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Introduction to Symmetry Analysis

Chapter 12 -Compressible Flow

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Symmetries of the Compressible Euler Equations

$$
\begin{aligned}
\Psi^1 &= u_t + uu_x + vu_y + wu_z + \frac{p_x}{\rho} = 0,\\
\Psi^2 &= v_t + uv_x + vv_y + wv_z + \frac{p_y}{\rho} = 0,\\
\Psi^3 &= w_t + uw_x + vw_y + ww_z + \frac{p_z}{\rho} = 0,\\
\Psi^4 &= \rho_t + u\rho_x + v\rho_y + w\rho_z + \rho(u_x + v_y + w_z) = 0,\\
\Psi^5 &= p_t + up_x + vp_y + wp_z + F[p,\rho](u_x + v_y + w_z) = 0.
\end{aligned}
\tag{12.1}
$$

$$
F[p,\rho] = -\rho\frac{\partial S/\partial\rho}{\partial S/\partial p}. \tag{12.2}
$$

Infinitesimal transformation

$$
\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, t, \boldsymbol{u}, p, \rho],\\
\tilde{t} &= t + s\tau[\boldsymbol{x}, t, \boldsymbol{u}, p, \rho],\\
\tilde{u}^i &= u^i + s\eta^i[\boldsymbol{x}, t, \boldsymbol{u}, p, \rho],\\
\tilde{p} &= p + s\zeta[\boldsymbol{x}, t, \boldsymbol{u}, p, \rho],\\
\tilde{\rho} &= \rho + s\sigma[\boldsymbol{x}, t, \boldsymbol{u}, p, \rho]
\end{aligned}
\tag{12.3}
$$

Group operator

$$
\begin{aligned}
X_{\{1\}} = {} & \xi^j\frac{\partial}{\partial x^j} + \tau\frac{\partial}{\partial t} + \eta^i\frac{\partial}{\partial u^i} + \zeta\frac{\partial}{\partial p} + \sigma\frac{\partial}{\partial \rho}\\
& + \eta^i_{\{j\}}\frac{\partial}{\partial u^i_j} + \eta^i_{\{t\}}\frac{\partial}{\partial u^i_t} + \zeta_{\{j\}}\frac{\partial}{\partial p_j} + \zeta_{\{t\}}\frac{\partial}{\partial p_t} + \sigma_{\{j\}}\frac{\partial}{\partial \rho_j} + \sigma_{\{t\}}\frac{\partial}{\partial \rho_t}.
\end{aligned}
\tag{12.4}
$$

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There are five equations in the system, and so five invariance conditions are involved in solving for the groups of (12.1). The invariance conditions written out are listed below:

$$
\begin{aligned}
X_{\{1\}}\Psi^1 &= X_{\{1\}}\left(u_t + uu_x + vu_y + wu_z + \frac{p_x}{\rho}\right) \\
&= u_x\eta^1 + u_y\eta^2 + u_z\eta^3 + \left(-\frac{p_x}{\rho^2}\right)\sigma \\
&\quad + u\eta^1_{\{1\}} + v\eta^1_{\{2\}} + w\eta^1_{\{3\}} + \eta^1_{\{t\}} + \frac{1}{\rho}\zeta_{\{1\}} = 0,
\end{aligned} \tag{12.5}
$$

$$
\begin{aligned}
X_{\{1\}}\Psi^2 &= X_{\{1\}}\left(v_t + uv_x + vv_y + wv_z + \frac{p_y}{\rho}\right) \\
&= v_x\eta^1 + v_y\eta^2 + v_z\eta^3 + \left(-\frac{p_y}{\rho^2}\right)\sigma \\
&\quad + u\eta^2_{\{1\}} + v\eta^2_{\{2\}} + w\eta^2_{\{3\}} + \eta^2_{\{t\}} + \frac{1}{\rho}\zeta_{\{2\}} = 0,
\end{aligned} \tag{12.6}
$$

$$
\begin{aligned}
X_{\{1\}}\Psi^3 &= X_{\{1\}}\left(w_t + uw_x + vw_y + ww_z + \frac{p_z}{\rho}\right) \\
&= w_x\eta^1 + w_y\eta^2 + w_z\eta^3 + \left(-\frac{p_z}{\rho^2}\right)\sigma \\
&\quad + u\eta^3_{\{1\}} + v\eta^3_{\{2\}} + w\eta^3_{\{3\}} + \eta^3_{\{t\}} + \frac{1}{\rho}\zeta_{\{3\}} = 0,
\end{aligned} \tag{12.7}
$$

$$
\begin{aligned}
X_{\{1\}}\Psi^4 &= X_{\{1\}}(\rho_t + u\rho_x + v\rho_y + w\rho_z + \rho(u_x + v_y + w_z)) \\
&= \eta^1\rho_x + \eta^2\rho_y + \eta^3\rho_z + \sigma(u_x + u_y + u_z) + \rho\left(\eta^1_{\{1\}} + \eta^2_{\{2\}} + \eta^3_{\{3\}}\right) \\
&\quad + u\sigma_{\{1\}} + v\sigma_{\{2\}} + w\sigma_{\{3\}} + \sigma_{\{t\}} = 0,
\end{aligned} \tag{12.8}
$$

$$
\begin{aligned}
X_{\{1\}}\Psi^5 &= X_{\{1\}}(p_t + up_x + vp_y + wp_z + F(p,\rho)(u_x + v_y + w_z)) \\
&= \eta^1 p_x + \eta^2 p_y + \eta^3 p_z \\
&\quad + \zeta F_p(u_x + u_y + u_z) + \sigma F_\rho(u_x + u_y + u_z) + F\left(\eta^1_{\{1\}} + \eta^2_{\{2\}} + \eta^3_{\{3\}}\right) \\
&\quad + u\zeta_{\{1\}} + v\zeta_{\{2\}} + w\zeta_{\{3\}} + \zeta_{\{t\}} = 0.
\end{aligned} \tag{12.9}
$$

The variables (u, v, w, p, ρ) satisfy the system (12.1), and this condition has to be imposed on (12.5) to (12.9). To accomplish this we use (12.1) to define a set of replacement rules to be inserted in each of the five invariance conditions

(12.5) to (12.9). A reasonable choice would be

$$
\begin{aligned}
u_t &\to -\left(uu_x + vu_y + wu_z + \frac{p_x}{\rho}\right),\\
v_t &\to -\left(uv_x + vv_y + wv_z + \frac{p_y}{\rho}\right),\\
w_t &\to -\left(uw_x + vw_y + ww_z + \frac{p_z}{\rho}\right),\\
\rho_t &\to -(u\rho_x + v\rho_y + w\rho_z + \rho(u_x + v_y + w_z)),\\
p_t &\to -(up_x + vp_y + wp_z + F[p, \rho](u_x + v_y + w_z)).
\end{aligned}
\tag{12.10}
$$

