Introduction to Symmetry Analysis

Chapter 9 - Partial Differential Equations

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Finite transformation of partial derivatives

$$T^s : \begin{Bmatrix} \tilde{x}^j = F^j[\boldsymbol{x}, \boldsymbol{y}, s], \; j = 1, \ldots, n \\ \tilde{y}^i = G^i[\boldsymbol{x}, \boldsymbol{y}, s], \; i = 1, \ldots, m \end{Bmatrix}.$$

The once extended finite transformation is

$$\begin{aligned} \tilde{x}^j &= F^j[\boldsymbol{x}, \boldsymbol{y}, s], \qquad j = 1, \ldots, n, \\ \tilde{y}^i &= G^i[\boldsymbol{x}, \boldsymbol{y}, s], \qquad i = 1, \ldots, m, \\ \tilde{y}^i_j &= G^i_{\{j\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s], \end{aligned}$$

where $\boldsymbol{y}_1$ is the vector of all possible first partial derivatives and where

$$G^i_{\{j\}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s) = D_\beta G^i (D_j F^\beta)^{-1}.$$

The p-th extended finite group is

$$\begin{aligned}
\tilde{x}^j &= F^j[\boldsymbol{x}, \boldsymbol{y}, s], \qquad j = 1, \ldots, n, \\
\tilde{y}^i &= G^i[\boldsymbol{x}, \boldsymbol{y}, s], \qquad i = 1, \ldots, m, \\
\tilde{y}^i_{j_1} &= G^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, s], \\
&\vdots \\
\tilde{y}^i_{j_1 j_2 \ldots j_p} &= G^i_{\{j_1 j_2 \ldots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p, s],
\end{aligned}$$

where

$$G^i_{\{j_1 j_2 \ldots j_p\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p, s] = D_\beta G^i_{\{j_1 j_2 \ldots j_{p-1}\}}(D_{j_p} F^\beta)^{-1}.$$

Variable count

$$(x, y, y_1, y_2, \ldots, y_p)$$

$$q = n + m \sum_{k=0}^{p} \frac{(n+k-1)!}{k!(n-1)!}.$$

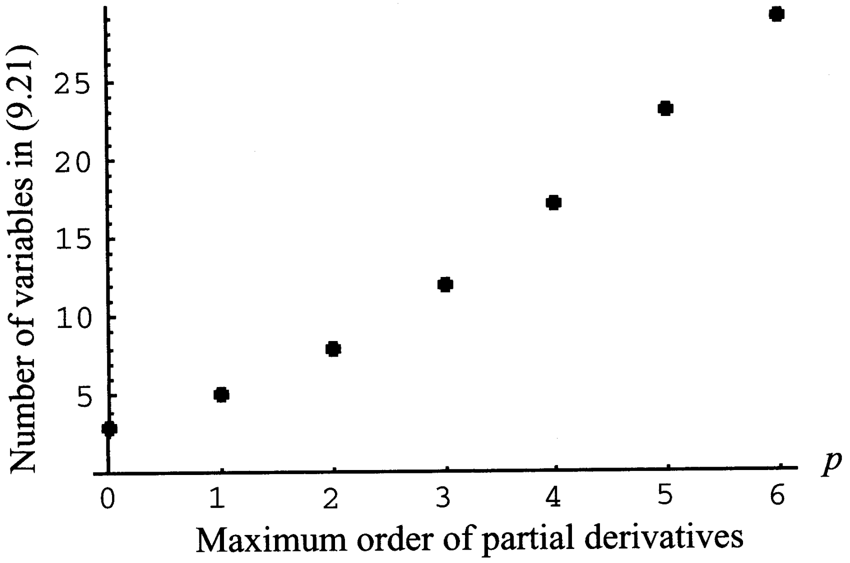

Fig. 9.1. Number of variables versus derivative order for $m = 1, n = 2$.

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Infinitesimal transformation of the first partial derivative

$$T^s : \left\{ \begin{aligned} \tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}], \quad j = 1, \ldots, n \\ \tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}], \quad i = 1, \ldots, m \end{aligned} \right\},$$

$$\xi^j[\boldsymbol{x}, \boldsymbol{y}] = \left(\frac{\partial F^j}{\partial s} \right)_{s=0}, \qquad \eta^i[\boldsymbol{x}, \boldsymbol{y}] = \left(\frac{\partial G^i}{\partial s} \right)_{s=0}.$$

Substitute

$$\tilde{y}^i_j = (D_\beta(y^i + s\eta^i))(D_j(x^\beta + s\xi^\beta))^{-1}.$$

$$\tilde{y}^i_j = \left(y^i_\beta + sD_\beta\eta^i\right)\left(\delta^\beta_j + sD_j\xi^\beta\right)^{-1}.$$

$$\left(\delta^\beta_j + sD_j\xi^\beta\right)^{-1} \approx \delta^\beta_j - s(D_j\xi^\beta).$$

$$\tilde{y}^i_j = \left(y^i_\beta + sD_\beta\eta^i\right)\left(\delta^\beta_j - sD_j\xi^\beta\right).$$

$$\tilde{y}^i_j = y^i_j + s\left(D_j\eta^i - y^i_\beta D_j\xi^\beta\right),$$

$$\tilde{x}^j = x^j + s\xi^j[\boldsymbol{x},\boldsymbol{y}], \qquad j = 1,\ldots,n,$$
$$\tilde{y}^i = y^i + s\eta^i[\boldsymbol{x},\boldsymbol{y}], \qquad i = 1,\ldots,m,$$
$$\tilde{y}^i_j = y^i_j + s\eta^i_{\{j\}}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}_1],$$

The (p-1)th order extended infinitesimal transformation is

$$
\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\mathbf{x}, \mathbf{y}], \\
\tilde{y}^i &= y^i + s\eta^i[\mathbf{x}, \mathbf{y}], \\
\tilde{y}^i_{j_1} &= y^i_{j_1} + s\eta^i_{\{j_1\}}[\mathbf{x}, \mathbf{y}, \mathbf{y}_1], \\
\tilde{y}^i_{j_1 j_2} &= y^i_{j_1 j_2} + s\eta^i_{\{j_1 j_2\}}[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \mathbf{y}_2], \\
&\vdots \\
\tilde{y}^i_{j_1 j_2 \ldots j_{p-1}} &= y^i_{j_1 j_2 \ldots j_{p-1}} + s\eta^i_{\{j_1 j_2 \ldots j_{p-1}\}}[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{p-1}].
\end{aligned}
$$

$$
\boxed{\eta^i_{\{j_1 j_2 \ldots j_p\}} = D_{j_p}\eta^i_{\{j_1 j_2 \ldots j_{p-1}\}} - y^i_{j_1 j_2 \ldots j_{p-1}\alpha} D_{j_p}\xi^\alpha .}
$$

