\frac{1}{2}\left(\frac{1}{2}mv^2 + \frac{\gamma}{r}\right) =$$

$$\frac{1}{2}mv^2 - \frac{1}{2}E \neq 0$$

Since DR/Dt is not zero, R is not a conserved “vector”.

The dilation group X^5 is not a variational symmetry of the Kepler system

The system (15.45) is also invariant under the three-parameter Lie–Bäcklund transformation

$$X^6 = \left(2x^1 x_t^i - x^i x_t^1 - \left(x^k x_t^k\right)\delta_1^i\right)\frac{\partial}{\partial x^i},$$
$$X^7 = \left(2x^2 x_t^i - x^i x_t^2 - \left(x^k x_t^k\right)\delta_2^i\right)\frac{\partial}{\partial x^i}, \qquad (15.55)$$
$$X^8 = \left(2x^3 x_t^i - x^i x_t^3 - \left(x^k x_t^k\right)\delta_3^i\right)\frac{\partial}{\partial x^i}.$$

Let's construct the one-component conserved vectors corresponding to each of these groups. Let $x = x^1$, $y = x^2$, and $z = x^3$. Thus the once extended group corresponding to X^6 is

$$X^6_{\{1\}} = (-yy_t - zz_t)\frac{\partial}{\partial x} + (2xy_t - yx_t)\frac{\partial}{\partial y} + (2xz_t - zx_t)\frac{\partial}{\partial z}$$
$$+ \left(-y_t^2 - yy_{tt} - z_t^2 - zz_{tt}\right)\frac{\partial}{\partial x_t}$$
$$+ (x_t y_t + 2xy_{tt} - yx_{tt})\frac{\partial}{\partial y_t}$$
$$+ (x_t z_t + 2xz_{tt} - zx_{tt})\frac{\partial}{\partial z_t}. \qquad (15.56)$$

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The action of this operator on the Lagrangian is

$$X^{6}_{\{1\}}L = \frac{2\gamma}{r^3}\left(xyy_t - y^2x_t + xzz_t - z^2x_t\right), \tag{15.57}$$

where (15.45) has been used to eliminate the second-derivative terms. The result (15.57) can be written as

$$X^{6}_{\{1\}}L = -D_t\left(2\gamma\frac{x}{r}\right), \tag{15.58}$$

where

$$r = (x^2 + y^2 + z^2)^{1/2}. \tag{15.59}$$

The expression $2\gamma x/r$ is the vector B^j that appears in (15.16). Note that $\xi^j = 0$ for the groups (15.55). This process can be repeated for the remaining operators in (15.55) to give

$$\begin{aligned} X^{7}_{\{1\}}L &= -D_t\left(2\gamma\frac{y}{r}\right), \\ X^{8}_{\{1\}}L &= -D_t\left(2\gamma\frac{z}{r}\right). \end{aligned} \tag{15.60}$$

Now the conserved vector (15.33) can be constructed as follows:

$$
\begin{aligned}
-2Q^1 &= L\xi^j + \left(\eta^i - y^i_\alpha \xi^\alpha\right)\frac{\partial L}{\partial y^i_j} - B^j = \eta^i \frac{\partial L}{\partial y^i_j} + 2\gamma \frac{x^1}{r} \\
&= -m x^i_t \eta^i + 2\gamma \frac{x^1}{r} \\
&= \left(2x^1 x^i_t - x^i x^1_t - \left(x^k x^k_t\right)\delta^i_1\right)\left(-m x^i_t\right) + 2\gamma \frac{x^1}{r} \\
&= m\left(x^k x^k_t x^1_t + x^i x^i_t x^1_t - 2x^1 x^i_t x^i_t\right) + 2\gamma \frac{x^1}{r}.
\end{aligned}
\tag{15.61}
$$

Similarly,

$$
\begin{aligned}
-2Q^2 &= m\left(x^k x^k_t x^2_t + x^i x^i_t x^2_t - 2x^2 x^i_t x^i_t\right) + 2\gamma \frac{x^2}{r}, \\
-2Q^3 &= m\left(x^k x^k_t x^3_t + x^i x^i_t x^3_t - 2x^3 x^i_t x^i_t\right) + 2\gamma \frac{x^3}{r}.
\end{aligned}
\tag{15.62}
$$

In vector notation,

$$
\boldsymbol{Q} = m((\boldsymbol{u}\cdot\boldsymbol{u})\boldsymbol{x} - (\boldsymbol{x}\cdot\boldsymbol{u})\boldsymbol{u}) - \gamma \frac{\boldsymbol{x}}{r},
\tag{15.63}
$$

where $\boldsymbol{u} = \boldsymbol{x}_t$. One can show that $D_t(\boldsymbol{Q}) = 0$.