It is important to keep in mind two points when making the replacements:

- All five replacements in (12.10) must be made in *each* of (12.5) to (12.9).
- It is essential to isolate a single term in each of the governing equations, such as the time derivatives in (12.10), in order to make the replacement. Replacing a product such as, say, uw_x is incorrect, because u and w_x do not only appear as that particular product in (12.5) to (12.9). Moreover, u is one of the independent variables of the infinitesimals. To remove it where it might appear explicitly in the invariance conditions would produce an overly restricted system of determining equations. For the same reason, it would not be appropriate to solve for ρ and try to remove it from the invariance conditions. For some complicated nonlinear equations, isolating a single term may be extremely difficult, but such cases are relatively rare.

Group operators

Running the package **`IntroToSymmetry.m`** reveals that (12.1) is invariant under an 11-parameter group with the following operators:

(1) Invariance under translation in time:

$$X^1 = \frac{\partial}{\partial t}. \tag{12.11}$$

(2) Invariance under translation in x:

$$X^2 = \frac{\partial}{\partial x}. \tag{12.12}$$

(3) Invariance under translation in y:

$$X^3 = \frac{\partial}{\partial y}. \tag{12.13}$$

(4) Invariance under translation in z:

$$X^4 = \frac{\partial}{\partial z}. \tag{12.14}$$

(5) Rotation about the z-axis:

$$X^5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}. \tag{12.15}$$

(6) Rotation about the x-axis:

$$X^6 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}. \tag{12.16}$$

(7) Rotation about the y-axis:

$$X^7 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + w\frac{\partial}{\partial u} - u\frac{\partial}{\partial w}. \tag{12.17}$$

(8) Constant-speed translation in the x-direction:

$$X^8 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \tag{12.18}$$

(9) Constant-speed translation in the y-direction:

$$X^9 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial u}. \tag{12.19}$$

(10) Constant-speed translation in the z-direction:

$$X^{10} = t\frac{\partial}{\partial z} + \frac{\partial}{\partial u}. \tag{12.20}$$

(11) The fundamental dilation group of the equation:

$$X^{11} = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}. \tag{12.21}$$

Additional groups arise when the function $F[p, \rho]$ is restricted in some way. A few examples are given below.

Case 1: $F = f[\rho]$. For this case there is one additional operator:

$$X^{12} = \frac{\partial}{\partial p}. \tag{12.22}$$

Case 2: $F = f[p]$. In this case the new symmetry is

$$X^{12} = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} + 2\rho\frac{\partial}{\partial \rho}. \tag{12.23}$$

Case 3: $F = A\rho^{\sigma},\ \sigma \neq 0$. Two additional symmetries arise:

$$\begin{aligned} X^{12} &= (\sigma-1)t\frac{\partial}{\partial t} - (\sigma-1)u\frac{\partial}{\partial u} - (\sigma-1)v\frac{\partial}{\partial v} - (\sigma-1)w\frac{\partial}{\partial w} \\ &\quad - 2\rho\frac{\partial}{\partial \rho} - 2\sigma p\frac{\partial}{\partial p}, \\ X^{13} &= \frac{\partial}{\partial p}. \end{aligned} \tag{12.24}$$

Case 4: $F = Ap$. This form of F also brings in two additional symmetries:

$$\begin{aligned} X^{12} &= t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} + 2\rho\frac{\partial}{\partial \rho}, \\ X^{13} &= p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}. \end{aligned} \tag{12.25}$$

Case 5: $F = \frac{5}{3}p$. This choice of F corresponds to the isentropic flow of a monatomic gas with ratio of specific heats $\gamma = \frac{5}{3}$. In this case three additional group operators arise:

$$\begin{aligned} X^{12} &= t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} + 2\rho\frac{\partial}{\partial \rho}, \\ X^{13} &= p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}, \\ X^{14} &= t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + yt\frac{\partial}{\partial y} + zt\frac{\partial}{\partial z} \\ &\quad + (x-ut)\frac{\partial}{\partial u} + (y-vt)\frac{\partial}{\partial v} + (z-wt)\frac{\partial}{\partial w} \\ &\quad - 5pt\frac{\partial}{\partial p} - 3\rho t\frac{\partial}{\partial \rho}. \end{aligned} \tag{12.26}$$

Sudden expansion of a gas cloud into a vacuum

Assume homentropic (homogeneously isentropic) flow of a perfect gas

$$\frac{p}{p_r} = \left(\frac{\rho}{\rho_r}\right)^{\gamma}; \ \nabla S = 0$$

$$\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} + \rho^{\gamma-2} \frac{\partial \rho}{\partial x^i} = 0,$$

$$\frac{\partial \rho}{\partial t} + u^j \frac{\partial \rho}{\partial x^j} + \rho \frac{\partial u^j}{\partial x^j} = 0, \qquad (12.30)$$

$$i = 1, \ldots, n, \qquad \text{sum over} \quad j = 1, \ldots, n,$$

Instead of beginning with the problem statement and then searching for an invariant group, let's back into this problem by beginning with the group X^{14} appended to the time translation group X^1, and then see what physical problem fits naturally into the chosen symmetries. For a perfect gas in n space dimensions, the group $t_0^2 X^1 + X^{14}$ takes the form

$$t_0^2 X^1 + X^{14} = (t_0^2 + t^2)\frac{\partial}{\partial t} + x^j t \frac{\partial}{\partial x^j} + (x^j - u^j t)\frac{\partial}{\partial u^j} - (n+2)pt\frac{\partial}{\partial p} - n\rho t \frac{\partial}{\partial \rho}, \qquad (12.31)$$

where t_0^2 is an arbitrary constant that will eventually play the role of an effective origin in time. The characteristic equations of (12.31) are

$$\frac{dt}{t_0^2 + t^2} = \frac{dx^j}{x^j t} = \frac{du^j}{x^j - u^j t} = \frac{dp}{(n+2)pt} = \frac{d\rho}{n\rho t}. \tag{12.32}$$

The last two terms generate the invariant

$$\psi = p\rho^{-(n+2)/n}. \tag{12.33}$$

The system (12.30) is invariant under the group (12.31) (more particularly the group X^{14}) if and only if

$$\gamma = \frac{n+2}{n}. \tag{12.34}$$

We therefore expect similarity solutions which are invariant under this group for one-, two-, and three-dimensional flow only for $\gamma = 3, 2,$ and $\frac{5}{3}$ respectively.