The number of terms in the infinitesimal versus the derivative order

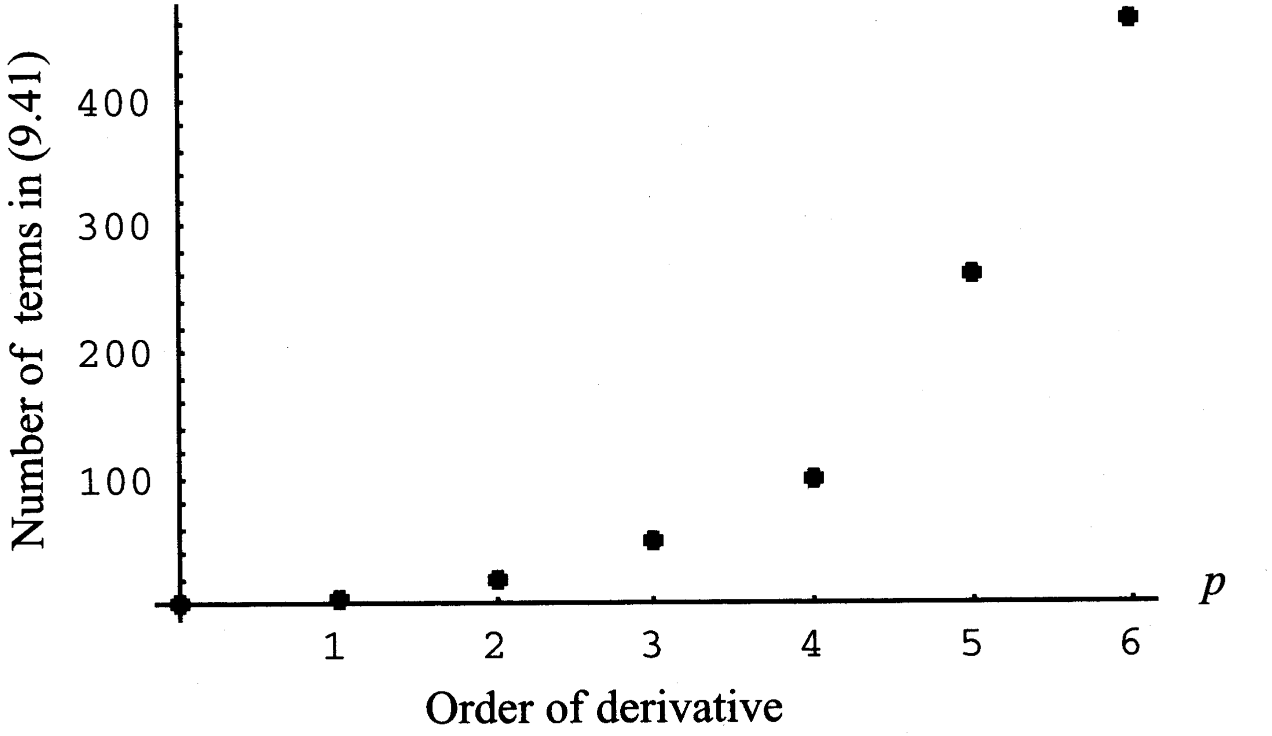

Fig. 9.2. Plot of the number of terms in the pth-order infinitesimal for the case $m = 1$, $n = 2$.

Theorem 9.2. *The pth-order system of partial differential equations* $\psi^i = \Psi^i[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p] = 0$ *is a vector of locally analytic functions of the differential variables* $\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p$. *Expand* $\Psi^i[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p]$ *in a Lie series*

$$\Psi^i[\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{y}}_1, \tilde{\mathbf{y}}_2, \ldots, \tilde{\mathbf{y}}_p]$$

$$= \Psi^i[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p] + sX_{\{p\}}\Psi^i + \frac{s^2}{2!}X_{\{p\}}\left(X_{\{p\}}\Psi^i\right) + \cdots, \quad (9.51)$$

where $X_{\{p\}}$ *is the pth extended group operator*

$$X_{\{p\}} = \xi^j \frac{\partial}{\partial x^j} + \eta^i \frac{\partial}{\partial y^i} + \eta^i_{\{j_1\}} \frac{\partial}{\partial y^i_{j_1}} + \eta^i_{\{j_1 j_2\}} \frac{\partial}{\partial y^i_{j_1 j_2}} + \cdots + \eta^i_{\{j_1 j_2 \ldots j_p\}} \frac{\partial}{\partial y^i_{j_1 j_2 \ldots j_p}}. \quad (9.52)$$

The system Ψ^i *is invariant under the group* (ξ^j, η^i) *if and only if*

$$\boxed{X_{\{p\}}\Psi^i = 0, \qquad i = 1, \ldots, m.} \quad (9.53)$$

The characteristic equations corresponding to (9.53) are

$$\boxed{\frac{dx^j}{\xi^j} = \frac{dy^i}{\eta^i} = \frac{dy^i_{j_1}}{\eta^i_{\{j_1\}}} = \frac{dy^i_{j_1 j_2}}{\eta^i_{\{j_1 j_2\}}} = \cdots = \frac{dy^i_{j_1 j_2 \ldots j_p}}{\eta^i_{\{j_1 j_2 \ldots j_p\}}}.} \quad (9.54)$$

Isolating the determining equations of the group - the Lie algorithm

Step 1. Note that there is one invariance condition for each equation in the system $\Psi^i = 0$. The invariance condition (9.53) is generally a rather long sum. Each term in (9.53) is of the form AB where A is some partial derivative of ξ^j or η^i and B is in general a product of partial derivatives of the various y^i. Begin by gathering together terms that multiply the same combinations of derivatives of the y^i.

Step 2. Attend to those combinations of derivatives of y^i that appear in the original system Ψ^i. By assumption, the y^i solve the original system, and so *all* the relations $\Psi^i = 0$ must be imposed on *each* invariance condition. This is commonly done by solving for some isolated derivative in each of the Ψ^i and replacing the corresponding term(s) in the invariance condition(s). The invariance conditions are then rearranged by gathering together common products of derivatives.

Step 3. All the coefficients multiplying various combinations of derivatives of the y^i are set equal to zero. These are the *determining equations* of the group. For a system of equations the determining equations from each invariance condition are concatenated together. This is the complete system of determining equations.

Step 4. The result of steps 2 and 3 is a (usually) overdetermined set of linear PDEs in the unknown infinitesimals (ξ^j, η^i). Initial considerations of these PDEs, many of which may be redundant, permit a number of them to be eliminated, and ultimately relatively few play a role in determining the (ξ^j, η^i).

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The classical point group of the heat equation

$$\phi = \Phi[x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}] = u_t - u_{xx} = 0.$$

$$\tilde{x} = x + s\xi[x, t, u],$$

$$\tilde{t} = t + s\tau[x, t, u],$$

$$\tilde{u} = u + s\eta[x, t, u],$$

$$\tilde{u}_{\tilde{x}} = u_x + s\eta_{\{x\}}[x, t, u, u_x, u_t],$$

$$\tilde{u}_{\tilde{t}} = u_t + s\eta_{\{t\}}[x, t, u, u_x, u_t],$$

$$\tilde{u}_{\tilde{x}\tilde{x}} = u_{xx} + s\eta_{\{xx\}}[x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}],$$

$$\tilde{u}_{\tilde{x}\tilde{t}} = u_{xt} + s\eta_{\{xt\}}[x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}],$$

$$\tilde{u}_{\tilde{t}\tilde{t}} = u_{tt} + s\eta_{\{tt\}}[x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}].$$

Invariance Condition

$$\begin{aligned}
X_{\{2\}}\Phi &= 0 \\
&= \xi\frac{\partial\Phi}{\partial x} + \tau\frac{\partial\Phi}{\partial t} + \eta\frac{\partial\Phi}{\partial u} + \eta_{\{x\}}\frac{\partial\Phi}{\partial u_x} + \eta_{\{t\}}\frac{\partial\Phi}{\partial u_t} \\
&\quad + \eta_{\{xx\}}\frac{\partial\Phi}{\partial u_{xx}} + \eta_{\{xt\}}\frac{\partial\Phi}{\partial u_{xt}} + \eta_{\{tt\}}\frac{\partial\Phi}{\partial u_{tt}}.
\end{aligned}$$

$$\eta_{\{t\}} - \eta_{\{xx\}} = 0$$

$$\begin{aligned}
\eta_{\{x\}} &= \eta_x + u_x\eta_u - u_x(\xi_x + u_x\xi_u) - u_t(\tau_x + u_x\tau_u), \\
\eta_{\{t\}} &= \eta_t + u_t\eta_u - u_x(\xi_t + u_t\xi_u) - u_t(\tau_t + u_t\tau_u).
\end{aligned}$$

$$\eta_{\{xx\}} = D_xD_x\eta - 2u_{xx}D_x\xi - 2u_{xt}D_x\tau - u_xD_xD_x\xi - u_tD_xD_x\tau.$$

$$\eta_{\{xx\}} = \eta_{xx} + u_{xx}\eta_u - u_{xx}u_t\tau_u - 2u_x u_{xt}\tau_u$$
$$- 3u_x u_{xx}\xi_u + u_x^2\eta_{uu} - u_x^2 u_t\tau_{uu}$$
$$- u_x^3\xi_{uu} - 2u_{xt}\tau_x - 2u_{xx}\xi_x + 2u_x\eta_{xu}$$
$$- 2u_x u_t\tau_{xu} - 2u_x^2\xi_{xu} - u_t\tau_{xx} - u_x\xi_{xx}.$$