Using the vector identity $\boldsymbol{u} \times (\boldsymbol{x} \times \boldsymbol{u}) = \boldsymbol{x}(\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{x})$, the conserved vector (15.63) can be written as

$$\boldsymbol{Q} = \boldsymbol{u} \times \boldsymbol{M} - \gamma \frac{\boldsymbol{x}}{r}. \tag{15.64}$$

This vector originates in a Lie–Bäcklund symmetry of the Kepler equations and is called in the literature Laplace's vector or the Runge–Lenz vector. The vector $\boldsymbol{Q}$ lies in the plane of the orbit and points along the major axis from the origin (which in the reduced-mass problem is a focus of the orbit) toward the perihelion (the point of closest approach to the origin). The magnitude of $\boldsymbol{Q}$ is proportional to the eccentricity of the orbit:

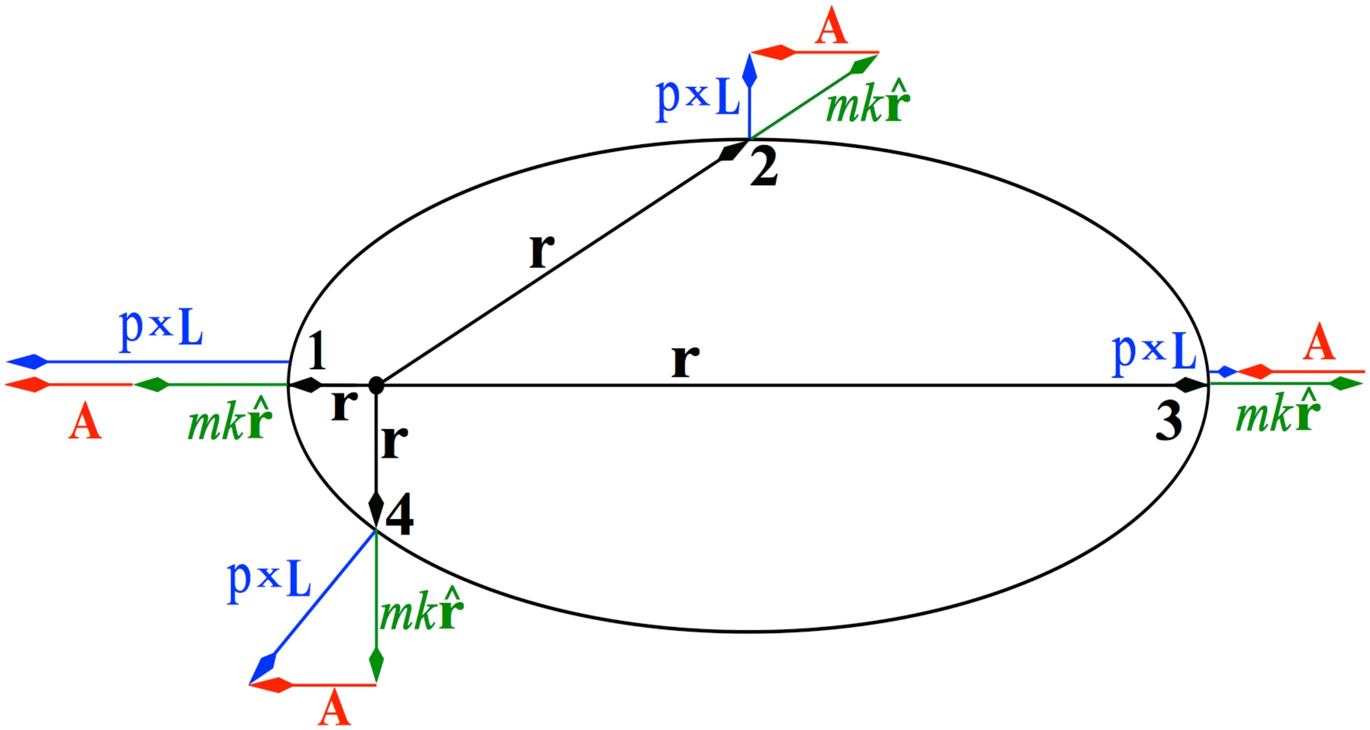
$$|\boldsymbol{Q}| = \gamma e = \gamma \left(1 + \frac{2H\Gamma^2}{m\alpha^2}\right)^{1/2}. \tag{15.65}$$

In summary, there are a total of seven conserved quantities for this problem:

$$\begin{aligned} E &= \frac{1}{2}mv^2 - \frac{\gamma}{r}, \\ \boldsymbol{M} &= \boldsymbol{r} \times \boldsymbol{P}, \\ \boldsymbol{Q} &= \boldsymbol{u} \times \boldsymbol{M} - \gamma \frac{\boldsymbol{x}}{r}. \end{aligned} \tag{15.66}$$

In addition to these constants of the motion, there is the scaling symmetry X^5 that gives Kepler's third law. Although X^5 does not produce a conservation law for the Kepler system, such a possibility is not precluded simply because the symmetry is a dilation. See Exercise 15.3.

The Laplace vector



$$m = \frac{m_1 m_2}{m_1 + m_2}$$

$$k = G\left(m_1 + m_2\right)$$

$$\gamma = G m_1 m_2$$

$$A = p \times L - mk\hat{r}$$

$$\bar{A} = \bar{Q} = \bar{p} \times \bar{L} - mk\hat{r} = \bar{u} \times \bar{M} - \gamma \frac{\bar{x}}{r}$$

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Problem 15.3

15.3 In Chapter 4, Example 4.4 we solved the two-body problem in a central force field with a general potential function $V[r]$. The equations of motion are

$$mx^i_{tt} + \frac{x^i}{r}\left(\frac{\partial V}{\partial r}\right) = 0. \tag{15.76}$$

The solution for the radius is expressed implicitly in terms of the time

$$t = \int_{r_0}^{r} \frac{dr}{\left(\frac{2}{m}(H - V[r]) - \frac{\Gamma^2}{m^2 r^2}\right)^{1/2}}, \tag{15.77}$$

and the angle is determined from conservation of angular momentum:

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{\Gamma dr}{r^2\left(2m(H - V[r]) - \frac{\Gamma^2}{r^2}\right)^{1/2}}. \tag{15.78}$$

For a general $V[r]$ the particle is constrained to move in an annular disk between two radii, $r_{\min}$ and $r_{\max}$. Eventually the particle motion fills the region between the two radii. Only when $V = -\gamma/r$ does the trajectory execute a closed path, and this is the situation for the Kepler problem. The quantity $\Gamma^2/(m^2r^2)$ is called the centrifugal energy [15.3] and becomes infinite as the radius of the orbit goes to zero, preventing the particle from falling in to the origin for any initial condition with a finite angular momentum. The term $\left(\frac{2}{m}\right)V[r] + \Gamma^2/(m^2r^2)$ in the integrands in (15.77) and (15.78) is called the effective potential energy, and the only way the particle can fall to the origin is if this term becomes negative or zero as the radius goes to zero. In this case the attractive force at the origin is strong enough to overcome the centrifugal energy. Study the symmetries and conservation laws for the system (15.76) for $V = -\gamma/r^n$. Show that for the case $n = 2$ the dilation group generates a variational symmetry. Compare your result with Example 15.2, and interpret it physically. What is special about $n = 2$?

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Problem 15.3

$$L = \frac{1}{2}m\left(\left(x_t^1\right)^2 + \left(x_t^2\right)^2 + \left(x_t^3\right)^2\right) + \frac{\gamma}{\left(\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2\right)}$$

$$mx_{tt}^i + 2\gamma\frac{x^i}{r^4} = 0$$

Dilation group $\tilde{x}^i = e^a x^i \qquad \tilde{t} = e^b t$

$$m\tilde{x}_{tt}^i + 2\gamma\frac{\tilde{x}^i}{\tilde{r}^4} = e^{a-2b}mx_{tt}^i + e^{-3a}2\gamma\frac{x^i}{r^4} = 0 \Rightarrow b = 2a$$

$$\tilde{x}^i = e^a x^i \qquad \tilde{t} = e^{2a}t$$

$$\eta^i = x^i \qquad \xi = 2t$$

Several interesting papers related to a potential that varies like 1/r^2:

"On the dynamical and geometrical symmetries of Keplerian motion"

"Invariant variation problems Emmy Noether"

"Anomalies in quantum mechanics: The 1/r^2 potential"

"A gravitational diffusion model without dark matter"

https://johncarlosbaez.wordpress.com/2015/08/30/the-inverse-cube-force-law/

https://en.wikipedia.org/wiki/Newton%27s_theorem_of_revolving_orbits

Is there a conservation law connected to invariance under dilation for the potential $V = -\gamma/r^2$?

$$\tilde{x}^i = e^a x^i \qquad \tilde{t} = e^{2a} t$$

$$\eta^i = x^i \qquad \xi = 2t$$

$$X = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + 2t \frac{\partial}{\partial x}$$

$$L = \frac{1}{2} m \left(\left(x_t^1\right)^2 + \left(x_t^2\right)^2 + \left(x_t^3\right)^2 \right) + \frac{\gamma}{\left(\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 \right)}$$

Show

$$\Gamma = L\xi + \left(\eta^i - x_t^i \xi\right) \frac{\partial L}{\partial x_t^i} = -2tE + \bar{P} \cdot \bar{x}$$

$$\Gamma \to R \qquad \boxed{\mathrm{R} = -2tE + \bar{P} \cdot \bar{x}}$$

Problem 15.3 constants of the motion

$$E = \frac{1}{2}mv^2 - \frac{\gamma}{r^2}$$

$$R = -2tE + \overline{P} \cdot \overline{x} = -2tE + \frac{m}{2}\frac{dr^2}{dt}$$

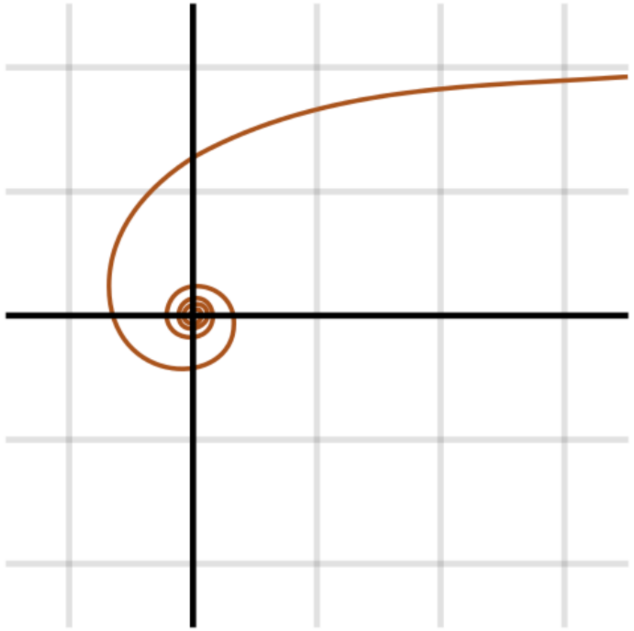
$$\Gamma = mr^2 \frac{d\theta}{dt}$$

Solve

$$r^2 = r_0^2 + \frac{2R}{m}t + \frac{2E}{m}t^2$$

$$\Gamma = mr^2 \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{\Gamma}{m\left(r_0^2 + \frac{2R}{m}t + \frac{2E}{m}t^2\right)}$$



Depending on the sign of E the mass spirals inward or outward.

