Introduction to Symmetry Analysis

Chapter 13 -Similarity Rules for Turbulent Shear Flows

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Reynolds number invariance

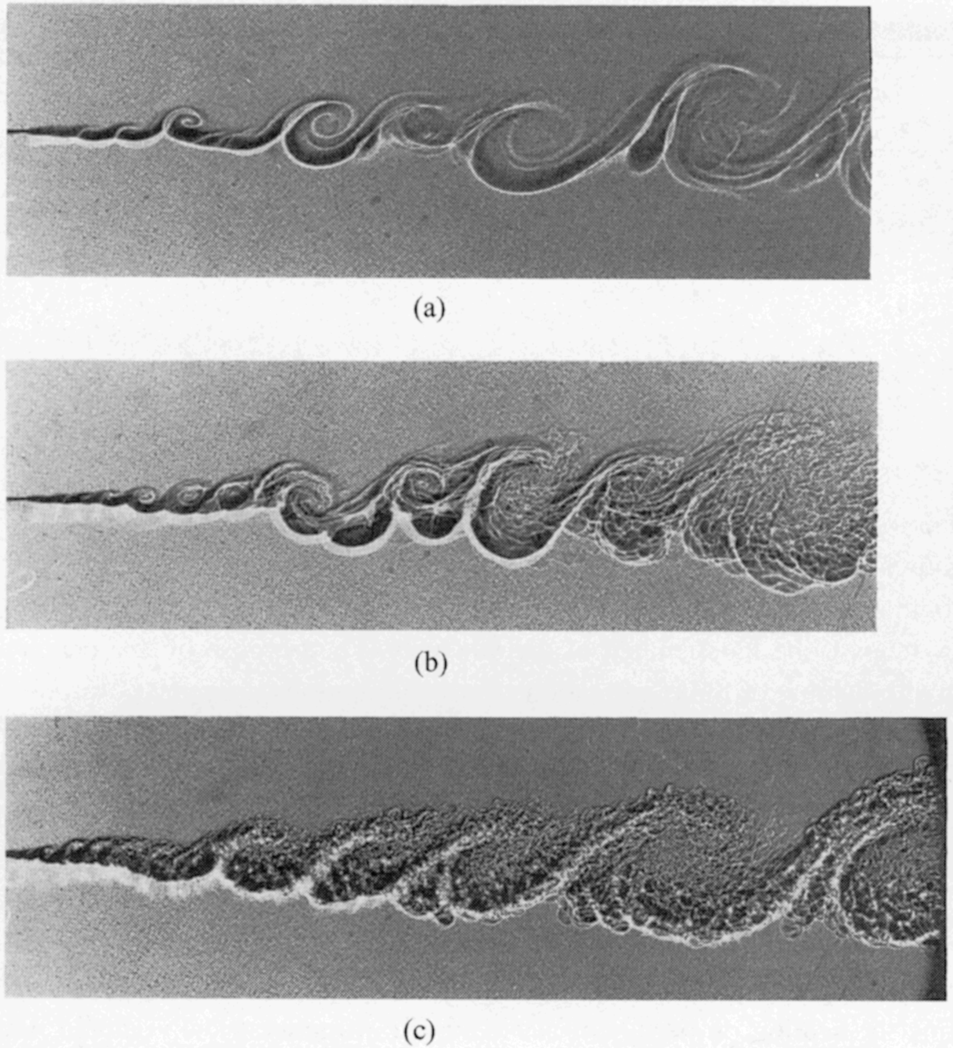

Fig. 13.1. Effects of Reynolds number on a plane mixing layer between helium (upper stream) and nitrogen (lower stream) from Brown and Roshko [13.1] and Roshko[13.2]. The Reynolds number in (a) is approximately 1.3×10^4 centimeter^{-1}. The thickness of the layer at the right side of the picture is approximately 2 cm. The speed of the lower stream is 10 m/s. Test-section pressures in atmospheres are: (a) 2, (b) 4, (c) 8. Dynamic pressures in the upper and lower streams are the same: $\rho_1 u_1^2 = \rho_2 u_2^2$.

Image above: The above photograph shows the turbulence field behind the Horns Rev offshore wind turbines. Horns Rev is located in the North Sea, 14 kilometers west of Denmark. Photographer Christian Steiness. From (http://wattsupwiththat.com/2011/04/28/the-wind-turbine-albedo-effect/).

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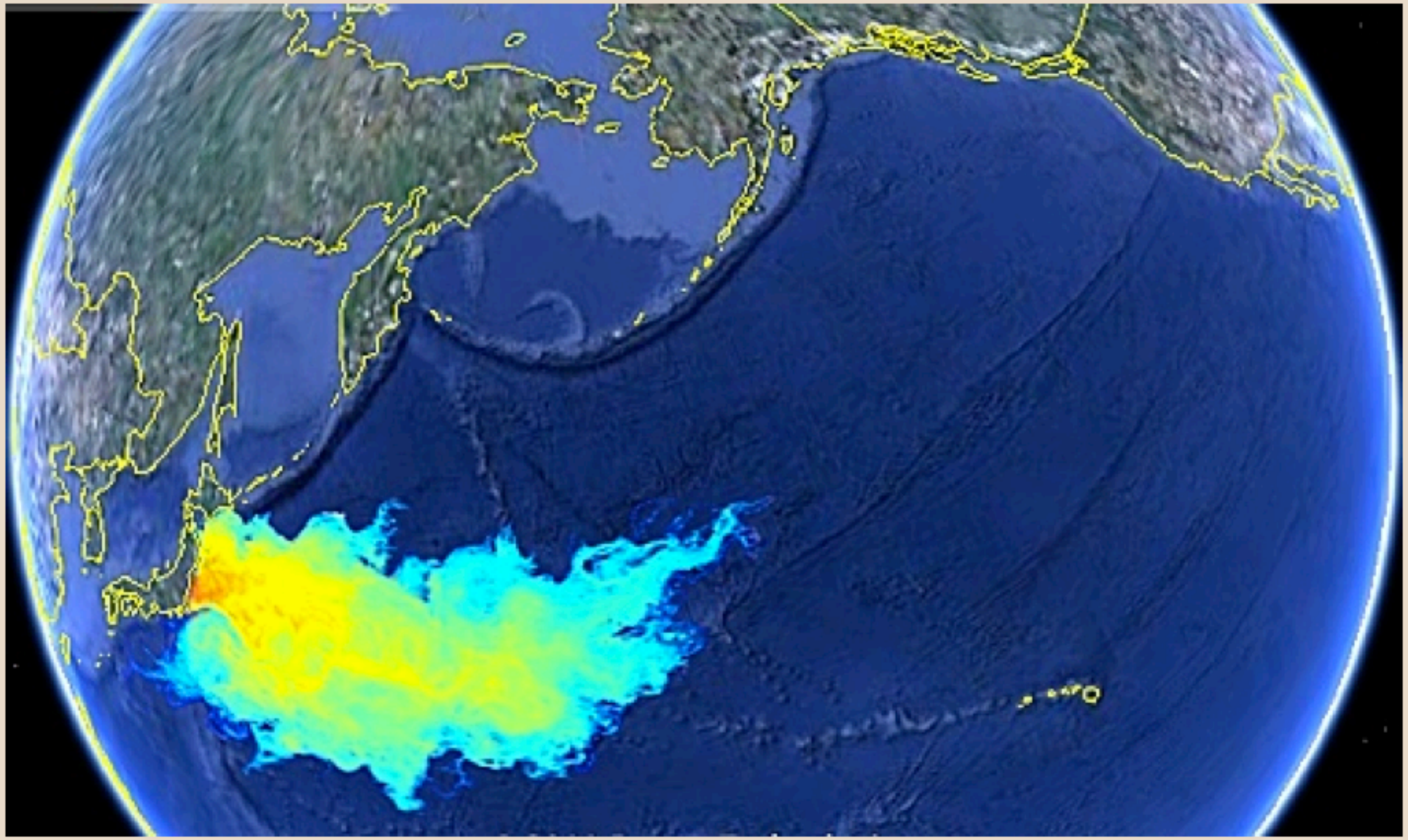
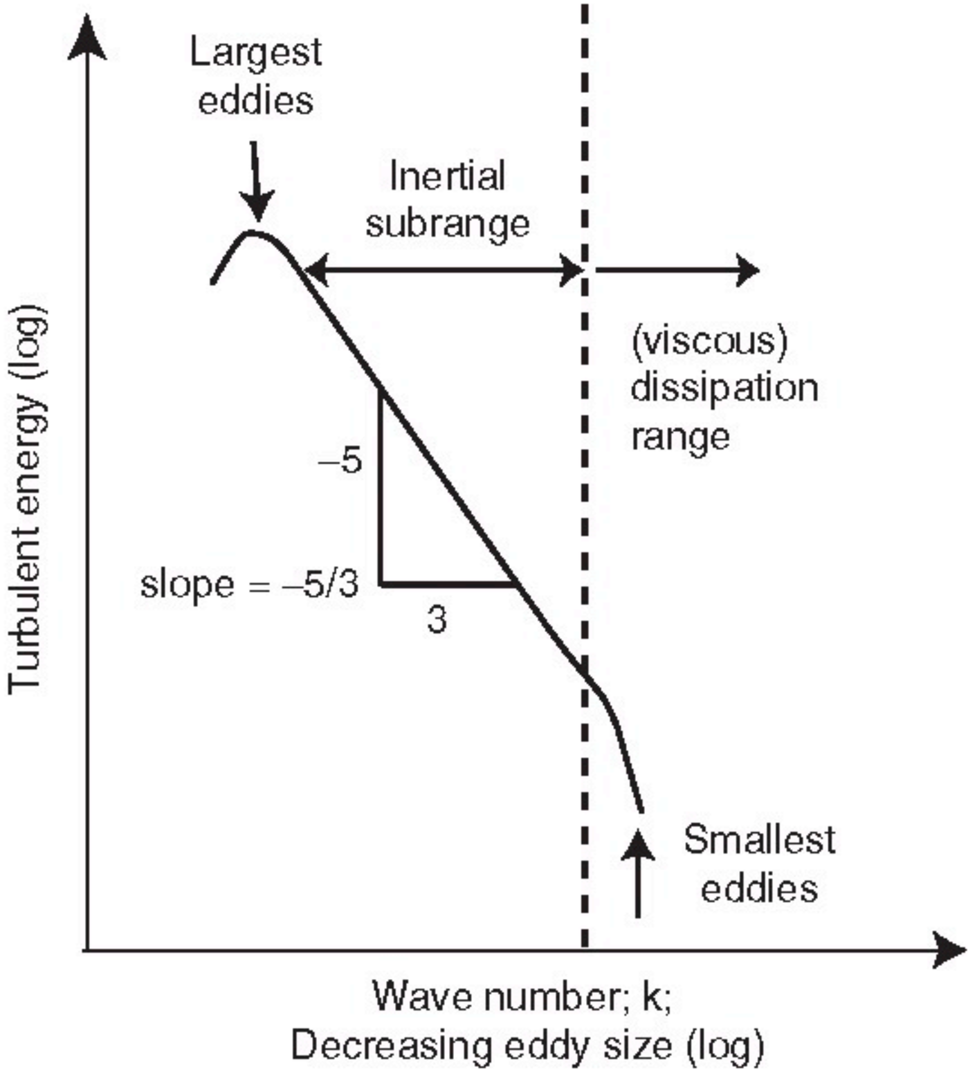