The fully expanded invariance condition is

$$\eta_{\{t\}} - \eta_{\{xx\}} = \eta_t - \eta_{xx} - u_{xx}\eta_u + u_t\eta_u + u_{xx}u_t\tau_u$$
$$- u_t^2\tau_u + 2u_x u_{xt}\tau_u + 3u_x u_{xx}\xi_u - u_x u_t\xi_u$$
$$- u_x^2\eta_{uu} + u_x^2 u_t\tau_{uu} + u_x^3\xi_{uu} + 2u_{xt}\tau_x$$
$$+ 2u_{xx}\xi_x - 2u_x\eta_{xu} + 2u_x u_t\tau_{xu} + 2u_x^2\xi_{xu}$$
$$+ u_t\tau_{xx} + u_x\xi_{xx} - u_t\tau_t - u_x\xi_t = 0.$$

Apply the constraint that the solution must satisfy the heat equation and gather terms

$$\eta_{\{t\}} - \eta_{\{xx\}} = (\eta_t - \eta_{xx}) + 2u_x u_{xt}(\tau_u) + 2u_x u_t(\xi_u + \tau_{xu})$$
$$+ u_x^2(2\xi_{xu} - \eta_{uu}) + u_x^2 u_t(\tau_{uu}) + u_x^3(\xi_{uu}) + 2u_{xt}(\tau_x)$$
$$+ u_t(\tau_{xx} + 2\xi_x - \tau_t) + u_x(\xi_{xx} - \xi_t - 2\eta_{xu}) = 0$$

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The *determining equations* of the point group of the heat equation

$$\begin{aligned}
\eta_t - \eta_{xx} &= 0, & \tau_{uu} &= 0, \\
\tau_u &= 0, & \xi_{uu} &= 0, \\
\xi_u + \tau_{xu} &= 0, & \tau_x &= 0, \\
2\xi_{xu} - \eta_{uu} &= 0, & \tau_{xx} + 2\xi_x - \tau_t &= 0, \\
\xi_{xx} - \xi_t - 2\eta_{xu} &= 0.
\end{aligned}$$

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Series solution of the determining equations

$$\begin{aligned}
\xi = {} & a^{10} + a^{11}x + a^{12}t + a^{13}u + a^{14}x^2 + a^{15}xt \\
& + a^{16}xu + a^{17}t^2 + a^{18}ut + a^{19}u^2 \\
& + a^{110}x^3 + a^{111}x^2t + a^{112}x^2u + a^{113}xt^2 + a^{114}xtu \\
& + a^{115}xu^2 + a^{116}t^3 + a^{117}t^2u + a^{118}tu^2 + a^{119}u^3, \\
\tau = {} & a^{20} + a^{21}x + a^{22}t + a^{23}u + a^{24}x^2 + a^{25}xt \\
& + a^{26}xu + a^{27}t^2 + a^{28}ut + a^{29}u^2 \\
& + a^{210}x^3 + a^{211}x^2t + a^{212}x^2u + a^{213}xt^2 + a^{214}xtu \\
& + a^{215}xu^2 + a^{216}t^3 + a^{217}t^2u + a^{218}tu^2 + a^{219}u^3, \\
\eta = {} & b^{10} + b^{11}x + b^{12}t + b^{13}u + b^{14}x^2 + b^{15}xt \\
& + b^{16}xu + b^{17}t^2 + b^{18}ut + b^{19}u^2 \\
& + b^{110}x^3 + b^{111}x^2t + b^{112}x^2u + b^{113}xt^2 + b^{114}xtu \\
& + b^{115}xu^2 + b^{116}t^3 + b^{117}t^2u + b^{118}tu^2 + b^{119}u^3.
\end{aligned}$$

The classical six parameter group of the heat equation

$$
\begin{aligned}
\xi &= a^{10} + b^{111}t + a^{24}x + b^{112}(xt),\\
\tau &= a^{20} + a^{24}(2t) + b^{112}(t^2),\\
\eta &= \left(-b^{112}\left(\frac{x^2}{4} + \frac{t}{2}\right) - \frac{b^{111}}{2}x + b^{110}\right)u + g(x,t),\\
&\quad \text{where} \quad g_{xx} - g_t = 0.
\end{aligned}
\tag{9.68}
$$

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Use the software to work out the groups of the heat equation in 3D

Heat_equation_in_3D copy.nb

In this example we use the package IntroToSymmetry.m to work out the point group of the heat equation in three dimensions

Ut-k*(Uxx+Uyy+Uzz)=0.

Clear all symbols in the current context.

```
In[1]:= ClearAll[Evaluate[Context[] <> "*"]]
```

First read in the package which is located in User Home Folder/Library/*Mathematica*/Applications/SymmetryAnalysis.

```
In[2]:= Needs["SymmetryAnalysis`IntroToSymmetry`"]
```

Enter the input equation as a string. Don't include the == 0 at the end.

```
In[3]:= inputequation =
  "D[u[x,y,z,t],t]-k*D[u[x,y,z,t],x,x]-k*D[u[x,y,z,t],y,y]-k*D[u[x,y,z,t],z,z]";
```

The function u[x,y,z,t] is a solution of the equation and this constraint must be applied in the form of a rule to the invariance condition. Be careful to check signs.

```
In[4]:= rulesarray =
   {"D[u[x,y,z,t],t]→k*D[u[x,y,z,t],x,x]+k*D[u[x,y,z,t],y,y]+k*D[u[x,y,z,t],z,z]"};
```

Enter the list of independent variables.

```
In[5]:= independentvariables = {"x", "y", "z", "t"};
```

Enter the list of dependent variables.

```
In[6]:= dependentvariables = {"u"};
```

Enter the list of function and constant names that must be preserved when the equation is converted to generic variables.

```
In[7]:= frozennames = {"k"};
```

Heat_equation_in_3D copy.nb

Enter the maximum derivative order of the input equation(s).

In[8]:=
```
p = 2;
```

The maximum derivative order that the infinitesimals are assumed to depend on is specified by the input parameter r. This parameter is only nonzero when the user is looking for Lie contact groups or Lie-Backlund groups. For the usual case where one is searching for point groups set r=0.

In[9]:=
```
r = 0;
```

When searching for Lie-Backlund groups (r=1 or greater) one can, without loss of generality, leave the independent variables untransformed. The corresponding infinitesimals (the xse's) are set to zero by setting xseon=0. If one is searching for point groups then set xseon=1. The choice xseon=1 is also an option when looking for Lie-Backlund groups and this can be useful when looking for contact symmetries.

In[10]:=
```
xseon = 1;
```

When searching for Lie-Backlund groups it is necessary to differentiate the input equation with respect to each of the independent variables producing derivatives of order p+r. These higher order differential consequences are appended to the set of rules applied to the invariance condition. This process is carried out automatically when internalrules=1. For point groups the equation or equation system is the only rule or set of rules needed and one sets internalrules=0.