Problem 15.3

If $\left.\frac{dr^2}{dt}\right|_{t=0} = 0$ then $R=0$

$$E = \frac{1}{2}mv^2 - \frac{\gamma}{r^2}$$

$$R = 0$$

$$\frac{dr^2}{dt} = \frac{4E}{m}t$$

$$r^2 = r_0^2 + \frac{2E}{m}t^2$$

$$\Gamma = mr^2\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{\Gamma}{m\left(r_0^2 + \frac{2E}{m}t^2\right)}$$

$$\Gamma = mr_0^2\left.\frac{d\theta}{dt}\right|_{t=0}$$

If $\left.\frac{dr^2}{dt}\right|_{t=0} = 0$ and $E=0$

$$\frac{1}{2}mv^2 = \frac{\gamma}{r^2}$$

$$R = 0$$

$$\frac{dr^2}{dt} = 0$$

$$r^2 = r_0^2 = const$$

$$\Gamma = mr_0^2\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{\Gamma}{mr_0^2} = const$$

$$\Gamma = mr_0^2\left.\frac{d\theta}{dt}\right|_{t=0}$$

The mass follows a metastable circular orbit

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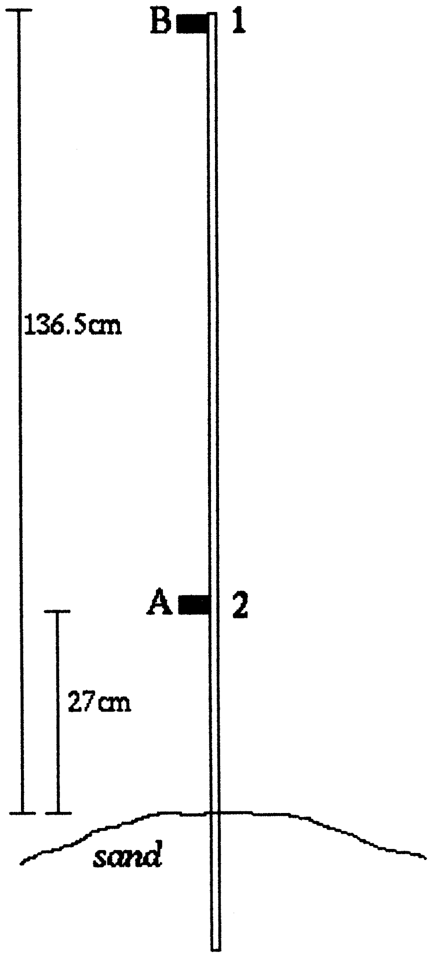
Dispersion of Flexural Waves