Image above: 8/11/11 simulation of radioactive seawater dispersed from Fukushima nears Hawaii. From (http://www.xydo.com/toolbar/27327691-asr_ltd_-_fukushima_radioactive_seawater_plume_dispersal_simulation). Note - users can zoom and rotate orientation of simulation.

Turbulent kinetic energy spectrum



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The Reynolds averaged Navier-Stokes equations

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}', \qquad p = \bar{p} + p', \tag{13.1}$$

$$\begin{gathered} \frac{\partial \bar{u}^i}{\partial x^i} = 0, \\ \frac{\partial \bar{u}^i}{\partial t} + \frac{\partial}{\partial x^j}(\bar{u}^j \bar{u}^i) + \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x^i} - \frac{1}{\rho}\frac{\partial \tau^{ij}}{\partial x^j} - 2\nu \frac{\partial \bar{s}^{ij}}{\partial x^j} = 0, \end{gathered} \tag{13.2}$$

Rate-of-strain $$\bar{s}^{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}^i}{\partial x^j} + \frac{\partial \bar{u}^j}{\partial x^i}\right). \tag{13.3}$$

$$\frac{\tau^{ij}}{\rho} = -\overline{u'^i u'^j}. \tag{13.4}$$

Away from a wall $$-\overline{u'^i u'^j} \gg 2\nu \bar{s}^{ij}. \tag{13.5}$$

For free shear flows the viscous stresses are often neglected compared to the Reynolds stresses.

$$\begin{gathered} \frac{\partial \bar{u}^j}{\partial x^j} = 0, \\ \frac{\partial \bar{u}^i}{\partial t} + \frac{\partial}{\partial x^j}(\bar{u}^j \bar{u}^i) + \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x^i} - \frac{1}{\rho}\frac{\partial \tau^{ij}}{\partial x^j} = 0. \end{gathered} \tag{13.6}$$

Integral length and velocity scales

$u_0 =$ integral velocity scale characterizing the overall motion,
$\delta =$ integral length scale characterizing the overall motion.

$$u' = \sqrt{\frac{u_1'^2 + u_2'^2 + u_3'^2}{2}}. \tag{13.7}$$

$$u' \propto u_0, \tag{13.8}$$

$$\varepsilon = 2\nu \overline{s'^{ij} s'^{ji}}, \tag{13.9}$$

$$s'^{ij} = \frac{1}{2}\left(\frac{\partial u'^i}{\partial x^j} + \frac{\partial u'^j}{\partial x^i}\right) \tag{13.10}$$

Dissipation scales with production

$$\varepsilon \propto \overline{u'^i u'^j} \frac{\partial \bar{u}^i}{\partial x^j}. \tag{13.11}$$

$$\varepsilon \propto \frac{u_0^3}{\delta}. \tag{13.12}$$

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Invariant group of the Euler equations

$$
\begin{gathered}
\tilde{x}^i = e^s x^i, \qquad \tilde{t} = e^{s/k} t, \qquad \tilde{\bar{u}}^i = e^{s(1-1/k)} \bar{u}^i, \\
\tilde{\tau}^{ij} = e^{s(2-2/k)} \tau^{ij}, \qquad \tilde{\bar{p}} = e^{s(2-2/k)} \bar{p},
\end{gathered} \tag{13.13}
$$

$$
\begin{aligned}
&\frac{\partial \tilde{\bar{u}}^i}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}^j} \tilde{\bar{u}}^j \tilde{\bar{u}}^i + \frac{\partial \tilde{\bar{p}}}{\partial \tilde{x}^i} - \frac{\partial \tilde{\tau}^{ij}}{\partial \tilde{x}^j} \\
&= \left(\frac{\partial \bar{u}^i}{\partial t} + \frac{\partial}{\partial x^j} \bar{u}^j \bar{u}^i + \frac{\partial \bar{p}}{\partial x^i} - \frac{\partial \tau^{ij}}{\partial x^j} \right) e^{a(1-2/k)} = 0.
\end{aligned} \tag{13.14}
$$

One parameter flows

$$\hat{M} = L^m T^{-n}. \tag{13.15}$$

Stationary plane jet. The integral momentum flux J/ρ is approximately constant at any streamwise position:

$$\frac{J}{\rho} = \int_{-\infty}^{\infty} \tilde{u}^2 \, d\tilde{y} = e^{a(3-2/k)} \int_{-\infty}^{\infty} u^2 \, dy. \tag{13.16}$$

The integral is invariant under dilation only for $k = \frac{2}{3}$.

Vortex ring. The hydrodynamic impulse, I/ρ, is the conserved integral for this flow (cf. Chapter 11, Section 11.5.1):

$$\frac{I}{\rho} = \frac{3}{2} \int \tilde{u} \, d\tilde{x} \, d\tilde{y} \, d\tilde{z} = e^{a(4-1/k)} \frac{3}{2} \int u \, dx \, dy \, dz. \tag{13.17}$$

In this case the integral is invariant for $k = \frac{1}{4}$.

Temporal similarity rules

$$\frac{dx^i}{x^i} = k\frac{dt}{t} = \left(\frac{k}{k-1}\right)\frac{du^i}{u^i} = \left(\frac{k}{2k-2}\right)\frac{dp}{p} = \left(\frac{k}{2k-2}\right)\frac{d\tau^{ij}}{\tau^{ij}} \quad (13.18)$$

$$\xi^i = \frac{x^i}{\delta[t]}, \qquad U^i = \frac{u^i}{u_0[t]}, \qquad P = \frac{p}{u_0[t]^2}, \qquad T^{ij} = \frac{\tau^{ij}}{u_0[t]^2}. \quad (13.19)$$

The time-dependent length and velocity scales in (13.19) are

$$\boxed{\delta[t] \propto M^{1/m}(t-t_0)^k, \qquad u_0[t] \propto M^{1/m}(t-t_0)^{k-1},} \quad (13.20)$$

where t_0 is the effective origin in time. The group parameter k is determined by the units of the governing parameter M:

$$\boxed{k = n/m.} \quad (13.21)$$

$$\frac{\bar{u}^i}{u_0[t]} = U^i\left[\frac{\boldsymbol{x}}{\delta[t]}\right], \qquad \frac{\bar{p}}{u_0[t]^2} = P\left[\frac{\boldsymbol{x}}{\delta[t]}\right], \qquad \frac{\tau^{ij}}{u_0[t]^2} = T^{ij}\left[\frac{\boldsymbol{x}}{\delta[t]}\right]. \quad (13.22)$$

Reduced equations

When the similarity variables (13.22) are substituted into the Reynolds equations (13.2), the result is that time drops out of the equations and the number of independent variables is reduced from four to three:

$$\frac{\partial U^j}{\partial \xi^j} = 0,$$

$$(k-1)U^i + (U^j - k\xi^j)\frac{\partial U^i}{\partial \xi^j} + \frac{1}{\rho}\frac{\partial P}{\partial \xi^i} - \frac{1}{\rho}\frac{\partial}{\partial \xi^j}(T^{ij}) = 0. \quad (13.23)$$

This is the same equation encountered in Chapter 11, Section 11.5 [cf. Equation (11.86)] where we analyzed the round jet with $k = \frac{1}{2}$. The equations for particle paths,

$$\frac{dx^i}{dt} = u^i[\boldsymbol{x}, t], \quad (13.24)$$

transform to the autonomous system

$$\frac{d\xi^i}{d\tau} = U^i[\boldsymbol{\xi}] - k\xi^i. \quad (13.25)$$

In these one-parameter flows all lengths scale with the same power of time.