In[11]:=
```
internalrules = 0;
```

Now work out the determining equations of the Lie point group that leaves the equation invariant. The output is available as a table of strings called zdeterminingequations.

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Heat_equation_in_3D copy.nb

In[12]:=
```
Timing[FindDeterminingEquations[
  independentvariables, dependentvariables, frozennames, p, r, xseon,
  inputequation, rulesarray, internalrules]]
```

The function FindDetermining Equations has begun, the memory in use = 92937592, the time used = 1.289762`

The function FindDeterminingEquations is nearly complete. The invariance condition has been created with all rules applied. The final step in the generation of the determining equations is to sum together terms in the table of invariance condition terms (called infinitesimaltable) that are multiplied by the same combination of products of free y derivatives. The result is the table infinitesimaltablesums corresponding to matching y-derivative expressions. If the invariance condition is long as it often is this process could take a long time since it requires sorting through the table infinitesimaltable once for each possible combination of y derivative products. This is the rate limiting step in the function FindDeterminingEquations.Virtually all other steps are quite fast including the generation of the extended derivatives of the infinitesimals.

The determining equations have been expressed in terms of z-variables, the length of zdeterminingequations = 33, the byte count of zdeterminingequations = 5464, the memory in use = 94083200, the time used = 1.4196019999999998`

FindDeterminingEquations is done. The memory in use = 94084952, the time used = 1.420035`

FindDeterminingEquations has finished executing. You can look at the output in the table zdeterminingequations. Each entry in this table is a determining equation in string format expressed in terms of z-variables. Rules for converting between z-variables and conventional variables are contained in the table ztableofrules. To view the determining equations in terms of conventional variables use the command ToExpression[zdeterminingequations]/.ztableofrules. There are two other items the user may wish to look at; the equation converted to generic (x1,x2,...,y1,y2,...) variables is designated equationgenericvariables and the various derivatives of the equation that appear in the invariance condition can be viewed in the table invarconditiontable. Rules for converting between z-variables and generic variables are contained in the table ztableofrulesxy.

Out[12]= {0.145509, Null}

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Heat_equation_in_3D copy.nb

In[13]:=
```
equationgenericvariables
```

Out[13]=
```
D[y1[x1,x2,x3,x4],x4]-k*D[y1[x1,x2,x3,x4],x1,x1]-k*D[y1[x1,x2,x3,x4],x2,x2]-k*D[y1[x1,x2,x3,x4],x3,x3]
```

In[14]:=
```
invarconditiontable
```

Out[14]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 1, -k, 0, -k, 0, 0, -k, 0, 0, 0, 0}
```

The program expresses the determining equations in terms of zvariables. Here is the correspondence between z-variables and conventional variables.

In[15]:=
```
ztableofrules
```

Out[15]=
```
{z1 → x, z2 → y, z3 → z, z4 → t, z5 → u[x, y, z, t]}
```

Here are the determining equations expressed in terms of z-variables. The equations in the table can be distinguished by the == 0 at the end of each item.

In[16]:=
```
zdeterminingequations;
```

How many determining equations are there?

In[17]:=
```
Length[zdeterminingequations]
```

Out[17]= 33

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Heat_equation_in_3D copy.nb

Now solve the determining equations in terms of multivariable polynomials of a selected order. The Mathematica function Solve uses Gaussian elimination to solve a large number of linear equations for the polynomial coefficients. The time roughly follows

time/timeref=((number of equations)/(number of equationsref))^n

where the exponent is between 2.4 and 2.7. The Mathematica function Timing outputs the time required for the SolveDeterminingEquations function to execute.

```
In[18]:= Timing[SolveDeterminingEquations[
  independentvariables, dependentvariables, r, xseon, zdeterminingequations, order = 3]]

The variable powertablelength is the number of terms required for each multivariate polynomial used
  for the infinitesimals. This number is determined by the choice of polynomial order and the number of zvariables.
  The time needed to solve the determining equations increases as powertable increases. powertablelength = 56

The polynomial expansions have been substituted into the determining equations. It is now time to collect the coefficients of various powers of
  zvariables into a table called table of coefficientsall. This step uses the function CoefficientList and is a fairly time consuming procedure.

The memory in use = 95740104, The time = 1.485287`

The number of unknown polynomial coefficients = 280

The number of equations for the polynomial coefficients = 412

Now it we are ready to use the function Solve to find the nonzero
  polynomial coefficients corresponding to the symmetries of the input equation(s). This can take a while.

The memory in use = 96078808, The time = 1.498034`

Solve has finished.

The function SolveDeterminingEquations is finished executing.

The memory in use = 96682000, The time = 1.7201620000000002`

You can look at the output in the tables xsefunctions and etafunctions. Each entry in these tables is an infinitesimal function in string format
  expressed in terms of z-variables and the group parameters. The output can also be viewed with the group parameters stripped away by looking at
  the table infinitesimalgroups. In either case you may wish to convert the z-variables to conventional variables using the table ztableofrules.

Keep in mind that this function only finds solutions of the determining equations that are of polynomial form. The
  determining equations may admit solutions that involve transcendental functions and/or integrals. Note that arbitrary functions
  may appear in the infinitesimals and that these can be detected by running the package function SolveDeterminingEquations
  for several polynomial orders. If terms of ever increasing order appear, then an arbitrary function is indicated.

Out[18]= {0.281522, Null}
```

Heat_equation_in_3D copy.nb

Here are the infinitesimals for the independent variables expressed in terms of z-variables.

```
In[19]:= xsefunctions

Out[19]= {xse1[z1_, z2_, z3_, z4_, z5_]=a10 + (a420*z1)/2 + a14*z2 + a110*z3 - 2*b136*k*z4 + a430*z1*z4,
 xse2[z1_, z2_, z3_, z4_, z5_]=a20 - a14*z1 + (a420*z2)/2 + a210*z3 - 2*b138*k*z4 + a430*z2*z4,
 xse3[z1_, z2_, z3_, z4_, z5_]=a30 - a110*z1 - a210*z2 + (a420*z3)/2 - 2*b141*k*z4 + a430*z3*z4,
 xse4[z1_, z2_, z3_, z4_, z5_]=a40 + a420*z4 + a430*z4^2}
```

and the infinitesimals for the dependent variables.

```
In[20]:= etafunctions

Out[20]= {eta1[z1_, z2_, z3_, z4_, z5_]=b10 + b11*z1 + b12*z1^2 + b13*z1^3 + b14*z2 + b15*z1*z2 + b16*z1^2*z2 + b17*z2^2 +
   b18*z1*z2^2 + b19*z2^3 + b110*z3 + b111*z1*z3 + b112*z1^2*z3 + b113*z2*z3 + b114*z1*z2*z3 + b115*z2^2*z3 +
   b116*z3^2 + b117*z1*z3^2 + b118*z2*z3^2 + b119*z3^3 + (2*b116*k + 2*b12*k + 2*b17*k)*z4 + (2*b117*k + 6*b13*k +
   2*b18*k)*z1*z4 + (2*b118*k + 2*b16*k + 6*b19*k)*z2*z4 + (2*b112*k + 2*b115*k + 6*b119*k)*z3*z4 + b135*z5 + b136*z1*z5
   - (a430*z1^2*z5)/(4*k) + b138*z2*z5 - (a430*z2^2*z5)/(4*k) + b141*z3*z5 - (a430*z3^2*z5)/(4*k) - (3*a430*z4*z5)/2}
```

Express the xse functions in terms of x,y variables.