Daniel A. Russell, GMI Engineering & Management Institute

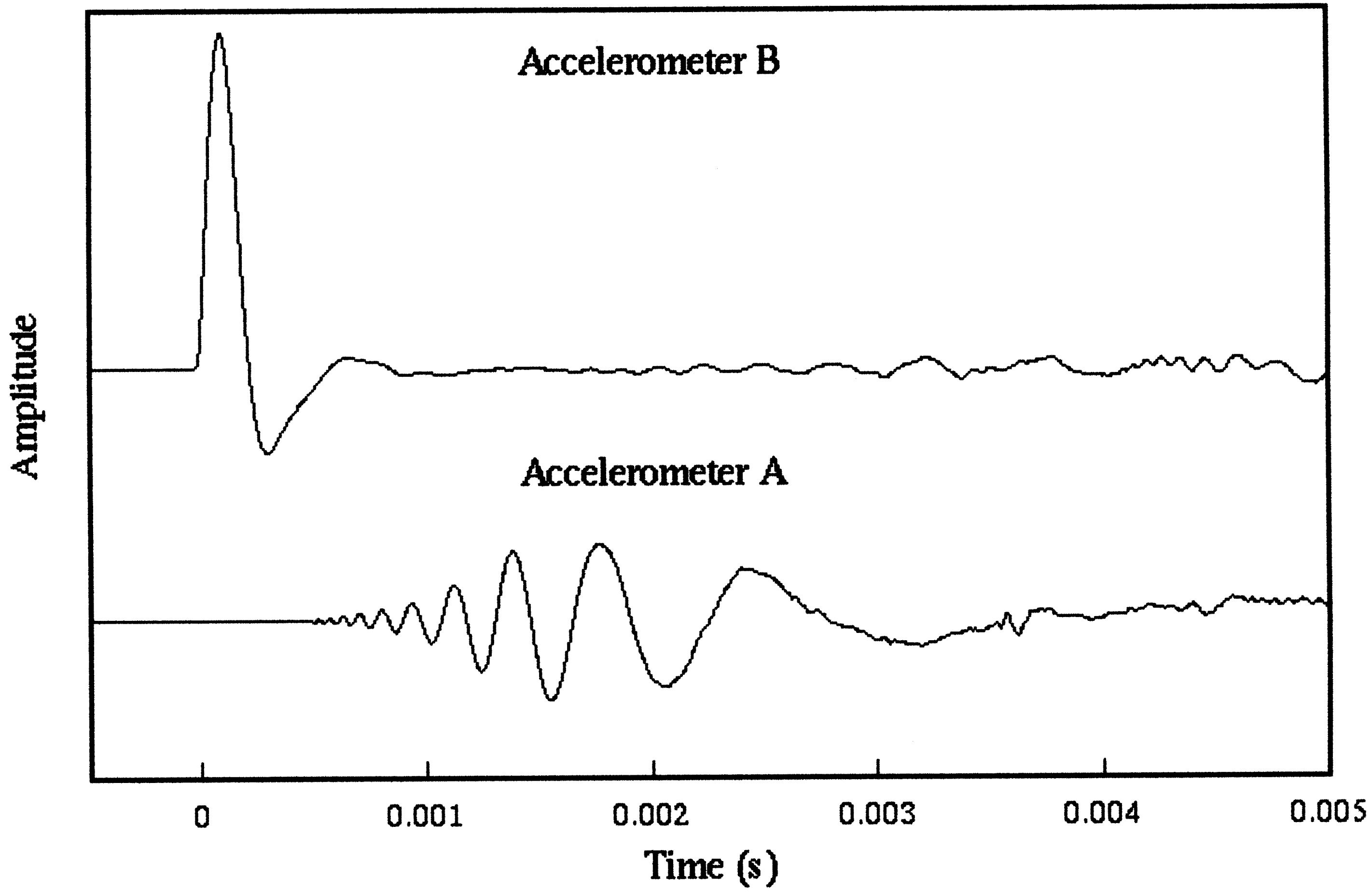


A simple experiment can be performed to demonstrate the dispersive nature of flexural waves

- An aluminum bar of cross-section 1/4 x1/2 inch and length 6 ft is vertically inserted in sand so that 4.48 ft (136.5 cm) is allowed to freely vibrate in a horizontal direction.
- The sand acts as a non-reflective boundary so that any flexural waves incident upon the sand will be completely absorbed.
- PCB accelerometers (1 gram) are attached at positions **A** and **B**
- Aluminum bar is excited with an impulse, using PCB hammer with plastic tip, at points **1** and **2**
- The time signals measured by the accelerometers are displayed on an oscilloscope (one that can store a trace)

Hammer impact at point 1

- Force pulse is very clean at location B
- Pulse disperses by the time it reaches location A --- higher frequency waves travel faster and arrive first --- lower frequency waves travel slower and arrive later



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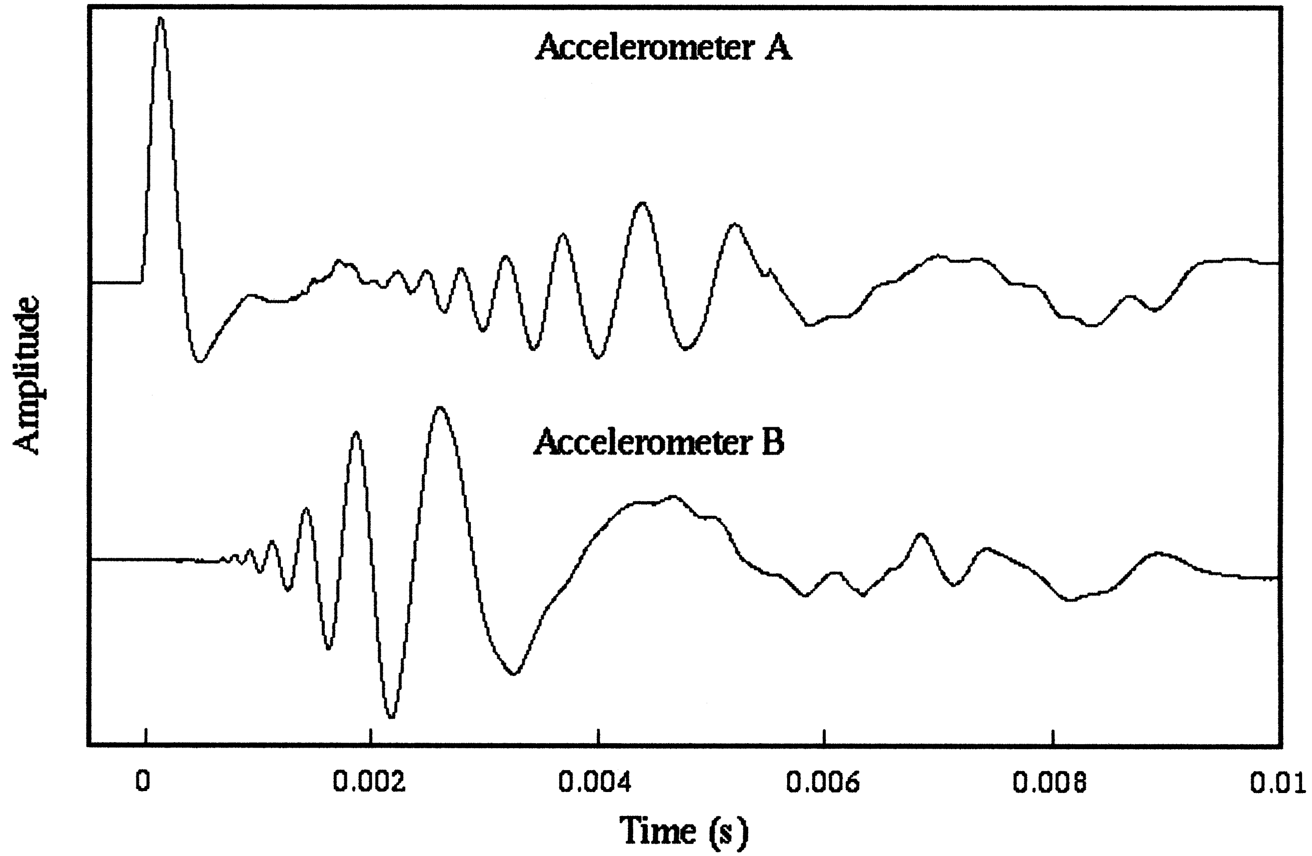
Hammer impact at point 2