Frames of reference

If an observer is selected to convect with a particular feature of the flow, then the observer will have to translate nonuniformly according to the power of time appropriate to the flow. Such a transformation is defined by

$$
\begin{aligned}
\tilde{x}^i &= x^i - \alpha^i M^{1/m}(t - t_0)^k, \\
\tilde{t} &= t, \\
\tilde{\bar{u}}^i &= \bar{u}^i - k\alpha^i M^{1/m}(t - t_0)^{k-1}, \\
\tilde{\bar{p}} &= \bar{p} + x^j k(k-1)\alpha^j M^{1/m}(t - t_0)^{k-2},
\end{aligned}
\tag{13.26}
$$

where the α^i determine the relative rates of motion of the observer in the three directions. We already know from the discussion in Chapter 11, Section 11.2, that the Navier–Stokes and Euler equations are invariant under the group (13.26). The Reynolds equations with the viscous term removed, (13.6), are as well. In terms of similarity variables, (13.26) becomes a simple translation,

$$
\begin{aligned}
\tilde{\xi}^i &= \xi^i - \alpha^i, \\
\tilde{\tau} &= \tau, \\
\tilde{U}^i &= U^i - k\alpha^i, \\
\tilde{P} &= P + \alpha^j \xi^j k(k-1).
\end{aligned}
\tag{13.27}
$$

In similarity coordinates, the equations for particle paths transform as follows:

$$
\begin{aligned}
\frac{d\tilde{\xi}^i}{d\tilde{\tau}} &= \frac{d\xi^i}{d\tau}, \\
\tilde{U}^i[\tilde{\boldsymbol{\xi}}] - k\tilde{\xi}^i &= U^i[\boldsymbol{\xi}] - k\xi^i.
\end{aligned}
\tag{13.28}
$$

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Spatial similarity rules - jets

$$(x - x_0) \propto M^{1/m}(t - t_0)^k. \tag{13.29}$$

$$\boxed{\delta \propto (x - x_0), \qquad u_0 \propto M^{1/n}(x - x_0)^{1-1/k}.} \tag{13.30}$$

Spatial similarity rules - wakes

$$\frac{D}{\rho} = C_D\left(\frac{1}{2}U_\infty^2\right)(\pi R^2). \tag{13.31}$$

$$\frac{D}{\rho} \propto \int_A U(U_\infty - U)\, dA, \tag{13.32}$$

$$\frac{D}{\rho U_\infty} \propto \int_A (U_\infty - U)\, dA. \tag{13.33}$$

$$(x - x_0) = U_\infty(t - t_0). \tag{13.34}$$

$$\boxed{\delta \propto M^{1/m}U_\infty^{-k}(x - x_0)^k, \qquad u_0 \propto M^{1/m}U_\infty^{1-k}(x - x_0)^{k-1}.} \tag{13.35}$$

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Reynolds number scaling

$$R_\delta = \frac{U_0\delta}{\nu} \propto \frac{M^{2/m}}{\nu}(t - t_0)^{2k-1}. \tag{13.36}$$

The idea of an eddy viscosity

$$-\overline{u'v'} = \nu_\tau \frac{\partial \bar{u}}{\partial y}. \tag{13.37}$$

$$\nu_\tau \propto u_0\,\delta. \tag{13.38}$$

$$Re_\tau = \frac{u_0\delta}{\nu_\tau} \propto \text{constant} \tag{13.39}$$

Table 13.1. *Various one-parameter shear flows and the units of the associated governing parameter.*

Flow	Invariant	M	Units	k
	Jetlike flows			
Plane mixing layer	Velocity difference	U_0	LT^{-1}	1
Plane jet	2-D momentum flux	$U_0^2\delta$	L^3T^{-2}	$\frac{2}{3}$
Round jet	3-D momentum flux	$U_0^2\delta^2$	L^4T^{-2}	$\frac{1}{2}$
Radial jet	3-D momentum flux	$U_0^2\delta^2$	L^4T^{-2}	$\frac{1}{2}$
Vortex pair	2-D impulse	$U_0\delta^2$	L^3T^{-1}	$\frac{1}{3}$
Vortex ring	3-D impulse	$U_0\delta^3$	L^4T^{-1}	$\frac{1}{4}$
Plane plume	2-D buoyancy flux	U_0^3	L^3T^{-3}	1
Round plume	3-D buoyancy flux	$U_0^3\delta$	L^4T^{-3}	$\frac{3}{4}$
Plane thermal	2-D buoyancy	$U_0^2\delta$	L^3T^{-2}	$\frac{2}{3}$
Round thermal	3-D buoyancy	$U_0^2\delta^2$	L^4T^{-2}	$\frac{1}{2}$
Line vortex	Circulation	$U_0\delta$	L^2T^{-1}	$\frac{1}{2}$
Diverging channel	Area flux	$U_0\delta$	L^2T^{-1}	$\frac{1}{2}$
Vortex-sheet rollup	Apex $\alpha; n = 1/(2-\alpha/\pi)$	$U_0^2\delta^{2-n}$	$L^{3-n}T^{-1}$	$1/(3-n)$
	Wakelike flows			
Plane wake	(2-D drag)$/U_\infty$	$U_0\delta$	L^2T^{-1}	$\frac{1}{2}$
Round wake	(3-D drag)$/U_\infty$	$U_0\delta^2$	L^3T^{-1}	$\frac{1}{3}$
Plane jet in cross flow	(2-D mom. flux)$/U_\infty$	$U_0\delta$	L^2T^{-1}	$\frac{1}{2}$
Round jet in cross flow	(3-D mom. flux)$/U_\infty$	$U_0\delta^2$	L^3T^{-1}	$\frac{1}{3}$
Plane plume in cross flow	(2-D buoy. flux)$/U_\infty$	U_0^2	L^2T^{-2}	1
Round plume in cross flow	(3-D buoy. flux)$/U_\infty$	$U_0^2\delta$	L^3T^{-2}	$\frac{2}{3}$
Grid turb. initial decay	Saffman invariant	$U_0^2\delta^3$	L^5T^{-2}	$\frac{2}{5}$
Grid turb. initial decay	Loitsianski invariant	$U_0^2\delta^5$	L^7T^{-2}	$\frac{2}{7}$