```
In[21]:= ToExpression[xsefunctions] /. ztableofrules
```

Out[21]=

$$\left\{a10 - 2\,b136\,k\,t + \frac{a420\,x}{2} + a430\,t\,x + a14\,y + a110\,z,\ a20 - 2\,b138\,k\,t - a14\,x + \frac{a420\,y}{2} + a430\,t\,y + a210\,z,\ a30 - 2\,b141\,k\,t - a110\,x - a210\,y + \frac{a420\,z}{2} + a430\,t\,z,\ a40 + a420\,t + a430\,t^2\right\}$$

Express the eta function in terms of these variables.

```
In[22]:= ToExpression[etafunctions] /. ztableofrules
```

Out[22]=

$$\Big\{b10 + (2\,b116\,k + 2\,b12\,k + 2\,b17\,k)\,t + b11\,x + (2\,b117\,k + 6\,b13\,k + 2\,b18\,k)\,t\,x + b12\,x^2 + b13\,x^3 + b14\,y + (2\,b118\,k + 2\,b16\,k + 6\,b19\,k)\,t\,y + b15\,x\,y + b16\,x^2\,y + b17\,y^2 + b18\,x\,y^2 + b19\,y^3 + b110\,z + (2\,b112\,k + 2\,b115\,k + 6\,b119\,k)\,t\,z + b111\,x\,z + b112\,x^2\,z + b113\,y\,z + b114\,x\,y\,z + b115\,y^2\,z + b116\,z^2 + b117\,x\,z^2 + b118\,y\,z^2 + b119\,z^3 + b135\,u[x, y, z, t] - \frac{3}{2}\,a430\,t\,u[x, y, z, t] + b136\,x\,u[x, y, z, t] - \frac{a430\,x^2\,u[x, y, z, t]}{4\,k} + b138\,y\,u[x, y, z, t] - \frac{a430\,y^2\,u[x, y, z, t]}{4\,k} + b141\,z\,u[x, y, z, t] - \frac{a430\,z^2\,u[x, y, z, t]}{4\,k}\Big\}$$

List the groups.

```
In[23]:= infinitesimalgroupsxy1 = infinitesimalgroups /. {z1 → x, z2 → y, z3 → z, z4 → t, z5 → u}
```

Out[23]=

$$\Big\{\{\{1, 0, 0, 0\}, \{0\}\}, \{\{y, -x, 0, 0\}, \{0\}\}, \{\{z, 0, -x, 0\}, \{0\}\}, \{\{0, 1, 0, 0\}, \{0\}\}, \{\{0, z, -y, 0\}, \{0\}\}, \{\{0, 0, 1, 0\}, \{0\}\}, \{\{0, 0, 0, 1\}, \{0\}\}, \left\{\left\{\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, t\right\}, \{0\}\right\}, \left\{\left\{t\,x, t\,y, t\,z, t^2\right\}, \left\{-\frac{3\,t\,u}{2} - \frac{u\,x^2}{4\,k} - \frac{u\,y^2}{4\,k} - \frac{u\,z^2}{4\,k}\right\}\right\}, \{\{0, 0, 0, 0\}, \{1\}\}, \{\{0, 0, 0, 0\}, \{x\}\}, \{\{0, 0, 0, 0\}, \{2\,k\,t + x^2\}\}, \{\{0, 0, 0, 0\}, \{6\,k\,t\,x + x^3\}\}, \{\{0, 0, 0, 0\}, \{y\}\}, \{\{0, 0, 0, 0\}, \{x\,y\}\}, \{\{0, 0, 0, 0\}, \{2\,k\,t\,y + x^2\,y\}\}, \{\{0, 0, 0, 0\}, \{2\,k\,t + y^2\}\}, \{\{0, 0, 0, 0\}, \{2\,k\,t\,x + x\,y^2\}\}, \{\{0, 0, 0, 0\}, \{6\,k\,t\,y + y^3\}\}, \{\{0, 0, 0, 0\}, \{z\}\}, \{\{0, 0, 0, 0\}, \{x\,z\}\}, \{\{0, 0, 0, 0\}, \{2\,k\,t\,z + x^2\,z\}\}, \{\{0, 0, 0, 0\}, \{y\,z\}\}, \{\{0, 0, 0, 0\}, \{x\,y\,z\}\}, \{\{0, 0, 0, 0\}, \{2\,k\,t\,z + y^2\,z\}\}, \{\{0, 0, 0, 0\}, \{2\,k\,t + z^2\}\}, \{\{0, 0, 0, 0\}, \{2\,k\,t\,x + x\,z^2\}\}, \{\{0, 0, 0, 0\}, \{2\,k\,t\,y + y\,z^2\}\}, \{\{0, 0, 0, 0\}, \{6\,k\,t\,z + z^3\}\}, \{\{0, 0, 0, 0\}, \{u\}\}, \{\{-2\,k\,t, 0, 0, 0\}, \{u\,x\}\}, \{\{0, -2\,k\,t, 0, 0\}, \{u\,y\}\}, \{\{0, 0, -2\,k\,t, 0\}, \{u\,z\}\}\Big\}$$

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```
In[24]:= Column[infinitesimalgroupsxy1]

Out[24]=
{{1, 0, 0, 0}, {0}}
{{y, -x, 0, 0}, {0}}
{{z, 0, -x, 0}, {0}}
{{0, 1, 0, 0}, {0}}
{{0, z, -y, 0}, {0}}
{{0, 0, 1, 0}, {0}}
{{0, 0, 0, 1}, {0}}
{{x/2, y/2, z/2, t}, {0}}
{{t x, t y, t z, t^2}, {-3 t u/2 - u x^2/(4 k) - u y^2/(4 k) - u z^2/(4 k)}}
{{0, 0, 0, 0}, {1}}
{{0, 0, 0, 0}, {x}}
{{0, 0, 0, 0}, {2 k t + x^2}}
{{0, 0, 0, 0}, {6 k t x + x^3}}
{{0, 0, 0, 0}, {y}}
{{0, 0, 0, 0}, {x y}}
{{0, 0, 0, 0}, {2 k t y + x^2 y}}
{{0, 0, 0, 0}, {2 k t + y^2}}
{{0, 0, 0, 0}, {2 k t x + x y^2}}
{{0, 0, 0, 0}, {6 k t y + y^3}}
{{0, 0, 0, 0}, {z}}
{{0, 0, 0, 0}, {x z}}
{{0, 0, 0, 0}, {2 k t z + x^2 z}}
{{0, 0, 0, 0}, {y z}}
{{0, 0, 0, 0}, {x y z}}
{{0, 0, 0, 0}, {2 k t z + y^2 z}}
{{0, 0, 0, 0}, {2 k t + z^2}}
{{0, 0, 0, 0}, {2 k t x + x z^2}}
{{0, 0, 0, 0}, {2 k t y + y z^2}}
{{0, 0, 0, 0}, {6 k t z + z^3}}
{{0, 0, 0, 0}, {u}}
{{-2 k t, 0, 0, 0}, {u x}}
{{0, -2 k t, 0, 0}, {u y}}
{{0, 0, -2 k t, 0}, {u z}}
```

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Drop the groups corresponding to superposition of an arbitrary solution of the equation.

```
In[25]:= infinitesimalgroupsxy =
           Drop[infinitesimalgroupsxy1, {10, 29}];

In[26]:= Column[infinitesimalgroupsxy]
```

Out[26]=

$\{\{1, 0, 0, 0\}, \{0\}\}$
$\{\{y, -x, 0, 0\}, \{0\}\}$
$\{\{z, 0, -x, 0\}, \{0\}\}$
$\{\{0, 1, 0, 0\}, \{0\}\}$
$\{\{0, z, -y, 0\}, \{0\}\}$
$\{\{0, 0, 1, 0\}, \{0\}\}$
$\{\{0, 0, 0, 1\}, \{0\}\}$
$\{\{\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, t\}, \{0\}\}$
$\{\{t x, t y, t z, t^2\}, \{-\frac{3 t u}{2} - \frac{u x^2}{4 k} - \frac{u y^2}{4 k} - \frac{u z^2}{4 k}\}\}$
$\{\{0, 0, 0, 0\}, \{u\}\}$
$\{\{-2 k t, 0, 0, 0\}, \{u x\}\}$
$\{\{0, -2 k t, 0, 0\}, \{u y\}\}$
$\{\{0, 0, -2 k t, 0\}, \{u z\}\}$