In a way (12.34) is a remarkable result. It is the same one that comes from the kinetic theory of gases, where n is the number of degrees of freedom of an individual gas molecule, yet there is nothing in (12.30) to suggest the corpuscular nature of the medium governed by (12.30). It almost seems that the equations anticipate the existence of monatomic gases with $n = 3$. Essentially, (12.34) expresses the dilation symmetry in the pressure and density common to both theories.

12.3.2 Solutions

What kind of solutions come out of this group? Solving the characteristic equations (12.32) leads to the following invariants:

$$\begin{aligned} \alpha^i &= \frac{x^i}{\left(t_0^2+t^2\right)^{1/2}}, \\ U^i[\alpha] &= u^i\left(t_0^2+t^2\right)^{1/2} - \frac{x_t^i}{\left(t_0^2+t^2\right)^{1/2}}, \\ P[\alpha] &= p\left(t_0^2+t^2\right)^{(n+2)/2}, \\ R[\alpha] &= \rho\left(t_0^2+t^2\right)^{n/2}. \end{aligned} \tag{12.35}$$

Now substitute (12.35) into (12.30) with $\gamma = (n+2)/n$. The result is

$$\begin{aligned} \alpha^i + U^j \frac{\partial U^i}{\partial \alpha^j} + R^{(2/n)-1}\frac{\partial R}{\partial \alpha^i} &= 0, \\ \frac{\partial (RU^j)}{\partial \alpha^j} &= 0, \end{aligned} \tag{12.36}$$

$$i = 1, \ldots, n, \qquad \text{sum over} \quad j = 1, \ldots, n.$$

The similarity variables (12.35) lead to the expected reduction in the number of independent variables; time is eliminated from the problem.

As was pointed out above, every once in a while group analysis can lead directly to interesting nontrivial solutions of the equations of motion. Let's consider the simplest possible case with $U^i[\alpha] = 0$. The self-similar continuity equation is satisfied and so (12.36) reduces to

$$\alpha^i + R^{(2/n)-1}\frac{\partial R}{\partial \alpha^i} = 0, \qquad i = 1, \ldots, n, \quad \text{sum over} \ \ j = 1, \ldots, n. \tag{12.37}$$

12.3.2.1 *Case 1:* $n = 1$, $\gamma = 3$

The solution is

$$R = (C^2 - \alpha^2)^{1/2}, \tag{12.38}$$

where C is a constant of integration. The density and velocity profiles in physical coordinates are

$$\rho = \frac{1}{(t_0^2 + t^2)^{1/2}} \left(C^2 - \frac{x^2}{t_0^2 + t^2} \right)^{1/2}, \qquad u = \frac{xt}{t_0^2 + t^2}. \tag{12.39}$$

as shown in Figure 12.2.

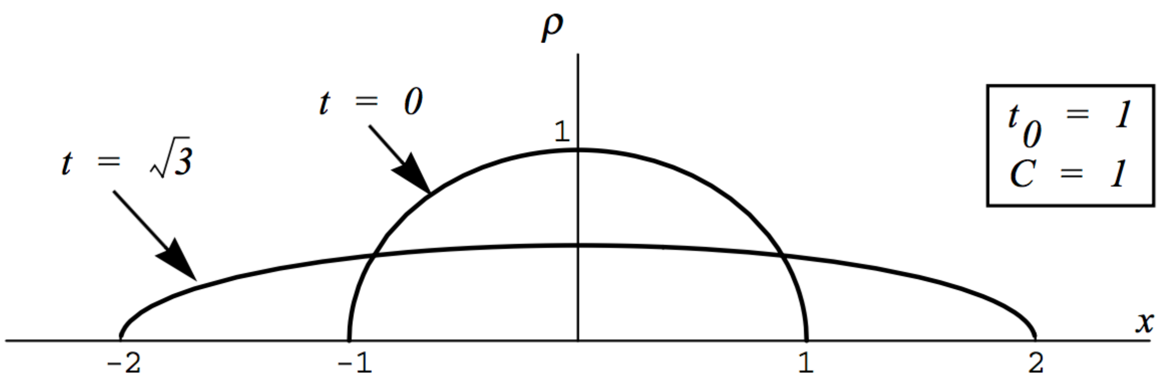

Fig. 12.2. Density profiles for a 1-D gas expanding into vacuum.

Gasdynamic – shallow water flow analogy $\gamma = 2$

12.3.2.2 Case 2: $n = 2$, $\gamma = 2$

The self-similar density is

$$R = \frac{C^2}{2} - \frac{\alpha_1^2 + \alpha_2^2}{2}, \qquad (12.40)$$

where C is again a constant of integration. The density and velocity profiles in physical coordinates are

$$\rho = \frac{1}{2(t_0^2 + t^2)}\left(C^2 - \frac{x^2 + y^2}{t_0^2 + t^2}\right), \qquad (12.41)$$
$$u = \frac{xt}{t_0^2 + t^2}, \qquad v = \frac{yt}{t_0^2 + t^2}.$$

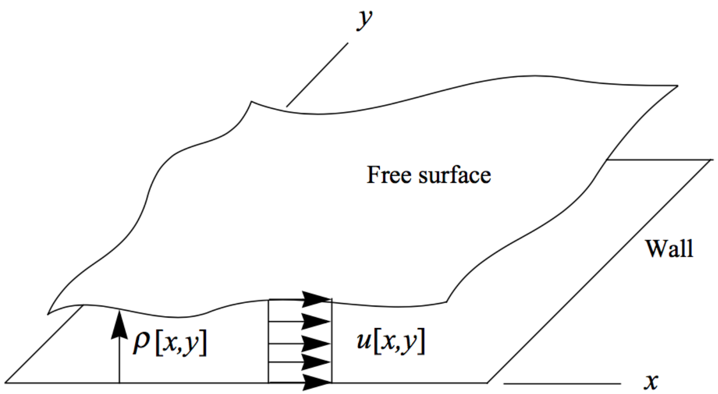

Fig. 12.1. Flow sketch for shallow-water analogy.