Scaling of a turbulent vortex ring

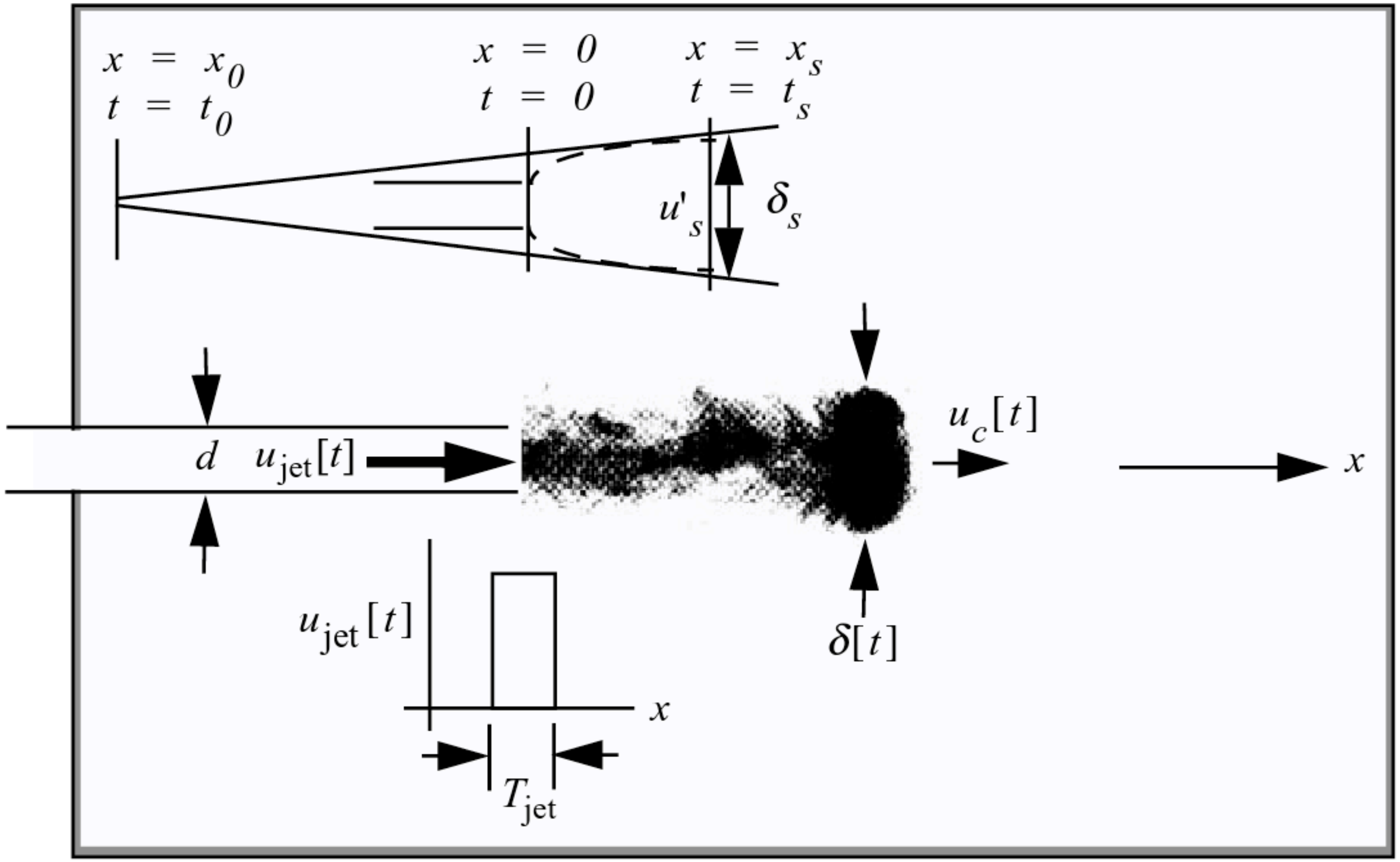
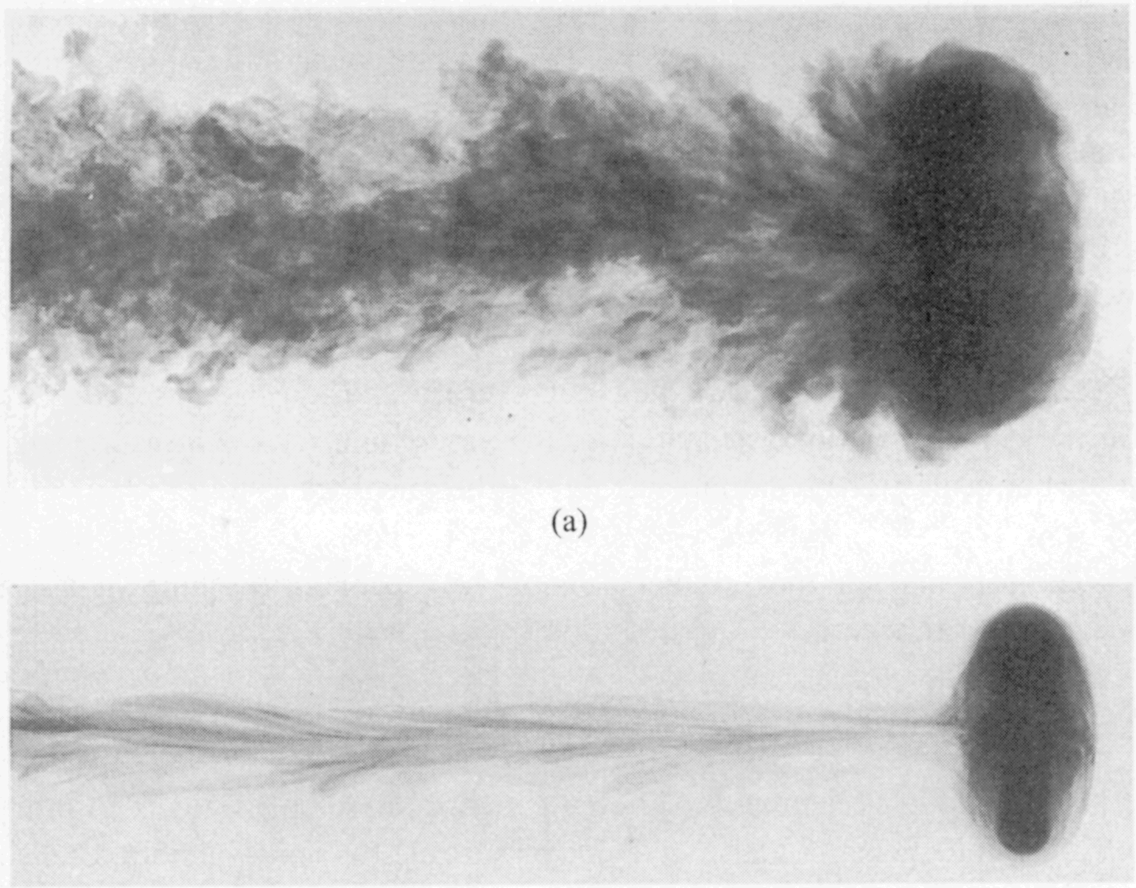


Fig. 13.3. Vortex-ring apparatus with experimental parameters. The sketch in the upper part of the figure defines parameters used to determine the effective origin of the ring.

High and Low Reynolds number vortex rings



(a)

(b)

Fig. 13.2. Turbulent and laminar vortex rings produced by an impulsive force, from the paper by Glezer and Coles [13.17]. Initial Reynolds number Γ_0/ν is (a) 27,000, (b) 7,500.

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Integral of the motion - the hydrodynamic impulse

$$\frac{3}{2}\int_V u\, dx\, dy\, dz = \int_0^t \int_V \frac{I}{\rho}\delta[x]\delta[y]\delta[z]\delta[t]\, dx\, dy\, dz\, dt = \frac{I}{\rho},$$

$$\delta[t] \propto (I/\rho)^{1/4}(t-t_0)^{1/4}, \qquad u_0[t] \propto (I/\rho)^{1/4}(t-t_0)^{-3/4},$$

$$\tilde{x}^i = e^a x^i, \qquad \tilde{t} = e^{4a} t, \qquad \tilde{\bar{u}}^i = e^{-3a}\bar{u}^i,$$
$$\tilde{\tau}^{ij} = e^{-6a}\tau^{ij}, \qquad \tilde{\bar{p}} = e^{-6a}\bar{p}.$$

$$\frac{U^i}{(I/\rho)^{1/4}(t-t_0)^{-3/4}} = G\left[\frac{\boldsymbol{x}-\boldsymbol{x}_0}{(I/\rho)^{1/4}(t-t_0)^{1/4}}\right].$$

$$\xi = \frac{x-x_0}{(I/\rho)^{1/4}(t-t_0)^{1/4}}, \qquad \eta = \frac{y}{(I/\rho)^{1/4}(t-t_0)^{1/4}}$$

Streamlines
and
Particle Paths

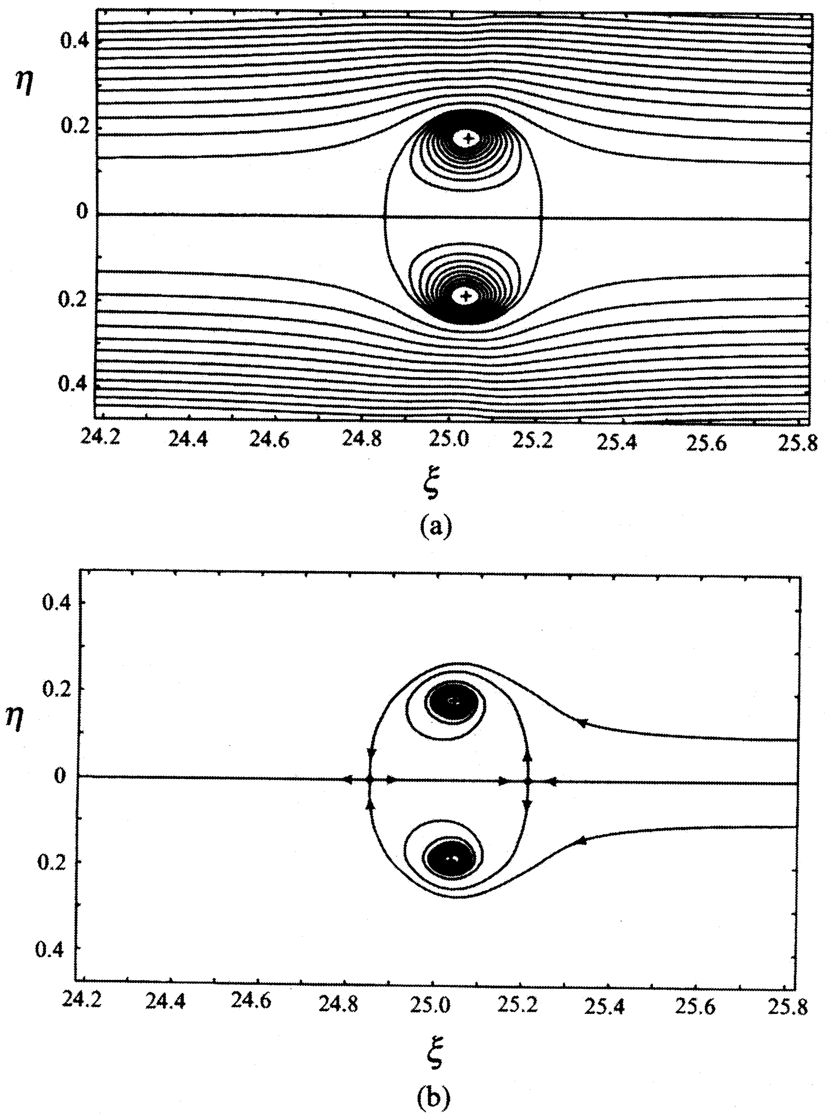

Fig. 13.4. Experimental results from [13.17]: (a) streamline pattern of the ensemble mean velocity field referred to an observer translating to the right with the ring, (b) particle paths of the ensemble mean velocity field.