Check the solution in the determining equations.

```
In[27]:= Simplify[ToExpression[zdeterminingequations]]

Out[27]= {True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True,
  True, True, True, True, True, True, True, True, True, True, True, True, True, True, True}
```

Impulsive source solutions of the heat equation

$$u_t - \kappa u_{xx} = 0$$

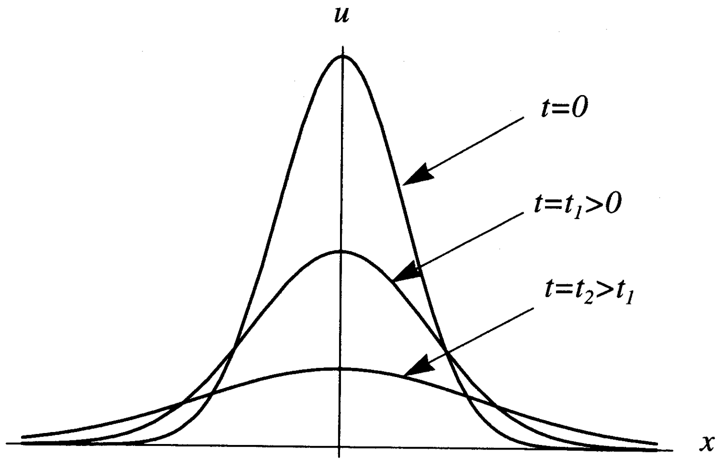

Fig. 9.3. Diffusion of heat from an impulsive source.

Boundary conditions

$$u[\pm\infty, t + t_0] = 0, \qquad t + t_0 \geq 0.$$

$$A = \int_0^\infty x^\alpha u[x, t_0]\, dx.$$

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Is the integral conserved?

$$\frac{d}{dt}\left(\int_0^\infty x^\alpha u\,dx\right) = \kappa \int_0^\infty x^\alpha u_{xx}\,dx$$

$$= \kappa(x^\alpha u_x - \alpha x^{\alpha-1} u)|_0^\infty + \alpha(\alpha-1)\left(\kappa \int_0^\infty x^{\alpha-1} u\,dx\right).$$

This problem is invariant under the three parameter group of dilations in the dependent and independent variables and translation in time.

$$\tilde{x} = e^{a} x, \qquad \tilde{t} = e^{2a} t + (e^{2a} - 1) t_0, \qquad \tilde{u} = e^{-(1+\alpha)a} u.$$

$$\tilde{x} = \infty \quad \Rightarrow \quad e^{a} x = \infty \quad \Rightarrow \quad x = \infty.$$

$$\tilde{u} = 0 \quad \Rightarrow \quad e^{-(1+\alpha)a} u = 0 \quad \Rightarrow \quad u = 0.$$

$$\tilde{t} + t_0 = 0 \quad \Rightarrow \quad e^{2a}(t + t_0) = 0 \quad \Rightarrow \quad t + t_0 = 0.$$

$$A = \int_0^\infty \tilde{x}^\alpha \tilde{u}\, d\tilde{x} = \int_0^\infty e^{a\alpha} x^\alpha e^{-a(1+\alpha)} u e^{a}\, dx = \int_0^\infty x^\alpha u\, dx.$$

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Similarity variables are the invariants of the infinitesimal transformation.

$$\tilde{x} = x + sx, \qquad \tilde{t} = t + s(2t + 2t_0), \qquad \tilde{u} = u - s(1+\alpha)u$$

$$\frac{dx}{x} = \frac{dt}{2t + 2t_0} = \frac{du}{-(1+\alpha)u}.$$

$$\zeta = \frac{x}{(2\kappa(t + t_0))^{1/2}}, \qquad U = \frac{u}{A}(2\kappa(t + t_0))^{(1+\alpha)/2}.$$

$$u = A(2\kappa(t + t_0))^{-(1+\alpha)/2}U(\zeta),$$

$$\int_0^\infty \zeta^\alpha U[\zeta]\, d\zeta = 1.$$

Now substitute the similarity form of the solution into the heat equation.
The result is a second order ODE of Sturm-Liouville type.

$$U_{\zeta\zeta} + \zeta U_\zeta + (1+\alpha)U = 0, \qquad U(\pm\infty) = 0.$$

$$U[\zeta] = \left(\sqrt{\frac{2}{\pi}\frac{1}{\alpha!}}\right)e^{-\zeta^2/2}H_\alpha[\zeta],$$

$$H_\alpha[\zeta] = (-1)^\alpha e^{\zeta^2/2}\frac{d^\alpha}{d\zeta^\alpha}\left(e^{-\zeta^2/2}\right).$$

$$H_0 = 1, \qquad H_1 = \zeta,$$
$$H_2 = \zeta^2 - 1, \qquad H_3 = \zeta^3 - 3\zeta,$$
$$H_4 = \zeta^4 - 6\zeta^2 + 3, \qquad H_5 = \zeta^5 - 10\zeta^3 + 15\zeta.$$

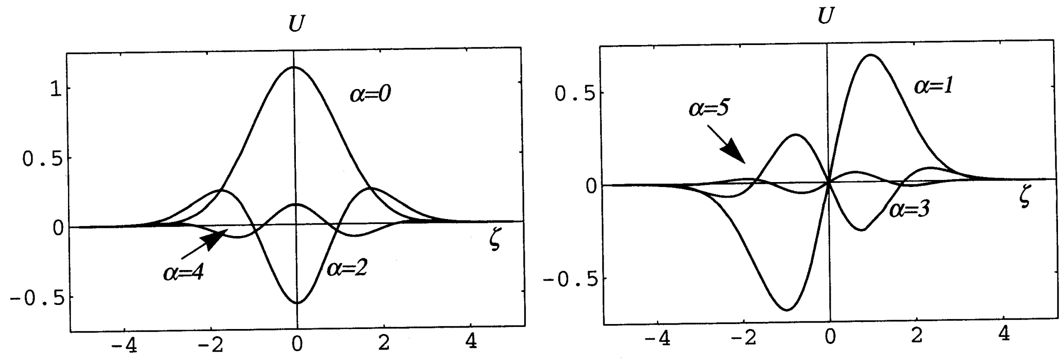

Fig. 9.4. Even and odd similarity solutions of the heat equation.