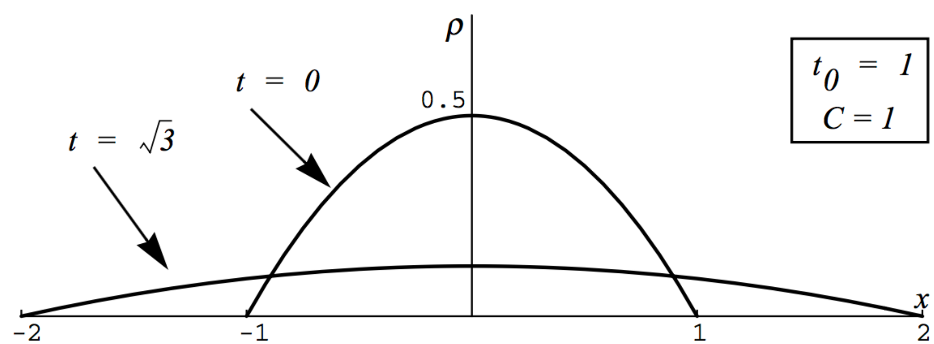

Fig. 12.3. Height profiles of a collapsing parabolic pile of water.

Following Reference [12.5], we can interpret the solution as a parabolic pile of water collapsing under its own weight. The solution is plotted in Figure 12.3.

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12.3.2.3 Case 3: $n = 3, \gamma = \frac{5}{3}$

The density solution is

$$R = \left(\frac{C^2}{3} - \frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}{3}\right)^{3/2}, \tag{12.42}$$

where C is a constant of integration. The density and velocity profiles for this case are

$$\begin{gathered} \rho = \frac{1}{3^{3/2}(t_0^2 + t^2)^{3/2}}\left(C^2 - \frac{x^2 + y^2 + z^2}{t_0^2 + t^2}\right)^{3/2}, \\ u = \frac{xt}{t_0^2 + t^2}, \qquad v = \frac{yt}{t_0^2 + t^2}, \qquad w = \frac{zt}{t_0^2 + t^2}, \end{gathered} \tag{12.43}$$

as shown in Figure 12.4.

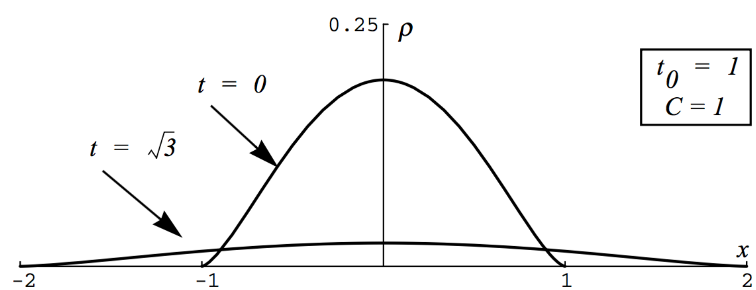

Fig. 12.4. Density profiles for a monatomic gas expanding into vacuum.

We have been a little cavelier in the use of dimensionless variables. Clearly there are parameters of the initial conditions that can be used to nondimensionalize variables such as the initial radius r_0 of the density distribution, and the initial pressure p_0 and density ρ_0 at the center of the distribution. See Exercise 12.1. The total energy contained in the gas cloud is

$$E = \int_0^{R_s}\left(\rho C_v T + \frac{1}{2}\rho u^2\right) 4\pi r^2\, dr = \int_0^{R_s}\left(\frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2\right) 4\pi r^2\, dr. \tag{12.44}$$

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Strong spherical blast wave

Radius of a sphere of ambient gas with the same total internal energy as the explosion energy

$$E = \tfrac{4}{3}\pi R_0^3 \rho_\infty C_v T_\infty$$

$$R_0 = \left(\frac{3}{4}\frac{\gamma - 1}{\pi}\frac{E}{p_\infty}\right)^{1/3}$$

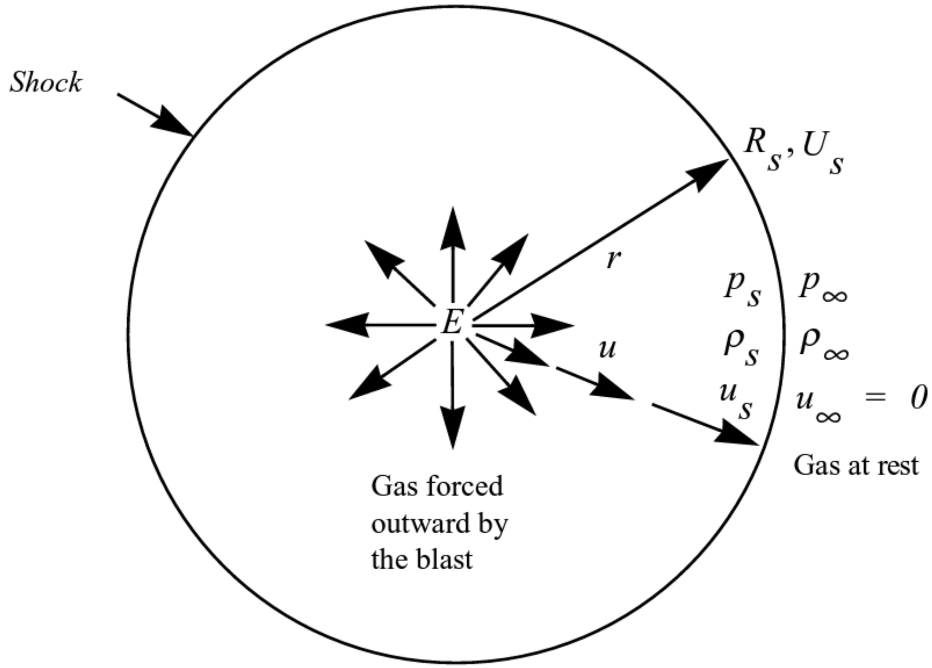

Fig. 12.5. Spherical point explosion.

Gas properties behind the shock come from normal shock theory with M>>1.

$$\rho_s = \frac{\gamma + 1}{\gamma - 1}\rho_\infty,$$

$$p_s = \frac{2}{\gamma + 1}\rho_\infty U_s^2,$$

$$u_s = \frac{2}{\gamma + 1}U_s,$$

As long as the radius of propagation of the shock satisfies $R_s \ll R_0$, the following assumptions hold:

(i) The thermal energy per unit volume of the ambient gas can be neglected compared to the energy per unit volume of the gas within the wave.

(ii) The pressure ratio across the shock is large: $p_S/p_\infty \gg 1$.

As a consequence of these two assumptions one can assume that the strong-shock limit can be used to characterize the jump in gas properties across the blast wave. Namely,

$$\rho_s = \frac{\gamma+1}{\gamma-1}\rho_\infty,$$
$$p_s = \frac{2}{\gamma+1}\rho_\infty U_s^2, \quad (12.47)$$
$$u_s = \frac{2}{\gamma+1}U_s,$$

where $U_s = dR_s/dt$ is the shock speed and u_s is the gas speed behind the shock.