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Recall that the incompressible Navier-Stokes equations are invariant under a group of arbitrary translations in space.

$$\tilde{x}^j = x^j + a^j[t],$$

$$\tilde{t} = t,$$

$$\tilde{u}^i = u^i + \frac{da^i}{dt},$$

$$\tilde{p} = p - x^j \frac{d^2 a^j}{dt^2} + g[t].$$

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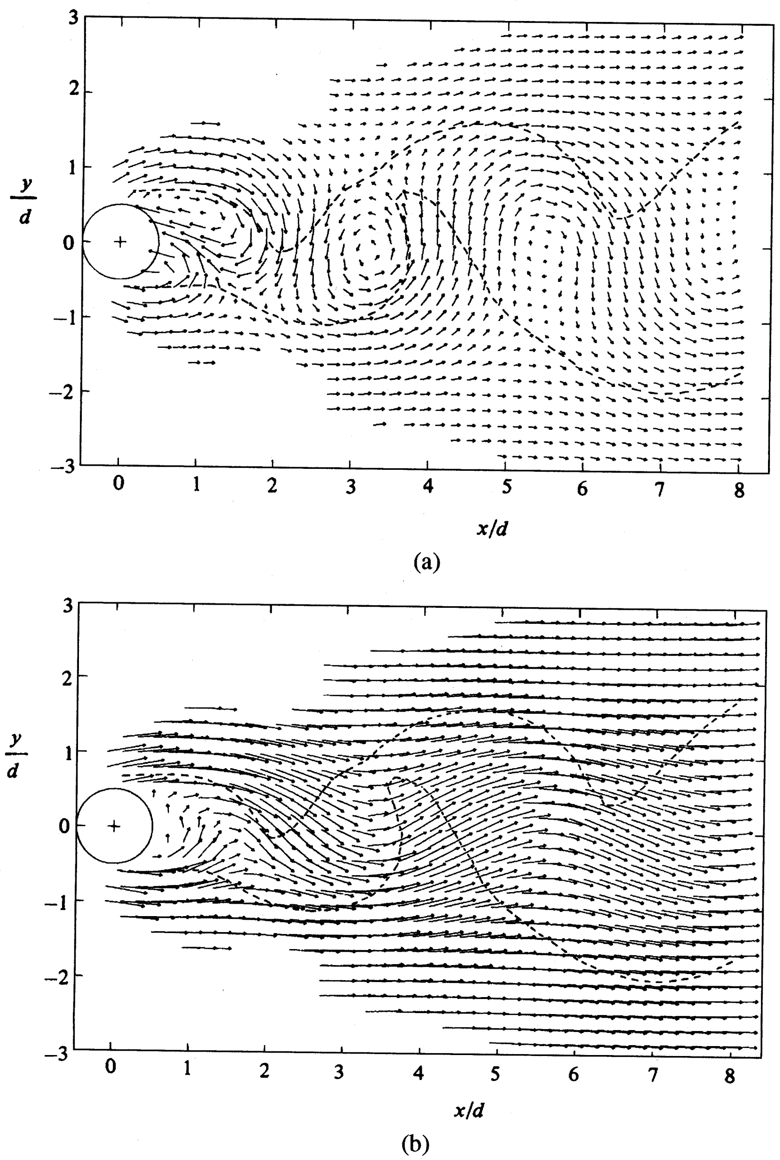

Fig. 11.1. Velocity vector field in the wake of a circular cylinder from Reference [11.6] as viewed by two observers: (a) frame of reference moving downstream at $0.755U_\infty$, (b) frame of reference fixed with respect to the cylinder. The dashed contour roughly corresponds to the instantaneous boundary of turbulence.

Instantaneous flow field in the wake of a circular cylinder as seen by two observers.

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Reduced equations

$$\frac{\partial U^j}{\partial \xi^j} = 0,$$

$$(k-1)U^i + (U^j - k\xi^j)\frac{\partial U^i}{\partial \xi^j} + \frac{1}{\rho}\frac{\partial P}{\partial \xi^i} - \frac{1}{\rho}\frac{\partial}{\partial \xi^j}(T^{ij}) = 0.$$

Particle paths

$$\frac{dx^i}{dt} = u^i[\boldsymbol{x}, t],$$

$$\frac{d\xi^i}{d\tau} = U^i[\boldsymbol{\xi}] - k\xi^i.$$

Frames of reference

$$\begin{aligned}
\tilde{x}^i &= x^i - \alpha^i M^{1/m}(t-t_0)^k,\\
\tilde{t} &= t,\\
\tilde{\bar{u}}^i &= \bar{u}^i - k\alpha^i M^{1/m}(t-t_0)^{k-1},\\
\tilde{\bar{p}} &= \bar{p} + x^j k(k-1)\alpha^j M^{1/m}(t-t_0)^{k-2},
\end{aligned}$$

Particle paths in similarity coordinates do not depend on the observer

$$
\begin{aligned}
\tilde{\xi}^i &= \xi^i - \alpha^i, \\
\tilde{\tau} &= \tau, \\
\tilde{U}^i &= U^i - k\alpha^i, \\
\tilde{P} &= P + \alpha^j \xi^j k(k-1).
\end{aligned}
$$

$$
\frac{d\tilde{\xi}^i}{d\tilde{\tau}} = \frac{d\xi^i}{d\tau},
$$

$$
\tilde{U}^i[\tilde{\boldsymbol{\xi}}] - k\tilde{\xi}^i = U^i[\boldsymbol{\xi}] - k\xi^i.
$$

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Streamlines
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Particle Paths

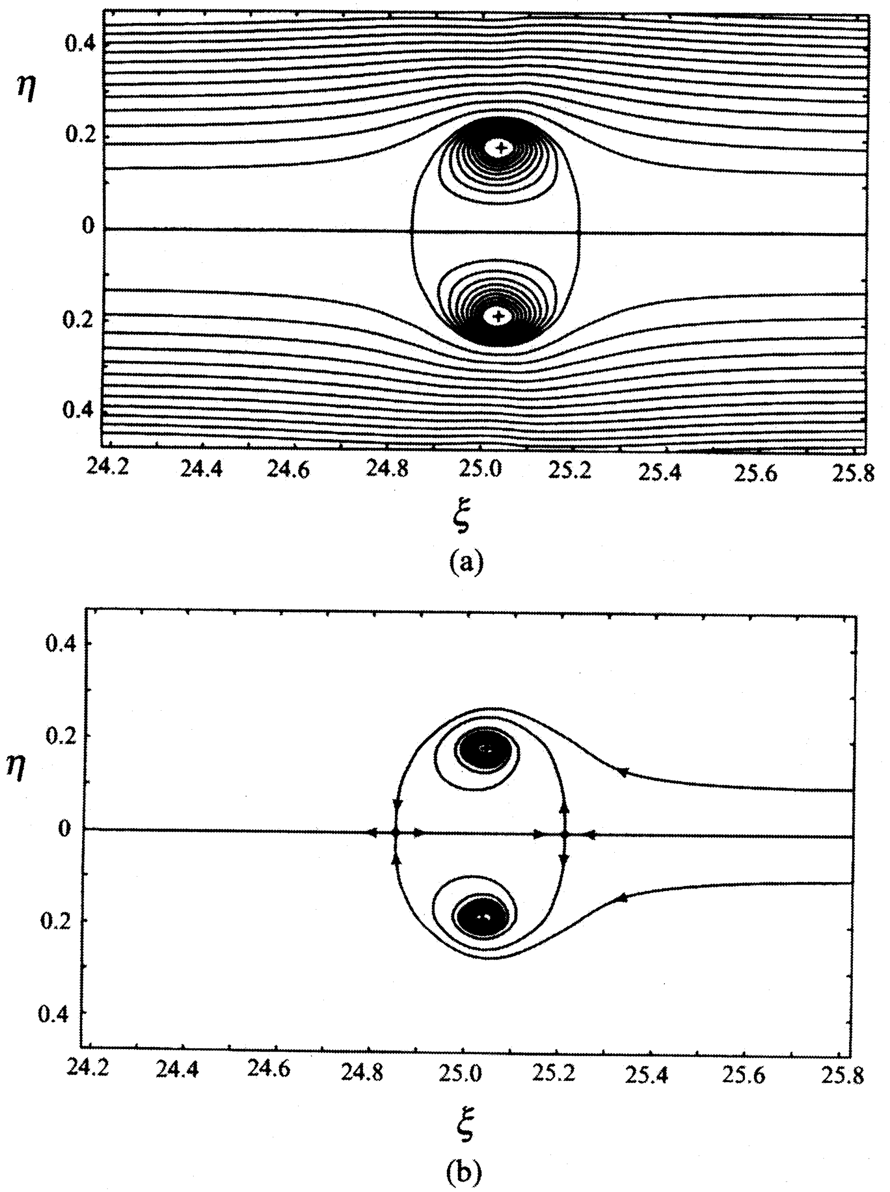

Fig. 13.4. Experimental results from [13.17]: (a) streamline pattern of the ensemble mean velocity field referred to an observer translating to the right with the ring, (b) particle paths of the ensemble mean velocity field.