- Initial force pulse is very clean at location A
- Pulse disperses by the time it reaches location B --- higher frequency waves travel faster and arrive first --- lower frequency waves travel slower and arrive later
- Flexural waves reflect from free end of beam and travel back down to location A --- dispersion is even more evident in reflected signal



Solutions

In dimensioned form the equation is

$$\frac{\partial^2 y}{\partial \tau^2} + \left(\frac{EI}{\rho A}\right)\frac{\partial^4 y}{\partial \chi^4} = 0$$

where

I - Moment of inertia about the neutral axis of bending $[I] = L^4$

A - Area of the cross section of the beam

E – Young's modulus $[E] = M/LT^2$

ρ – material density

Take the origin of coordinates at the center of the rod. Even solutions are of the form

$$y[\chi,\tau] = e^{i\omega\tau}\left(A\,\mathrm{Cosh}[k\chi] + B\,\mathrm{Cos}[k\chi]\right)$$

and odd solutions are

$$y[\chi,\tau] = e^{i\omega\tau}\left(C\,\mathrm{Sinh}[k\chi] + D\,\mathrm{Sin}[k\chi]\right)$$

The frequency and wave number cannot be selected independently. They are related by

$$\left(-\omega^2 + \frac{EI}{\rho A} k^4\right) y[\chi, t] = 0$$

thus

$$\omega^2 = \frac{EI}{\rho A} k^4$$

and the wave speed is

$$c = \frac{\omega}{k} = \left(\frac{EI}{\rho A}\right)^{1/2} k$$

The waves are highly dispersive with short wavelengths traveling much faster than long waves.

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Nondimensionalize using the characteristic wave number and frequency

$$x = k_0 \chi \qquad t = \omega_0 \tau$$

where

$$k_0 = \left(\frac{1}{I}\right)^{1/4} \qquad \omega_0 = \left(\frac{E}{\rho A}\right)^{1/2}$$

In dimensionless variables the equation becomes

$$y_{tt} + y_{xxxx} = 0$$

Solution by separation of variables

We wish to solve

$$y_{tt} + y_{xxxx} = 0$$

Combine invariance under translation in time and invariance under dilation of the dependent variable

$$X^a = \frac{\partial}{\partial t} \qquad X^b = y\frac{\partial}{\partial y}$$

To form the group operator

$$X = \frac{1}{\lambda}X^a + X^b = \frac{1}{\lambda}\frac{\partial}{\partial t} + y\frac{\partial}{\partial y}$$

With characteristic equations

$$\frac{dx}{0} = \lambda dt = \frac{dy}{y}$$

and invariants

$$\psi^1 = x \qquad \psi^2 = y/e^{\lambda t}$$

We can expect a solution of the form

$$y = e^{\lambda t}G[x]$$

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We are seeking time-periodic solutions of the equation. Let

$$\lambda = i\omega$$

The solution is of the form

$$y[x,t] = e^{i\omega t} G[x]$$

Substitute

$$y_{tt} + y_{xxxx} = e^{i\omega t}\left(G_{xxxx} - \omega^2 G\right)$$

The fourth order ODE

$$G_{xxxx} - \omega^2 G = 0$$

has the general solution

$$G(x) = Ae^{\omega^{1/2}x} + Be^{-\omega^{1/2}x} + CSin\left(\omega^{1/2}x\right) + DCos\left(\omega^{1/2}x\right)$$

Even and odd solutions are

$$y[x,t] = e^{i\omega t}\left(A\,\mathrm{Cosh}[kx] + B\,\mathrm{Cos}[kx]\right)$$

$$y[x,t] = e^{i\omega t}\left(C\,\mathrm{Sinh}[kx] + D\,\mathrm{Sin}[kx]\right)$$

where

$$\omega^2 = k^4$$

Superposition of solutions for various frequencies and associated wave numbers can be used to match the boundary conditions for a given problem.

The Lagrangian

The equation governing flexural waves in a beam is

$$y_{tt} + y_{xxxx} = 0$$

Solutions of this equation minimize the action integral

$$S = \int \left(-\frac{1}{2} y_t^2 + \frac{1}{2} y_{xx}^2 \right) dt dx$$

ie, solutions minimize the volume integral of the difference between kinetic and potential energy. The equation can be generated from the Lagrangian

$$L = -\frac{1}{2} y_t^2 + \frac{1}{2} y_{xx}^2$$

Substitute the Lagrangian into the Euler-Lagrange equations

$$\frac{\partial L}{\partial y} - D_t\left(\frac{\partial L}{\partial y_t}\right) - D_x\left(\frac{\partial L}{\partial y_x}\right) + D_{tt}\left(\frac{\partial L}{\partial y_{tt}}\right) + D_{tx}\left(\frac{\partial L}{\partial y_{tx}}\right) + D_{xx}\left(\frac{\partial L}{\partial y_{xx}}\right) = 0$$