Notice that the limiting density ratio across the shock is finite. Therefore, in contrast to the ambient pressure, the effect of the ambient density on the shock speed cannot be ignored. Thus the flow pattern generated by the blast wave is completely determined by only two physical parameters, the energy E and the ambient density ρ_∞. Dimensional analysis applied to these parameters, including the time and the shock radius, with dimensions

$$\hat{E} = ML^2/T^2, \qquad \hat{\rho} = M/L^3, \qquad \hat{R}_s = L, \qquad \hat{t} = T, \tag{12.48}$$

leads to

$$\left(\frac{\rho_\infty}{E}\right)^{1/5} \frac{R_s}{t^{2/5}} = \text{constant} = \alpha_s. \tag{12.49}$$

The shock speed is

$$U_s = \frac{dR_s}{dt} = \alpha_s \frac{2}{5}\left(\frac{E}{\rho_\infty}\right)^{1/5} t^{-3/5}. \tag{12.50}$$

The constant α_s remains to be determined. The flow conditions just behind the shock are

$$\begin{aligned}
\rho_s &= \frac{\gamma+1}{\gamma-1}\rho_\infty, \\
p_s &= \frac{2}{\gamma+1}\rho_\infty U_s^2 = \alpha_s^2\left(\frac{4}{25}\right)\left(\frac{2}{\gamma+1}\right)\left(\rho_\infty^{3/5}E^{2/5}\right)t^{-6/5}, \\
u_s &= \frac{2}{\gamma+1}U_s = \alpha_s\left(\frac{2}{5}\right)\left(\frac{2}{\gamma+1}\right)\left(\frac{E}{\rho_\infty}\right)^{1/5} t^{-3/5}.
\end{aligned} \tag{12.51}$$

In order to determine α_s we turn to the equations of motion and solve for the flow between the shock front and the origin of the explosion, $0 < r < R_s$. The

Governing equations

Inviscid, isentropic flow behind the shock

$$\rho_t + u\rho_r + \rho\left(u_r + \frac{2u}{r}\right) = 0,$$

$$u_t + uu_r + \frac{p_r}{\rho} = 0, \qquad (12.52)$$

$$p_t + up_r + \gamma p\left(u_r + \frac{2u}{r}\right) = 0.$$

The last equation in (12.52) is derived from the equation for conservation of entropy. When combined with the continuity equation it can be written as

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\ln[p/\rho^{\gamma}] = 0. \qquad (12.53)$$

Noting the formula (12.29) for the entropy of an ideal gas, we can write (12.53) as

$$\frac{DS}{Dt} = 0, \qquad (12.54)$$

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If we neglect the internal energy of the ambient fluid being continuously enclosed by the outwardly moving shock, then the total energy of the gas between the origin and the shock front is approximately constant,

$$E = \int_0^{R_s} \left(\rho C_v T + \frac{1}{2}\rho u^2 \right) 4\pi r^2 \, dr = \int^{R_s} \left(\frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \right) 4\pi r^2 \, dr. \tag{12.55}$$

Invariant group

See equation (12.44). The spherical symmetry of the problem, together with the result (12.49) from dimensional analysis and the conserved integral (12.55), suggests that the problem may be invariant under a dilation group. Let's try

$$\tilde{r} = e^a r, \qquad \tilde{t} = e^b t, \qquad \tilde{u} = e^c u, \qquad \tilde{p} = e^d p, \qquad \tilde{\rho} = e^f \rho. \quad (12.56)$$

Transforming (12.52) using (12.56) and requiring invariance reduces the group to the following:

$$\begin{gathered} \tilde{r} = e^a r, \qquad \tilde{t} = e^b t, \qquad \tilde{u} = e^{a-b} u, \\ \tilde{p} = e^d p, \qquad \tilde{\rho} = e^{d-2a+2b} \rho. \end{gathered} \quad (12.57)$$

The infinitesimal three-parameter group operator corresponding to (12.57) is

$$X = ar\frac{\partial}{\partial r} + bt\frac{\partial}{\partial t} + (a-b)u\frac{\partial}{\partial u} + (d)p\frac{\partial}{\partial p} + (d-2a+2b)\rho\frac{\partial}{\partial \rho}. \quad (12.58)$$

This is a sum of group operators for the basic compressible flow equations, (12.1). In particular, we add the dilation operator X^{11} in (12.21) to the operators X^{12} and X^{13} in (12.25) corresponding to case 4 ($F = A\rho$). That is,

$$X = aX^{11} - (a-b)X^{12} + dX^{13}. \quad (12.59)$$

Invariant group – cont'd

The constancy of α_s suggested by the results of dimensional analysis (12.49) implies that $b = \frac{5}{2}a$. The group is now simplified to

$$\tilde{r} = e^{a} r, \qquad \tilde{t} = e^{(5/2)a} t, \qquad \tilde{u} = e^{-(3/2)a} u, \qquad \tilde{p} = e^{d} p, \qquad \tilde{\rho} = e^{d+3a} \rho. \tag{12.62}$$

Using (12.62) to transform the energy integral leads to

$$E = \int_0^{R_s} \left(\frac{\tilde{p}}{\gamma - 1} + \frac{1}{2} \tilde{\rho} \tilde{u}^2 \right) 4\pi \tilde{r}^2 \, d\tilde{r} = e^{d+3a} \int_0^{R_s} \left(\frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2 \right) 4\pi r^2 \, dr, \tag{12.63}$$

and for invariance we require $d = -3a$. Finally, the one-parameter group that leaves the entire problem invariant is

$$\tilde{r} = e^{a} r, \qquad \tilde{t} = e^{(5/2)a} t, \qquad \tilde{u} = e^{-(3/2)a} u, \qquad \tilde{p} = e^{-3a} p, \qquad \tilde{\rho} = \rho. \tag{12.64}$$

The characteristic equations of (12.64) are

$$\frac{dr}{r} = \frac{2\,dt}{5t} = \frac{2\,du}{-3u} = \frac{dp}{-3p} = \frac{d\rho}{0}. \tag{12.65}$$

Similarity solution

The problem is invariant under a one-parameter dilation group.

$$\tilde{r} = e^{a} r, \qquad \tilde{t} = e^{(5/2)a} t, \qquad \tilde{u} = e^{-(3/2)a} u, \qquad \tilde{p} = e^{-3a} p, \qquad \tilde{\rho} = \rho$$