The particle path plot shown how the vortex ring entrains fluid as it grows.

What about fine scales?

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The turbulent kinetic energy equation

Momentum and continuity equations

$$\frac{\partial u^i}{\partial x^i} = 0$$

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x^j}\left(u^i u^j + \frac{p}{\rho}\delta^{ij} - 2\nu s^{ij}\right) = 0$$

$$s^{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^i}{\partial x^j}\right)$$

Kinetic energy equation - project the momentum equation onto the velocity vector

$$u^i\frac{\partial u^i}{\partial t} + u^i\frac{\partial}{\partial x^j}\left(u^i u^j + \frac{p}{\rho}\delta^{ij} - 2\nu s^{ij}\right) = 0$$

$$u^i\frac{\partial u^i}{\partial t} = \frac{\partial\left(\frac{u^i u^i}{2}\right)}{\partial t}$$

$$u^i\frac{\partial u^i u^j}{\partial x^j} = u^i u^j\frac{\partial u^i}{\partial x^j} + u^i u^i\frac{\partial u^j}{\partial x^j} = u^i u^j\frac{\partial u^i}{\partial x^j}$$

$$\frac{\partial\left(\frac{u^i u^i}{2}u^j\right)}{\partial x^j} = u^i u^j\frac{\partial u^i}{\partial x^j} + \frac{u^i u^i}{2}\frac{\partial u^j}{\partial x^j} = u^i u^j\frac{\partial u^i}{\partial x^j}$$

$$\boxed{\frac{\partial\left(\frac{u^i u^i}{2}\right)}{\partial t} + \frac{\partial}{\partial x^j}\left(\frac{u^i u^i}{2}u^j + \frac{p}{\rho}u^j - 2\nu u^i s^{ij}\right) + 2\nu\frac{\partial u^i}{\partial x^j}s^{ij} = 0}$$

Kinetic energy dissipation

$$2\nu\frac{\partial u^i}{\partial x^j}s^{ij} =$$

$$2\nu\left(\frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right) + \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} - \frac{\partial u^j}{\partial x^i}\right)\right)s^{ij} =$$

$$2\nu\left(s^{ij} + \omega^{ij}\right)s^{ij} = 2\nu s^{ij}s^{ij}$$

Decompose the flow into a mean and fluctuating part.

A $$\frac{\partial\left(\frac{\overline{u}^i\overline{u}^i}{2}+\frac{\overline{u'^i u'^i}}{2}\right)}{\partial t}+\frac{\partial}{\partial x^j}\left(\frac{\overline{u}^i\overline{u}^i}{2}\overline{u}^j+\frac{\overline{u'^i u'^i}}{2}\overline{u}^j+\overline{u'^i u'^i}\overline{u}^i+\frac{\overline{u'^i u'^i u'^j}}{2}+\overline{u}^j\frac{\overline{p}}{\rho}+\frac{\overline{u'^j p'}}{\rho}-2\nu\left(\overline{u}^i\overline{s}^{ij}+\overline{u'^i s'^{ij}}\right)\right)+2\nu\left(\overline{s}^{ij}\overline{s}^{ij}+\overline{s'^{ij} s'^{ij}}\right)=0$$

Project the mean momentum equation onto the mean velocity vector

$$\overline{u}^i\frac{\partial\overline{u}^i}{\partial t}+\overline{u}^i\overline{u}^j\frac{\partial\overline{u}^i}{\partial x^j}+\overline{u}^i\frac{\partial\overline{u'^i u'^j}}{\partial x^j}+\frac{\partial}{\partial x^j}\left(\frac{\overline{p}\overline{u}_j}{\rho}\right)-2\nu\overline{u}^i\frac{\partial\overline{s}^{ij}}{\partial x^j}=0$$

$$\overline{u}^i\frac{\partial\overline{s}_{ij}}{\partial x^j}=\frac{\partial}{\partial x^j}\left(\overline{u}^i\overline{s}^{ij}\right)-\overline{s}^{ij}\overline{s}^{ij}$$

$$\frac{\partial\left(\frac{\overline{u}^i\overline{u}^i}{2}\right)}{\partial t}+\overline{u}^j\frac{\partial\left(\frac{\overline{u}^i\overline{u}^i}{2}\right)}{\partial x^j}+\overline{u}^i\frac{\partial\overline{u'_i u'_j}}{\partial x^j}+\frac{\partial}{\partial x^j}\left(\frac{\overline{p}\overline{u}^j}{\rho}\right)-2\nu\frac{\partial}{\partial x^j}\left(\overline{u}^i\overline{s}^{ij}\right)+2\nu\overline{s}^{ij}\overline{s}^{ij}=0$$

B $$\frac{\partial\left(\frac{\overline{u}^i\overline{u}^i}{2}\right)}{\partial t}+\frac{\partial}{\partial x^j}\left(\frac{\overline{u}^i\overline{u}^i}{2}\overline{u}_j+\overline{u'_i u'_j}\overline{u}_i+\frac{\overline{p}\overline{u}_j}{\rho}-2\nu\overline{u}_i\overline{s}_{ij}\right)-\overline{u'_i u'_j}\frac{\partial\overline{u}_i}{\partial x^j}+2\nu\overline{s}_{ij}\overline{s}_{ij}=0$$

Subtract B from A .

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Subtract B from A .

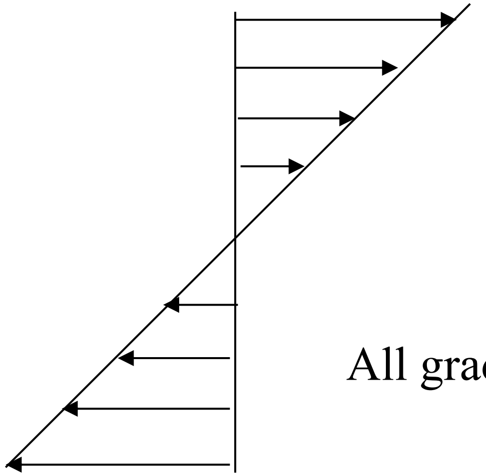
The turbulent kinetic energy (TKE) transport equation is

Turbulent kinetic energy production

$$\frac{\partial\left(\frac{\overline{u'^i u'^i}}{2}\right)}{\partial t} + \frac{\partial}{\partial x^j}\left(\frac{\overline{u'^i u'^i}}{2}\bar{u}^j + \frac{\overline{u'^i u'^i u'^j}}{2} + \frac{\overline{u'^j p'}}{\rho} - 2\nu\left(\overline{u'^i s'^{ij}}\right)\right) + \overline{u'^i u'^j}\frac{\partial \bar{u}^i}{\partial x^j} + 2\nu\left(\overline{s'^{ij} s'^{ij}}\right) = 0$$

Turbulent kinetic energy dissipation

Consider stationary homogeneous shear flow



$$\bar{u} = (ky, 0, 0)$$

All gradients of correlations are zero

$$\boxed{2\nu\left(\overline{s'^{ij} s'^{ij}}\right) = -\overline{u'^i u'^j}\frac{\partial \bar{u}^i}{\partial x^j}}$$

$$\varepsilon = 2\nu\overline{s'^{ij}\, s'^{ij}}$$

Dissipation of TKE equals production of TKE

The important implication of all this is that the dissipation term is comparable to all the other terms in the kinetic energy balance despite the fact that is is multiplied by the kinematic viscosity

Conclusion: fluctuating rates of strain are very large!

Fine scale motions

Using the scaling relation (13.11) that comes from the turbulent kinetic energy equation, we can write

$$\varepsilon \propto \frac{u_0^3}{\delta}, \tag{13.40}$$

which can be rearranged to read

$$\sqrt{s'^{ik}s'^{ki}} \propto \frac{u_0}{\delta}\left(\frac{u_0\delta}{2\nu}\right)^{1/2} \tag{13.41}$$

We now define a new length scale, λ, called the *Taylor microscale*, that, when associated with u_0, can account for turbulent kinetic energy dissipation [13.11], [13.13]:

$$\varepsilon \propto \nu\left(\frac{u_0^2}{\lambda^2}\right). \tag{13.42}$$

Combining (13.42) with (13.40) leads to the following estimates for the Taylor microscale:

$$\frac{\lambda}{\delta} \propto \frac{1}{(R_\delta)^{1/2}}, \qquad \lambda \propto (\nu(t-t_0))^{1/2}. \tag{13.43}$$

According to this estimate, there is always some eddying motion in the flow which has a characteristic length that varies like $\sqrt{\nu t}$ and is independent of the governing parameter M. In a similar vein note that the velocity gradients of the large-scale motion vary according to

$$\frac{u_0}{\delta} \propto \frac{1}{t-t_0} \qquad (t > t_0), \tag{13.44}$$

which is also independent of M. In a sense the large-scale gradients constitute a clock that can be used to date the evolution of the flow just as in the case of the laminar round jet.