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What about fractional values of α? In this case the even and odd solutions of (9.94) can be expressed in terms of confluent hypergeometric functions,

$$
\begin{aligned}
U_{\text{even}} &= Ce^{-\zeta^2/4} M\left[-\frac{\alpha}{2}, \frac{1}{2}, \frac{\zeta^2}{2}\right], \qquad \alpha = 0, 2, 4, \ldots, \\
U_{\text{odd}} &= C\zeta e^{-\zeta^2/4} M\left[-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, \frac{\zeta^2}{2}\right], \qquad \alpha = 1, 3, 5, \ldots,
\end{aligned}
\tag{9.98}
$$

where

$$
\begin{aligned}
M[a, b, s] &= 1 + \sum_{k=1}^{\infty} \frac{a_k}{b_k} \frac{s^k}{k!}, \\
a_k &= a(a+1)\cdots(a+k-1), \\
b_k &= b(b+1)\cdots(b+k-1).
\end{aligned}
\tag{9.99}
$$

However, these solutions cannot individually satisfy the integral constraint (9.93). For fractional α and large values of the argument, the asymptotic behavior of the solutions (9.98) is

$$
\lim_{\zeta \to \infty} U \approx (\ )\zeta^{\alpha} e^{-\zeta^2/2} + (\ )\zeta^{-\alpha-1} \tag{9.100}
$$

In this case the integral does not converge:

$$
\int_0^{\infty} \zeta^{\alpha} U[\zeta]\, d\zeta \approx \ln \zeta |_0^{\infty} \tag{9.101}
$$

Nevertheless it is possible to combine the solutions (9.98) so as to cancel the singularity. This leads to the construction of the parabolic cylinder functions

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$U = D_\alpha[\pm\zeta]$, where

$$D_\alpha[\pm\zeta] = \sqrt{\pi}2^{\alpha/2}e^{-\zeta^2/4}\left[\frac{M[-\frac{\alpha}{2},\frac{1}{2},\frac{\zeta^2}{2}]}{\Gamma[\frac{1}{2}-\frac{\alpha}{2}]} \mp \sqrt{2}\zeta\frac{M[-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},\frac{\zeta^2}{2}]}{\Gamma[-\frac{\alpha}{2}]}\right]. \tag{9.102}$$

See Abramowitz and Stegun [9.2]. The gamma function is

$$\Gamma[a] = \int_0^\infty s^{a-1}e^{-s}\,ds. \tag{9.103}$$

The branch cut for defining the parabolic cylinder functions in the complex plane is taken to be along the negative complex axis, $|\mathrm{Arg}[\theta]| < 3\pi/4$. For small values of ζ,

$$\lim_{\zeta\to 0} D_\alpha[\zeta] = \sqrt{\pi}2^{\alpha/2}e^{-\zeta^2/4}\left(\frac{1-\frac{\alpha}{2!}\zeta^2+\frac{\alpha(\alpha-2)}{4!}\zeta^4-\cdots}{\Gamma[\frac{1}{2}-\frac{\alpha}{2}]} - \sqrt{2}\frac{\zeta-\frac{\alpha-1}{3!}\zeta^3+\frac{(\alpha-1)(\alpha-3)}{5!}\zeta^5-\cdots}{\Gamma[-\frac{\alpha}{2}]}\right), \tag{9.104}$$

and for large ζ

$$\lim_{\zeta\to\infty} D_\alpha[\zeta] = \zeta^\alpha e^{-\zeta^2/4}\left(1-\frac{\alpha(\alpha-1)}{2\zeta^2}+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{8\zeta^4}-\cdots\right). \tag{9.105}$$

These functions yield solutions for fractional values of α with integrals of the form of (9.93) that converge. However, as can be seen from the combination of even and odd terms in (9.104), they are neither symmetric nor antisymmetric functions.

The solutions (9.92) for various integer α form a complete set of orthogonal functions. Therefore any smooth decaying solution that goes to zero sufficiently fast at infinity can be represented as a series expansion

$$u[x,t] = \sum_{\alpha=0}^{\infty} A_\alpha(2k(t+t_0))^{-(1+\alpha)/2}\left(\left(\sqrt{\frac{2}{\pi}\frac{1}{\alpha!}}\right)e^{-\zeta^2/2}H_\alpha[\zeta]\right). \tag{9.106}$$

The coefficients A_α in the series are determined from the various moments of the initial condition using the orthogonality of the expansion functions. It is clear from (9.106) that regardless of the initial condition, the terms that dominate the large-time, final decay of the temperature are the lowest nonzero modes

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$\alpha = 0,\ 1$. This is because these are the modes with the slowest decay. In the next section we will study a remarkable example of nonlinear heat conduction where the fractional-α solutions play a crucial role in a *symmetric* problem and where the final state of decay is not $\alpha = 0$ or $\alpha = 1$.

Before we leave this example, it is worthwhile saying a few words about the initial condition and about the effective origin in time that was included when we incorporated time translation with the group (9.83). The distribution of u at $t = 0$ is

$$u[x, 0] = \sum_{\alpha=0}^{\infty} A_\alpha (2kt_0)^{-(1+\alpha)/2} \left(\sqrt{\frac{2}{\pi}} \frac{1}{\alpha!} \right) e^{-x^2/4kt_0} H_\alpha \left[\frac{x}{\sqrt{2kt_0}} \right]. \qquad (9.107)$$

The parameter t_0 enables one to specify an initial distribution that is smooth and infinitely differentiable. In the limit $t_0 \to 0$ the $\alpha = 0$ term in the distribution (9.107) is a useful form of the Dirac delta function. Higher values of α correspond to the various derivatives of the Dirac delta function, and the integral of the $\alpha = 0$ term is the Heaviside function.

Flow in porous media - A modified problem of an instantaneous heat source.

Darcy's law

$$u = -\frac{k}{\mu}\nabla p, \qquad (9.108)$$

where k is the permeability of the medium, μ is the viscosity, and p is the pressure. The continuity equation is

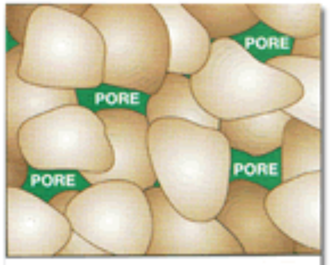
$$\frac{\partial \sigma \rho}{\partial t} + \nabla \cdot \rho \bar{u} = 0, \qquad (9.109)$$

where σ is the porosity of the medium and ρ is the density of the fluid. Now let

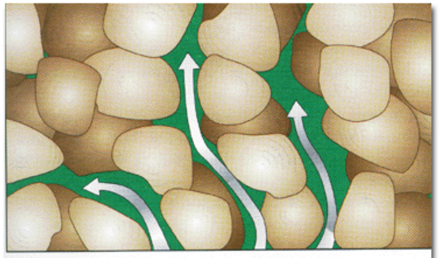
$$\rho = \rho_0\left(1 + \frac{p - p_0}{\lambda}\right). \qquad (9.110)$$

The compressibility of the fluid is generally very small, with values of $\lambda \approx 10^4$ kg/cm^2. When we substitute (9.110) into (9.109) and use (9.108), we find that

Pressure diffuses in a porous medium.



A pore is a small open space


Connected pores give a rock permeability.

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$$\frac{\partial p}{\partial t} - \kappa \nabla^2 p = 0, \tag{9.111}$$

$$\kappa = \frac{k\lambda}{\mu\sigma}. \tag{9.112}$$

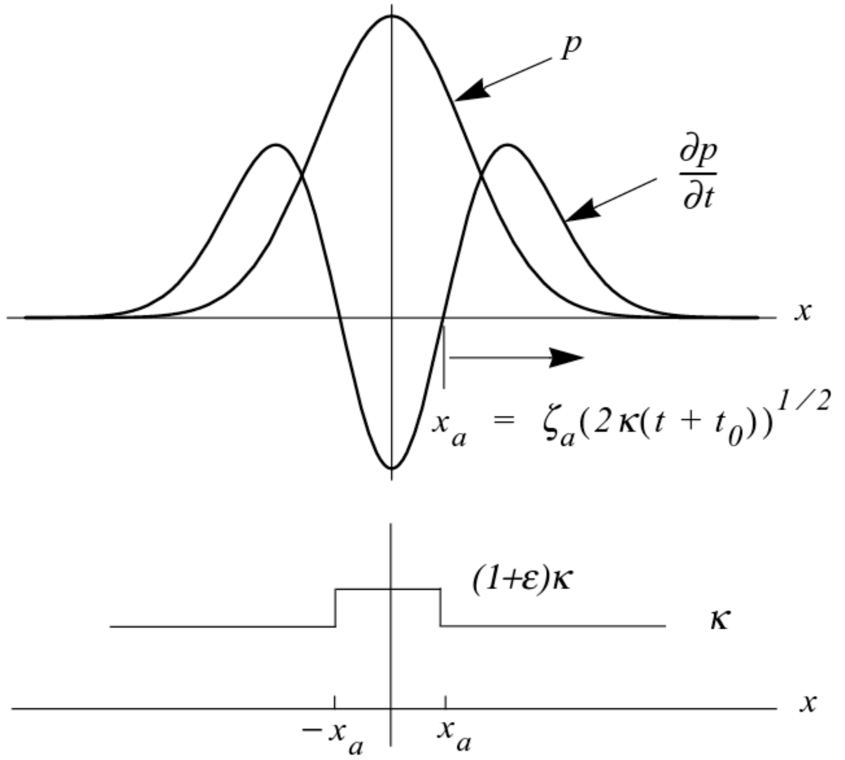

Fig. 9.5. Spatial variation in diffusivity for the modified problem.