Similarity variables

$$r = \left(\frac{E}{\rho_\infty} \right)^{1/5} t^{2/5} \ \alpha$$

$$u = \frac{2}{5} \left(\frac{2}{\gamma + 1} \right) \left(\frac{E}{\rho_\infty} \right)^{1/5} t^{-3/5} \ \alpha U(\alpha)$$

$$p = \frac{4}{25} \left(\frac{2}{\gamma + 1} \right) \left(\rho_\infty{}^{3} E^{2} \right)^{1/5} t^{-6/5} \ \alpha^{2} P(\alpha)$$

$$\rho = \rho_\infty \left(\frac{\gamma + 1}{\gamma - 1} \right) G(\alpha)$$

Solution

The governing equations (12.67) were solved exactly by von Neumann [12.11]. The velocity is related to the radial coordinate by

$$\frac{\alpha}{\alpha_s} = U^{-2/5}\left(\frac{\gamma U - \frac{\gamma+1}{2}}{\frac{\gamma-1}{2}}\right)^{\mu_1}\left(\frac{\frac{7-\gamma}{2}}{\frac{5}{2}(\gamma+1)-(3\gamma-1)U}\right)^{\mu_2}, \quad (12.68)$$

where

$$\mu_1 = \frac{\gamma-1}{2\gamma+1}, \qquad \mu_2 = \frac{13\gamma^2-7\gamma+12}{5(3\gamma-1)(2\gamma+1)}. \quad (12.69)$$

The density is determined as a function of velocity:

$$G = \left(\frac{\gamma U - \frac{\gamma+1}{2}}{\frac{\gamma-1}{2}}\right)^{\mu_3}\left(\frac{\frac{7-\gamma}{2}}{\frac{5}{2}(\gamma+1)-(3\gamma-1)U}\right)^{\mu_4}\left(\frac{\frac{\gamma-1}{2}}{\frac{\gamma+1}{2}-U}\right)^{\mu_5}, \quad (12.70)$$

where

$$\mu_3 = \frac{3}{2\gamma+1}, \qquad \mu_4 = \frac{13\gamma^2-7\gamma+12}{(2-\gamma)(3\gamma-1)(2\gamma+1)}, \qquad \mu_5 = \frac{1}{2-\gamma}. \quad (12.71)$$

Finally, the pressure is expressed in terms of the density and velocity functions:

$$\frac{P}{G} = \left(\frac{\frac{\gamma+1}{2}-U}{\gamma U - \frac{\gamma+1}{2}}\right)U^2. \quad (12.72)$$

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The ranges of these functions are

$$\begin{aligned} 0 \leq \alpha \leq \alpha_s, \\ \frac{\gamma+1}{2\gamma} \leq U[\alpha] \leq 1, \\ 0 \leq G[\alpha] \leq 1, \\ \infty \geq P[\alpha] \geq 1. \end{aligned} \tag{12.73}$$

Note that the physical velocity is zero at the origin ($\alpha = 0$) and the physical pressure has a finite limit. This can be seen from the limits,

$$\begin{aligned} &\lim_{\alpha \to 0} \left(\frac{\alpha}{\alpha_s}\right) U = 0, \\ &\lim_{\alpha \to 0} \left(\frac{\alpha}{\alpha_s}\right)^2 P = \left(\frac{\gamma+1}{2\gamma}\right)^{\frac{6}{5}} \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma-1}{2-\gamma}} \left(\frac{\gamma(7-\gamma)}{(\gamma+1)(2\gamma+1)}\right)^{\frac{(9-2\gamma)(13\gamma^2-7\gamma+12)}{5(2-\gamma)(3\gamma-1)(2\gamma+1)}} \end{aligned} \tag{12.74}$$

Shock front

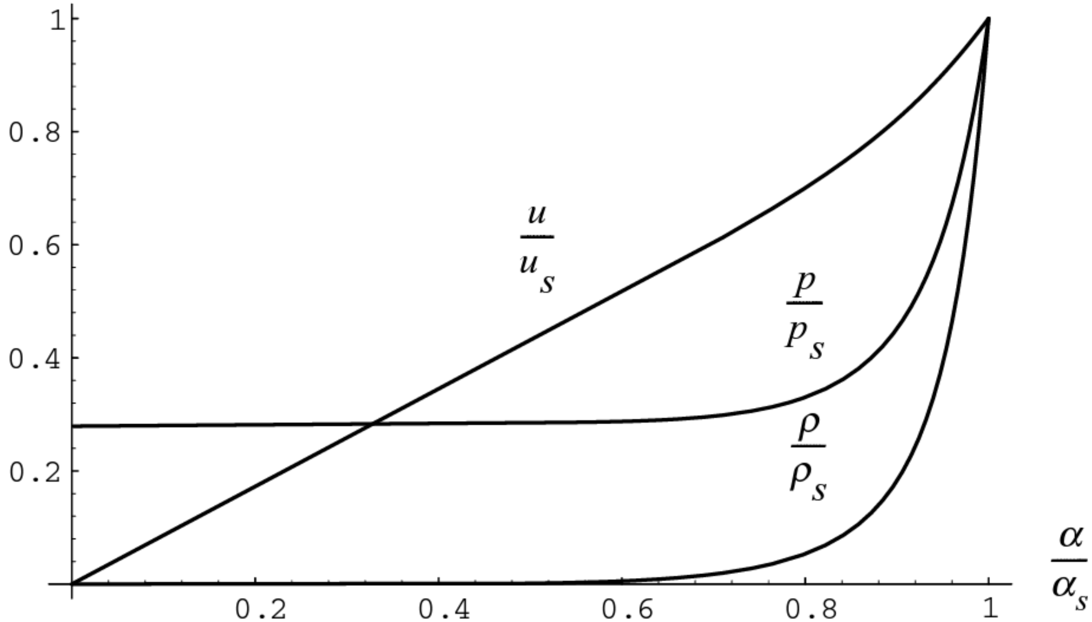

Fig. 12.6. Velocity, density, and pressure for $\gamma = 1.4$.

$$\rho_s = \frac{\gamma + 1}{\gamma - 1}\rho_\infty,$$

$$p_s = \frac{2}{\gamma + 1}\rho_\infty U_s^2 = \alpha_s^2\left(\frac{4}{25}\right)\left(\frac{2}{\gamma + 1}\right)(\rho_\infty^{3/5} E^{2/5})t^{-6/5}$$

$$u_s = \frac{2}{\gamma + 1}U_s = \alpha_s\left(\frac{2}{5}\right)\left(\frac{2}{\gamma + 1}\right)\left(\frac{E}{\rho_\infty}\right)^{1/5} t^{-3/5}.$$