Now let's define new length *and* velocity scales that can account for dissipation of TKE. These are the velocity and length scales defined by Kolmogorov [13.14]. See also the discussion of Kolmogorov theory in References [13.15] and [13.16]. The Kolmogorov scales can be regarded as motions that constitute the lower limit for instability – motions with a characteristic Reynolds number of order one. Let

$$\varepsilon \propto \nu\left(\frac{\upsilon^2}{\eta^2}\right), \qquad \frac{\upsilon\eta}{\nu} \approx 1. \tag{13.45}$$

Equation (13.45) can be used in conjunction with (13.12) to generate the following estimates of the Kolmorgorov velocity and length scales:

$$\frac{\eta}{\delta} \propto \frac{1}{(R_\delta)^{3/4}}, \qquad \eta \propto \nu^{3/4} M^{-1/2m}(t-t_0)^{3/4-k/2} \tag{13.46}$$

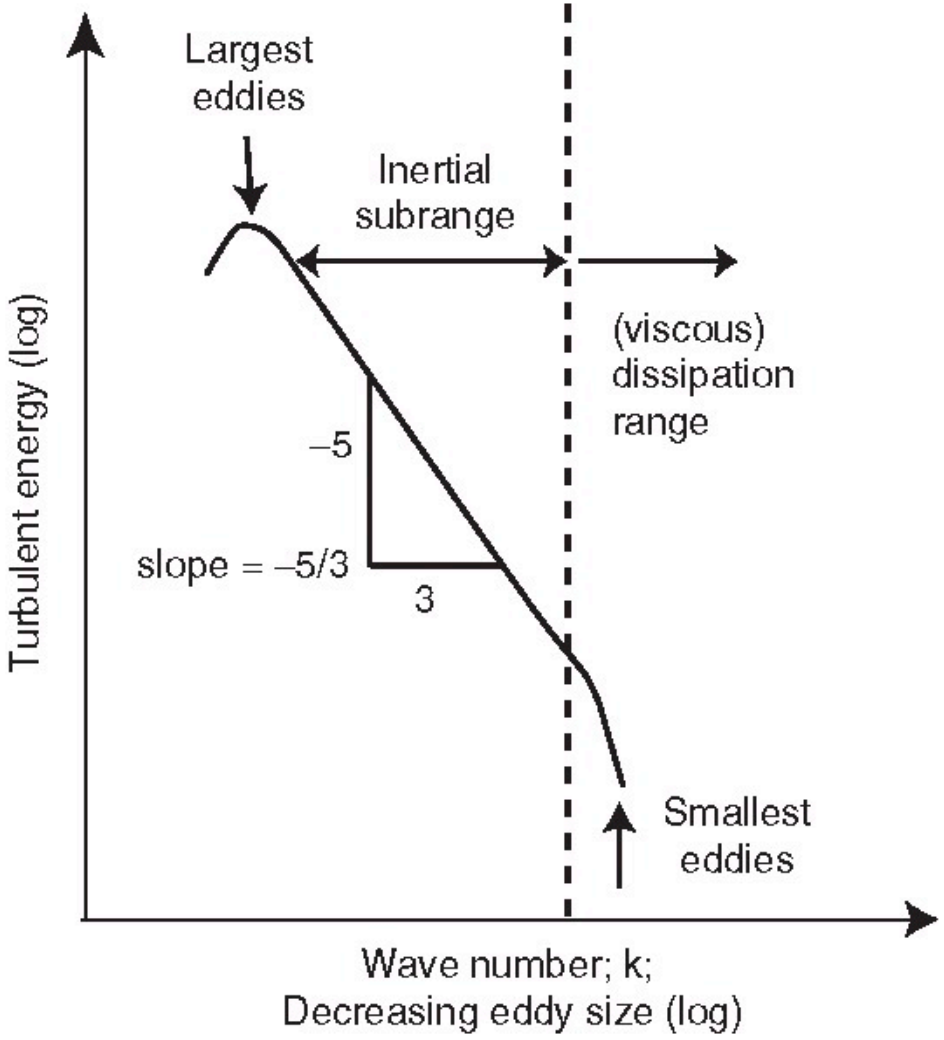
and

$$\frac{\upsilon}{u_0} \propto \frac{1}{(R_\delta)^{1/4}}, \qquad \upsilon \propto \nu^{1/4} M^{1/2m}(t-t_0)^{k/2-3/4}. \tag{13.47}$$

Small scale gradients

$$\frac{u_0}{\lambda} \propto \frac{\upsilon}{\eta} \propto \nu^{-1/2} M^{1/m}(t-t_0)^{k-3/2}. \tag{13.48}$$

Turbulent kinetic energy spectrum



Scaling the inertial subrange

Assume the governing parameter is

$$M = \varepsilon \propto u_0^3/\delta \tag{13.49}$$

with units $\hat{u}_0^3/\hat{\delta} = L^2T^{-3}$ and exponent $k = \frac{3}{2}$.

$$\delta \propto \varepsilon^{1/2}(t-t_0)^{3/2}, \qquad u_0 \propto \varepsilon^{1/2}(t-t_0)^{1/2} \tag{13.50}$$

$$R_\delta \propto (t-t_0)^2, \qquad \lambda \propto (t-t_0)^{1/2}, \qquad \eta \propto (t-t_0)^0. \tag{13.51}$$

The wavenumber of an eddy is essentially the inverse of its scale.

$$\kappa \propto 1/\delta. \tag{13.52}$$

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The TKE per unit wavenumber should scale as

$$E(\kappa) \propto \frac{u_0^2}{1/\delta} \propto \varepsilon^{3/2}(t - t_0)^{5/2}. \qquad (13.53)$$

$$t - t_0 \propto \frac{\delta^{2/3}}{\varepsilon^{1/3}} \propto \frac{\kappa^{-2/3}}{\varepsilon^{1/3}}. \qquad (13.54)$$

$$E(\kappa) \propto \varepsilon^{2/3}\kappa^{-5/3}. \qquad (13.55)$$

This is the scaling of TKE first postulated by Kolmogorov in 1941
and seems to agree with measurements in high Reynolds number flows.

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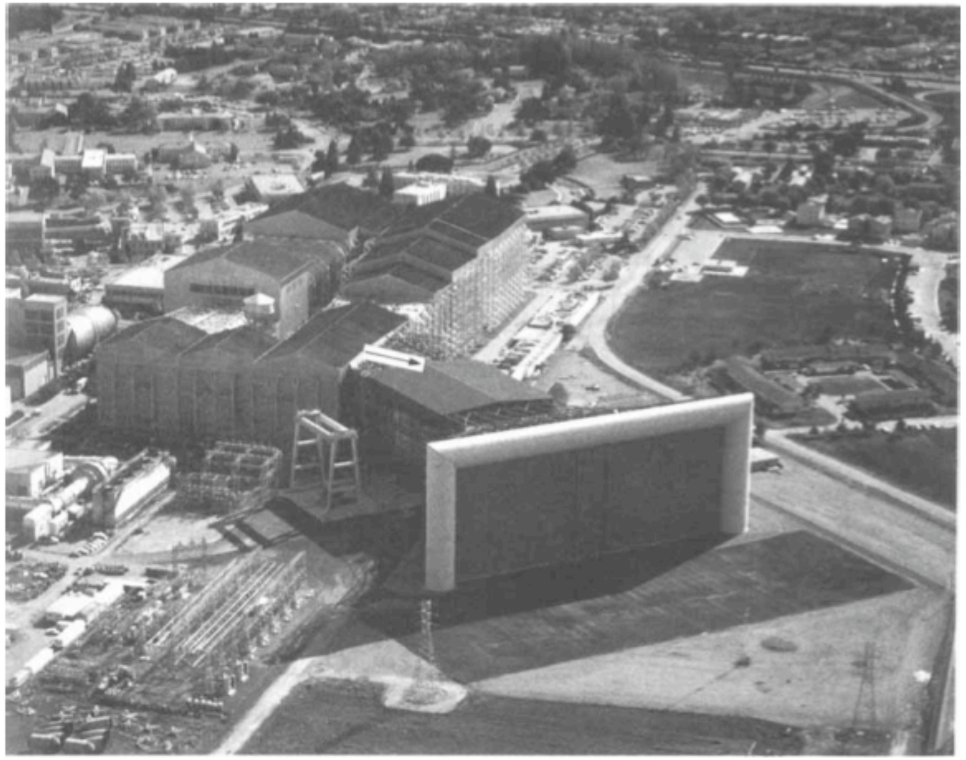

FIGURE 1. An aerial view of the Full-Scale Aerodynamics Facility at NASA Ames Research Center, showing the intake to the 80 × 120 foot test section. The arrow shows our measurement location in the attic.