The result is

$$-D_t\left(\frac{\partial L}{\partial y_t}\right) + D_{xx}\left(\frac{\partial L}{\partial y_{xx}}\right) = 0$$

$$-D_t\left(-y_t\right) + D_{xx}\left(y_{xx}\right) = 0$$

$$y_{tt} + y_{xxxx} = 0$$

Symmetries

The equation is invariant under a four parameter group of translations and dilations plus the infinite-dimensional group corresponding to linear superposition of solutions.

$$X^1 = \frac{\partial}{\partial t} \qquad X^2 = \frac{\partial}{\partial x} \qquad X^3 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} \qquad X^4 = y\frac{\partial}{\partial y} \qquad X^5 = \phi[t,x]\frac{\partial}{\partial y}$$

where $\phi[t,x]$ is a solution of

$$\phi_{tt} + \phi_{xxxx} = 0$$

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Conservation laws from symmetries

The relations used to generate components of the conserved vectors corresponding to variational symmetries of this equation are:

$$\theta^t = \mu\left(\frac{\partial L}{\partial y_t} - D_t\left(\frac{\partial L}{\partial y_{tt}}\right) - D_x\left(\frac{\partial L}{\partial y_{tx}}\right)\right) + D_t\mu\left(\frac{\partial L}{\partial y_{tt}}\right) + D_x\mu\left(\frac{\partial L}{\partial y_{tx}}\right)$$

$$\theta^x = \mu\left(\frac{\partial L}{\partial y_x} - D_t\left(\frac{\partial L}{\partial y_{xt}}\right) - D_x\left(\frac{\partial L}{\partial y_{xx}}\right)\right) + D_t\mu\left(\frac{\partial L}{\partial y_{xt}}\right) + D_x\mu\left(\frac{\partial L}{\partial y_{xx}}\right)$$

$$\Gamma^t = L\tau + \mu\left(\frac{\partial L}{\partial y_t} - D_t\left(\frac{\partial L}{\partial y_{tt}}\right) - D_x\left(\frac{\partial L}{\partial y_{tx}}\right)\right) + D_t\mu\left(\frac{\partial L}{\partial y_{tt}}\right) + D_x\mu\left(\frac{\partial L}{\partial y_{tx}}\right)$$

$$\Gamma^x = L\xi + \mu\left(\frac{\partial L}{\partial y_x} - D_t\left(\frac{\partial L}{\partial y_{xt}}\right) - D_x\left(\frac{\partial L}{\partial y_{xx}}\right)\right) + D_t\mu\left(\frac{\partial L}{\partial y_{xt}}\right) + D_x\mu\left(\frac{\partial L}{\partial y_{xx}}\right)$$

where the characteristic function is

$$\mu = \eta - \tau y_t - \xi y_x$$

Invariance under translation in time

Infinitesimals

$$\tau = 1$$
$$\xi = 0$$
$$\eta = 0$$

Characteristic function $\mu = -y_t$

Conserved vector

$$\theta^t = (-y_t)(-y_t) = y_t^2$$
$$\theta^x = (-y_t)(-y_{xxx}) - y_{xt}y_{xx} = y_t y_{xxx} - y_{xt}y_{xx}$$
$$\Gamma^t = L\tau + \theta^t = \left(-\frac{1}{2}y_t^2 + \frac{1}{2}y_{xx}^2\right) + y_t^2 = \frac{1}{2}y_t^2 + \frac{1}{2}y_{xx}^2$$
$$\Gamma^x = L\xi + \theta^x = y_t y_{xxx} - y_{xt}y_{xx}$$

Check to see if this vector is indeed conserved!

$$D_t\Gamma^t + D_x\Gamma^x = D_t\left(\frac{1}{2}y_t^2 + \frac{1}{2}y_{xx}^2\right) + D_x\left(y_t y_{xxx} - y_{xt}y_{xx}\right) =$$
$$y_t y_{tt} + y_{xx}y_{xxt} + y_{xt}y_{xxx} + y_t y_{xxxx} - y_{xxt}y_{xx} - y_{xt}y_{xxx} =$$
$$y_t y_{tt} + y_t y_{xxxx} = -\mu\left(y_{tt} + y_{xxxx}\right) = 0$$

The quantity

$$E = \frac{1}{2}{y_t}^2 + \frac{1}{2}{y_{xx}}^2$$

is the energy per unit length of the rod: kinetic energy + strain energy

Note that

$$\int_0^L \left(D_t\Gamma^t + D_x\Gamma^x\right)dx = \frac{d}{dt}\int_0^L \left(\frac{1}{2}{y_t}^2 + \frac{1}{2}{y_{xx}}^2\right)dx + \left(\Gamma^x(t,L) - \Gamma^x(t,0)\right) = 0$$

$$\Gamma^x = y_t y_{xxx} - y_{xt} y_{xx}$$

If the bending moment (y_{xx}) and shear force (y_{xxx}) vanish or if $y_t = 0$ at the ends of the beam, then the total energy of the beam is conserved.