Determination of α_s

$$\gamma = \frac{n+2}{n}, \tag{12.75}$$

$$E = \int_0^{R_s} \left(\frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \right) 4\pi r^2 \, dr, \tag{12.76}$$

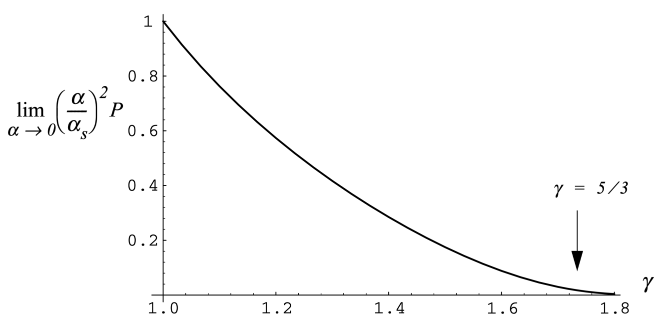

Fig. 12.7. Dependence of the pressure at the origin of the blast on γ.

which, in nondimensional terms, becomes

$$\frac{32\pi}{25(\gamma^2 - 1)} \int_0^{\alpha_s} (P + GU^2)\alpha^4 \, d\alpha = 1. \tag{12.77}$$

It is actually simpler to carry out the integration in (12.77) by integrating with respect to U (which is monotonic in α) and making use of (12.68) to replace α. Let $\alpha/\alpha_s = F[U]$. Now

$$\frac{32\pi\alpha_s^5}{25(\gamma^2 - 1)} \int_{(\gamma+1)/2\gamma}^{1} (P + RU^2)(F^4 F_U) \, dU = 1, \tag{12.78}$$

which, using (12.72), becomes

$$\frac{32\pi\alpha_s^5}{25(\gamma^2 - 1)} \int_{(\gamma+1)/2\gamma}^{1} \left(\frac{\gamma - 1}{\gamma U - \frac{\gamma+1}{2}} \right) GU^3 (F^4 F_U) \, dU = 1. \tag{12.79}$$

The relation (12.79) is integrated, allowing α_s to be evaluated. This process is carried out for various γ, and the result is as shown in Figure 12.8. Note that

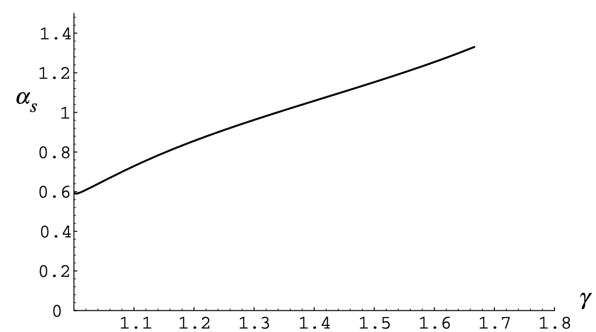

Fig. 12.8. The shock speed parameter as a function of γ.

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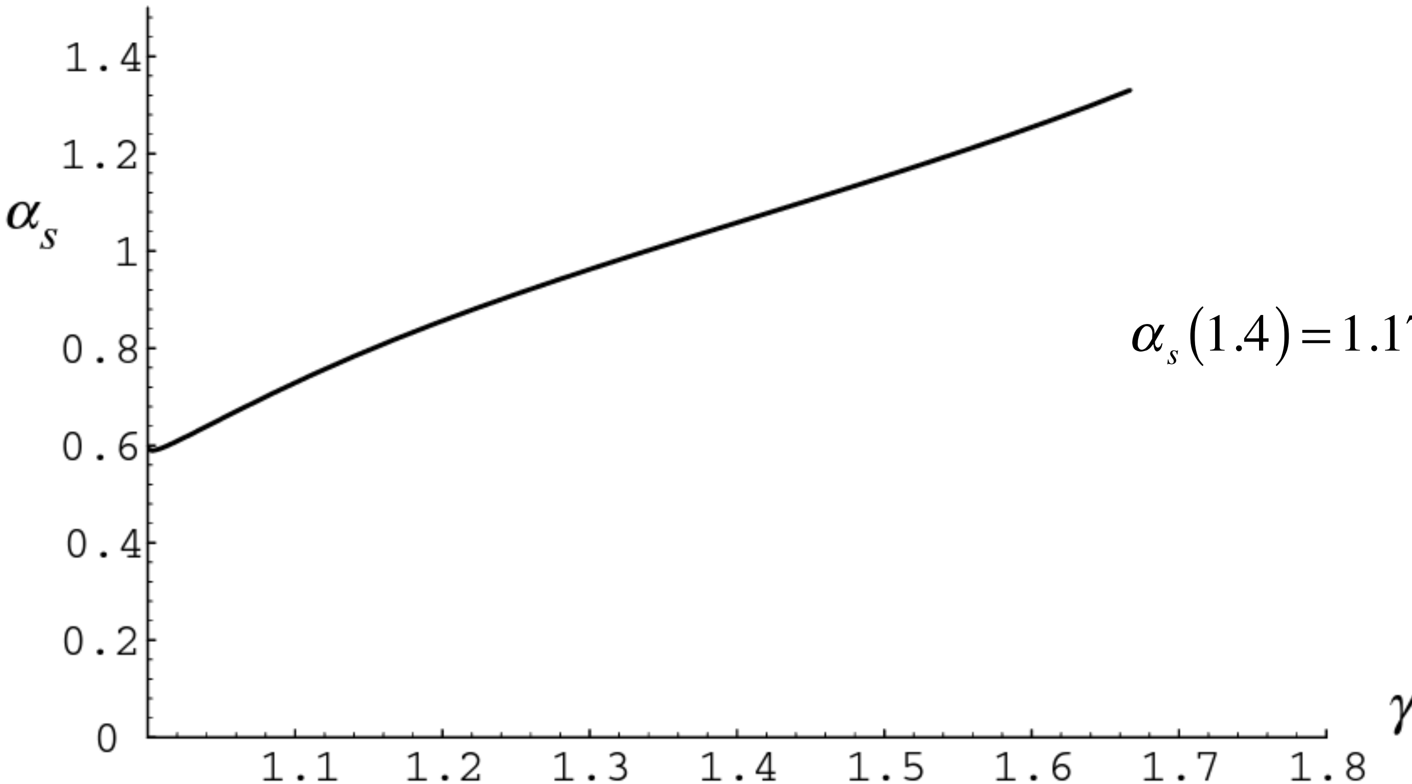
Solution for the unknown constant as a function of gamma



$\alpha_s(1.4) = 1.175$

Fig. 12.8. The shock speed parameter as a function of γ.