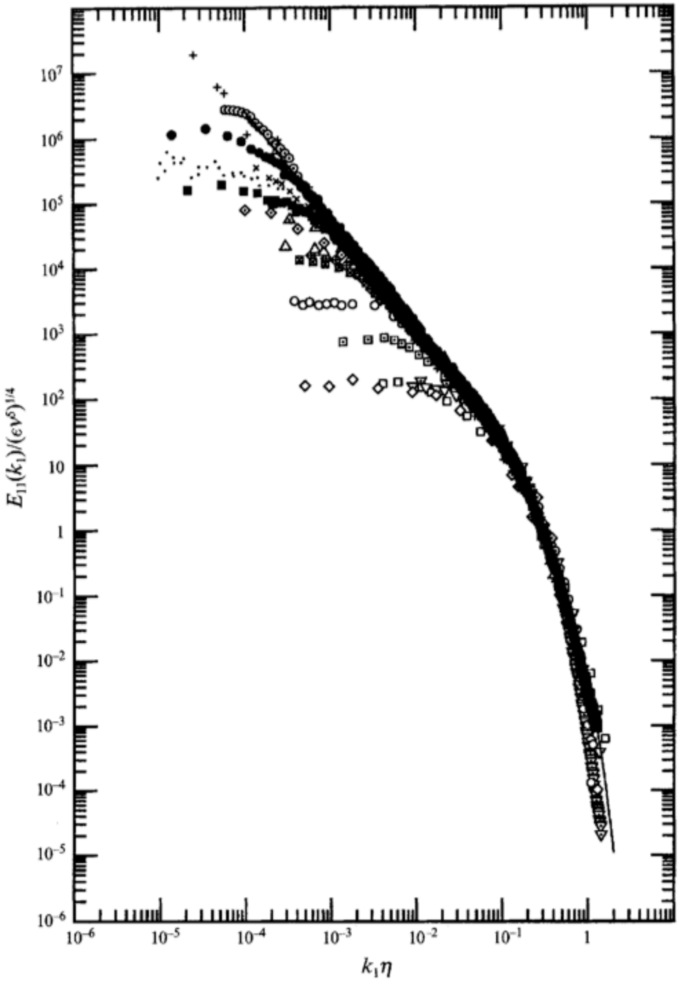

FIGURE 9. Kolmogorov's universal scaling for one-dimensional longitudinal power spectra. The present mid-layer spectra for both free-stream velocities are compared with data from other experiments. This compilation is from Chapman (1979), with later additions. The solid line is from Pao (1965). R_λ: □, 23 boundary layer (Tielman 1967); ◇, 23 wake behind cylinder (Uberoi & Freymuth 1969); ▽, 37 grid turbulence (Comte-Bellot & Corrsin 1971); ▽, 53 channel centreline (Kim & Antonia (DNS) 1991); ⊡, 72 grid turbulence (Comte-Bellot & Corrsin 1971); ○, 130 homogeneous shear flow (Champagne *et al.* 1970); ⊠, 170 pipe flow (Laufer 1954); ⊕, 282 boundary layer (Tielman 1967); ◇, 308 wake behind cylinder (Uberoi & Freymuth 1969); △, 401 boundary layer (Sanborn & Marshall 1965); △, 540 grid turbulence (Kistler & Vrebalovich 1966); ×, 780 round jet (Gibson 1963); ·, 850 boundary layer (Coantic & Favre 1974); +, ~ 2000 tidal channel (Grant *et al.* 1962); ⊙, 3180 return channel (CAHI Moscow 1991); ●, 1500 boundary layer (present data, mid-layer: $U_e = 50\ \mathrm{m\ s^{-1}}$); ■, 600 boundary layer (present data, mid-layer: $U_e = 10\ \mathrm{m\ s^{-1}}$).

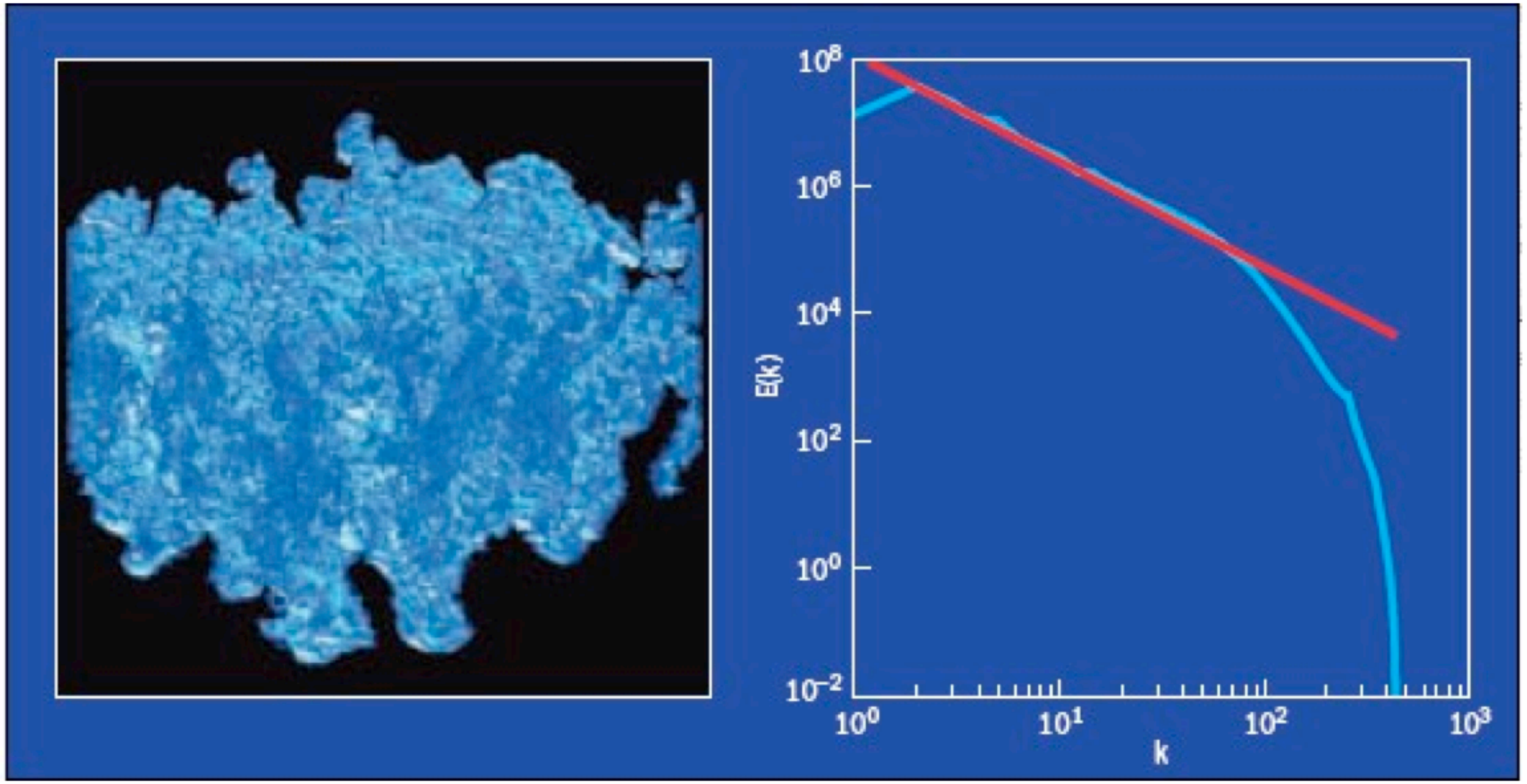

Figure 10. Image of a three-dimensional Rayleigh–Taylor unstable flame in a Type Ia supernova and the computed kinetic energy spectrum (blue curve) exhibiting the classical $k^{5/3}$ decay (red line).

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13.4 Flow past a flat plate of length L is shown in Figure 13.13. Assume an attached laminar Blasius boundary layer over the length of the plate. Show that the drag per unit span of the plate is proportional to $U_\infty^{3/2} L^{1/2}$. How would you expect the turbulence intensity u' to depend on U_∞ and L at a fixed point x in the far wake?

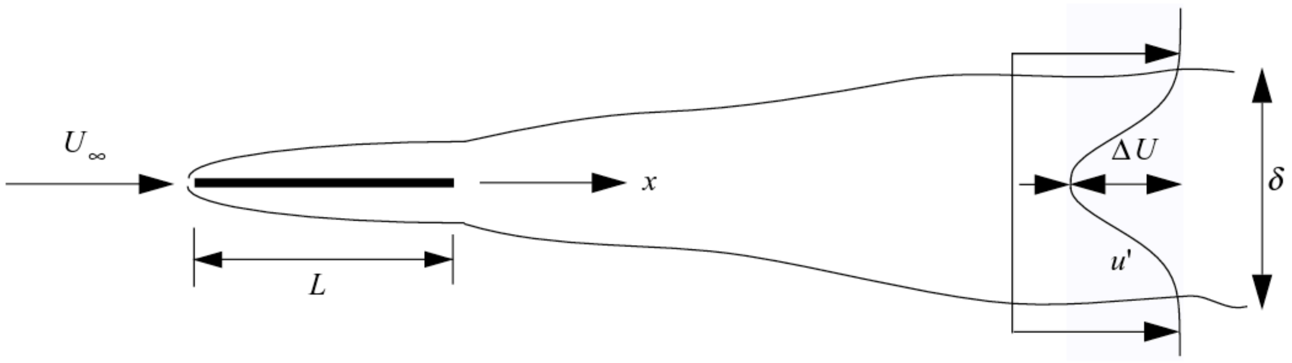

Fig. 13.13.

13.5 Solve the turbulent counterpart of Exercise 13.4. Assume an attached turbulent boundary layer over the length of the plate. The local skin friction coefficient can be taken as $C_f = 0.06(U_\infty x/\nu)^{-1/5}$. How would you expect the turbulence intensity u' to depend on U_∞ and L at a fixed point x in the far wake?