Invariance under translation in space

Infinitesimals $\tau = 0 \qquad \xi = 1 \qquad \eta = 0$

Characteristic function $\mu = -y_x$

Conserved vector

$$\theta^t = (-y_x)(-y_t) = y_x y_t$$
$$\theta^x = (-y_x)(-y_{xxx}) + (-y_{xx})(-y_{xx}) = y_x y_{xxx} - y_{xx}{}^2$$

$$\Gamma^t = L\tau + \theta^t = y_x y_t$$
$$\Gamma^x = L\xi + \theta^x = -\frac{1}{2} y_t{}^2 + y_x y_{xxx} - \frac{1}{2} y_{xx}{}^2$$

Check to see if this vector is indeed conserved!

$$D_t\Gamma^t + D_x\Gamma^x = D_t(y_x y_t) + D_x\left(-\frac{1}{2} y_t{}^2 + y_x y_{xxx} - \frac{1}{2} y_{xx}{}^2\right) =$$
$$y_{tt} y_x + y_t y_{xt} - y_t y_{tx} + y_{xx} y_{xxx} + y_x y_{xxxx} - y_{xx} y_{xxx} =$$
$$y_{tt} y_x + y_x y_{xxxx} = y_x(y_{tt} + y_{xxxx}) = 0$$

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The quantity

$$P = y_t y_x$$

can be regarded as the effective "momentum per unit length" of the rod.

Note that

$$\int_0^L \left(D_t \Gamma^t + D_x \Gamma^x\right) dx = \frac{d}{dt}\int_0^L \left(y_t y_x\right) dx + \left(\Gamma^x(t,L) - \Gamma^x(t,0)\right) = 0$$

$$\Gamma^x = -\frac{1}{2} y_t^{\,2} + y_x y_{xxx} - \frac{1}{2} y_{xx}^{\,2}$$

If the velocity, bending moment and shear force vanish at the ends of the beam, ie the time, second and third spatial derivatives are zero, then the total momentum is conserved.

Invariance under translation in y[x,t]

Infinitesimals $\tau = 0 \quad \xi = 0 \quad \eta = 1$

Characteristic function $\mu = 1$

Conserved vector

$$\theta^t = -y_t$$
$$\theta^x = -y_{xxx}$$

$$\Gamma^t = L\tau + \theta^t = -y_t$$
$$\Gamma^x = L\xi + \theta^x = -y_{xxx}$$

Check to see if this vector is indeed conserved.

$$D_t\Gamma^t + D_x\Gamma^x = D_t\left(-y_t\right) + D_x\left(-y_{xxx}\right) =$$
$$-y_{tt} - y_{xxxx} = 0$$

The quantity

$$m = -y_t$$

can be regarded as the effective "mass per unit length" of the rod.

Note that

$$\int_0^L \left(D_t\Gamma^t + D_x\Gamma^x\right)dx = \frac{d}{dt}\int_0^L \left(-y_t\right)dx + \left(\Gamma^x\left(t,L\right) - \Gamma^x\left(t,0\right)\right) = 0$$

$$\Gamma^x = -y_{xxx}$$

If the shear force vanishes at the ends of the beam, ie the third spatial derivative is zero, then the total mass is conserved.

Invariance under dilation

Infinitesimals $\tau = 4t \qquad \xi = 2x \qquad \eta = y$

Characteristic function $\mu = y - 4ty_t - 2xy_x$

Conserved vector

$$\theta^t = -yy_t + 4ty_t^{\,2} + 2xy_x y_t$$
$$\theta^x = (y - 4ty_t - 2xy_x)(-y_{xxx}) + (y_x - 4ty_{tx} - 2y_x - 2xy_{xx})(y_{xx})$$

$$\Gamma^t = 2ty_{xx}^{\,2} - yy_t + 2ty_t^{\,2} + 2xy_x y_t$$
$$\Gamma^x = -xy_t^{\,2} - yy_{xxx} + 4ty_t y_{xxx} + 2xy_x y_{xxx} - 4ty_{tx} y_{xx} - y_x y_{xx} - xy_{xx}^{\,2}$$

Check to see if this vector is indeed conserved!

$$D_t\Gamma^t + D_x\Gamma^x = -yy_{tt} + 4ty_t y_{tt} + 2xy_x y_{tt} +$$
$$-yy_{xxxx} + 4ty_t y_{xxxx} + 2xy_x y_{xxxx} =$$
$$(-y + 4ty_t + 2xy_x)(y_{tt} + y_{xxxx}) =$$
$$\mu(y_{tt} + y_{xxxx}) = 0$$

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The quantity

$$\Gamma^t = 2ty_{xx}{}^2 - yy_t + 2ty_t{}^2 + 2xy_x y_t = 4t\left(\frac{1}{2}y_t{}^2 + \frac{1}{2}y_{xx}{}^2\right) + y(-y_t) + 2x(y_x y_t)$$

$$\Gamma^t = my + 2Px + 4tE$$

is conserved.

Integrate along the beam.

$$\int_0^L \left(D_t\Gamma^t + D_x\Gamma^x\right)dx = \frac{d}{dt}\int_0^L (my + 2Px + 4tE)dx + \left(\Gamma^x(t,L) - \Gamma^x(t,0)\right) = 0$$

$$\Gamma^x = -xy_t{}^2 - yy_{xxx} + 4ty_t y_{xxx} + 2xy_x y_{xxx} - 4ty_{tx}y_{xx} - y_x y_{xx} - xy_{xx}{}^2$$

If the velocity, bending moment and shear force vanish at the ends of the beam, ie the time, second and third spatial derivatives are zero, then the integral below is conserved.

$$\frac{d}{dt}\int_0^L \left(my[t,x] + 2Px + 4tE\right)dx = 0$$