α_s varies relatively little for a range of γ between 1.1 and 1.4. Since there is considerable uncertainty in the actual value of γ inside the blast zone, this is the key feature of the problem that enabled Taylor to use the theory to estimate the energy of the first atomic bomb blast with some reasonable hope of accuracy.

We recall that

$$\left(\frac{\rho_\infty}{E}\right)^{1/5}\frac{R_s}{t^{2/5}} = \text{constant} = \alpha_s, \tag{12.80}$$

or

$$R_s = \alpha_s\left(\frac{E}{\rho_\infty}\right)^{1/5} t^{2/5}, \tag{12.81}$$

or

$$\ln R_s = \ln\left[\alpha_s\left(\frac{E}{\rho_\infty}\right)^{1/5}\right] + \frac{2}{5}\ln t. \tag{12.82}$$

When $\ln R_s$ is plotted versus $\ln t$ with α_s estimated from Figure 12.8 the result is a value for E.

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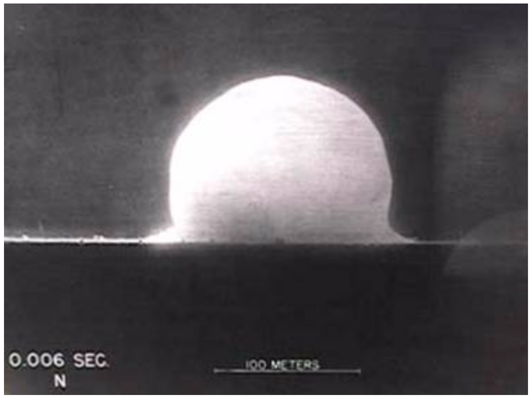
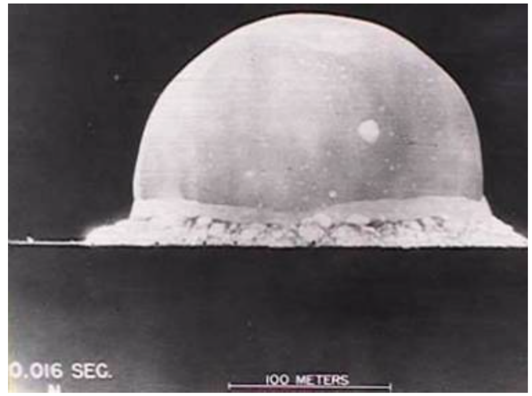
Shock wave produced by a very strong point explosion

Images from the trinity test, Alamogordo N.M., July 16, 1945 made public in 1947



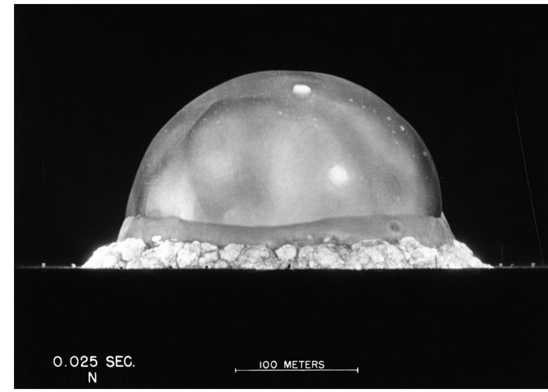
0.006 sec

100 meters



0.016 sec

100 meters

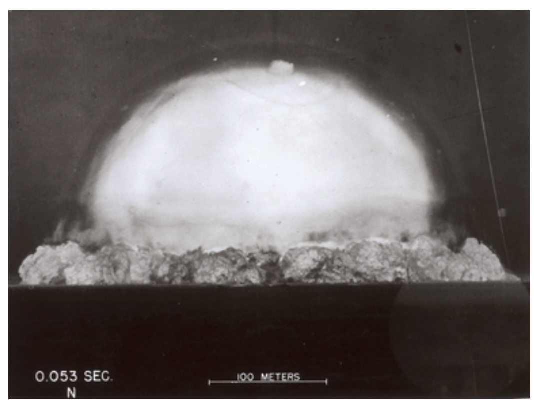


0.025 sec

100 meters

0.040 sec

100 meters



0.053 sec

100 meters

0.062 sec

100 meters

G.I Taylor UK 1941 and 1950, John Von Neumann USA 1941 and 1947 and Leonid Sedov USSR 1946

Knowing α_s determines the energy of the explosion

$$\left(\frac{\rho_\infty}{E}\right)^{1/5}\frac{R_s}{t^{2/5}} = \text{constant} = \alpha_s, \tag{12.80}$$

or

$$R_s = \alpha_s\left(\frac{E}{\rho_\infty}\right)^{1/5} t^{2/5}, \tag{12.81}$$

or

$$\ln R_s = \ln\left[\alpha_s\left(\frac{E}{\rho_\infty}\right)^{1/5}\right] + \frac{2}{5}\ln t. \tag{12.82}$$

When $\ln R_s$ is plotted versus $\ln t$ with α_s estimated from Figure 12.8 the result is a value for E.

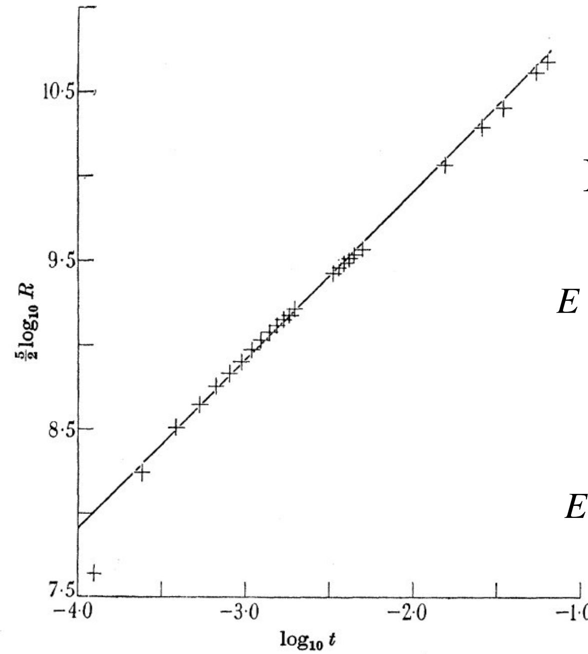


FIGURE 1. Logarithmic plot showing that $R^{\frac{5}{2}}$ is proportional to t.

Estimated energy

$E = 7.19\times 10^{13}\, J$

Actual energy

$E = 6.30\times 10^{13}\, J$