$$p_t - (1+\varepsilon)\kappa p_{xx} = 0, \qquad p_t \le 0,$$
$$p_t - \kappa p_{xx} = 0, \qquad p_t \ge 0,$$

$$p[\pm\infty, t+t_0] = 0, \qquad t+t_0 \ge 0.$$

$$p[x_{a-}, t+t_0] = p[x_{a+}, t+t_0],$$
$$p_x[x_{a-}, t+t_0] = p_x[x_{a+}, t+t_0].$$

$$A = \int_0^{\infty} x^{\alpha} p[x, t+t_0]\, dx$$

Based on our knowledge of the conventional problem in Section 9.4, we seek a solution where the integral

$$A = \int_0^{\infty} x^{\alpha} p[x, t + t_0]\, dx \tag{9.116}$$

is preserved. Following Section 9.4, the governing equation can be integrated by parts as follows:

$$\begin{aligned}
\frac{d}{dt}\left(\int_0^{\infty} x^{\alpha} p\, dx\right) &= (1+\varepsilon)\kappa \int_0^{x_a} x^{\alpha} p_{xx}\, dx + \kappa \int_{x_a}^{\infty} x^{\alpha} p_{xx}\, dx \\
&= (1+\varepsilon)\kappa (x^{\alpha} p_x - \alpha x^{\alpha-1} p)|_0^{x_a} \\
&\quad + \alpha(\alpha-1)\left((1+\varepsilon)\kappa \int_0^{x_a} x^{\alpha-1} p\, dx\right) \\
&\quad + \kappa (x^{\alpha} p_x - \alpha x^{\alpha-1} p)|_{x_a}^{\infty} + \alpha(\alpha-1)\left(\kappa \int_{x_a}^{\infty} x^{\alpha-1} p\, dx\right).
\end{aligned} \tag{9.117}$$

For $\alpha = 0,\ 1$, corresponding to the first even and odd solutions respectively, the integral becomes

$$\frac{d}{dt}\left(\int_0^\infty x^\alpha p\,dx\right) = \varepsilon\kappa\left(x_a^\alpha p_x\big|_{x=x_a} - \alpha x_a^{\alpha-1} p\big|_{x=x_a}\right). \qquad (9.118)$$

In this case the integral for these two values of α is clearly not preserved, and a similarity solution of the problem does not exist for $\varepsilon \neq 0$.

In spite of the fact that no solution exists for $\alpha = 0,\ 1$, we will continue to investigate whether a solution can exist for *some* value of α. In fact the problem really boils down to this: given $\varepsilon > 0$, does there exist a value of α that solves the problem defined by (9.113) and (9.114)?

So we push on and assume the existence of a solution that is invariant under the group (9.83), with similarity variables of the same form as in the uniform diffusivity case:

$$\zeta = \frac{x}{(2\kappa(t+t_0))^{1/2}}, \qquad p = A(2\kappa(t+t_0))^{-(1+\alpha)/2} P[\zeta]. \qquad (9.119)$$

Upon substitution of these variables, the governing equation in each domain is

$$\begin{aligned} (1+\varepsilon)P^-_{\zeta\zeta} + \zeta P^-_\zeta + (1+\alpha)P^- &= 0, \qquad 0 \le \zeta \le \zeta_\alpha, \\ P^+_{\zeta\zeta} + \zeta P^+_\zeta + (1+\alpha)P^+ &= 0, \qquad \zeta_a \le \zeta \le \infty, \end{aligned} \qquad (9.120)$$

where

$$\zeta_a = \frac{x_a}{(2\kappa(t+t_0))^{1/2}}. \qquad (9.121)$$

The solution is subject to the far-field condition

$$P^+[\pm\infty] = 0 \qquad (9.122)$$

and the integral invariant

$$\int_0^{x_a} \zeta^\alpha P^-[\zeta]\,d\zeta + \int_{x_a}^\infty \zeta^\alpha P^+[\zeta]\,d\zeta = 1. \qquad (9.123)$$

Equations (9.120), (9.122), and (9.123) constitute a nonlinear eigenvalue problem for the unknowns α and ζ_a.

At the internal boundary $\zeta = \zeta_a$ the following matching conditions apply:

$$\begin{aligned} P^-[\zeta_a] &= P^+[\zeta_a], \\ P^-_\zeta[\zeta_a] &= P^+_\zeta[\zeta_a], \end{aligned} \qquad (9.124)$$

and

$$\zeta_a P^-_\zeta[\zeta_a] + (1+\alpha)P^-[\zeta_a] = 0,$$
$$\zeta_a P^+_\zeta[\zeta_a] + (1+\alpha)P^+[\zeta_a] = 0. \tag{9.125}$$

The last condition comes from the fact that, by assumption, the time derivative of the pressure is zero at the internal boundary and therefore $p_{xx} = 0 \Rightarrow P_{\zeta\zeta} = 0$ at x_a. An even solution, valid for arbitrary α, inside the internal boundary is

$$P^-[\zeta] = C_1 e^{-\zeta^2/4(1+\varepsilon)}\left(D_\alpha\left[\frac{\zeta}{\sqrt{1+\varepsilon}}\right] + D_\alpha\left[-\frac{\zeta}{\sqrt{1+\varepsilon}}\right]\right)$$
$$= C_1\left(\frac{2}{\Gamma[\frac{1}{2}-\frac{\alpha}{2}]}\right)\sqrt{\pi}2^{\alpha/2}e^{-\zeta^2/2(1+\varepsilon)}M\left[-\frac{\alpha}{2},\frac{1}{2},\frac{\zeta^2}{2(1+\varepsilon)}\right],$$
$$0 \le \zeta \le \zeta_a, \tag{9.126}$$

where we have used the parabolic cylinder functions introduced in Section 9.3. A solution beyond the internal boundary that ensures convergence of the integral constraint (9.123) is

$$P^+[\zeta] = C_2 e^{-\zeta^2/4} D_\alpha[\zeta]$$
$$= C_2\sqrt{\pi}2^{\alpha/2}e^{-\zeta^2/2}\left(\frac{M[-\frac{\alpha}{2},\frac{1}{2},\frac{\zeta^2}{2}]}{\Gamma[\frac{1}{2}-\frac{\alpha}{2}]} - \sqrt{2}\zeta\frac{M[-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},\frac{\zeta^2}{2}]}{\Gamma[-\frac{\alpha}{2}]}\right). \tag{9.127}$$

See (9.101). Now apply the first matching condition in (9.124) to eliminate C_2:

$$C_1 e^{-\zeta_a^2/4(1+\varepsilon)}\left(D_\alpha\left[\frac{\zeta_a}{\sqrt{1+\varepsilon}}\right] + D_\alpha\left[-\frac{\zeta_a}{\sqrt{1+\varepsilon}}\right]\right) = C_2 e^{-\zeta_a^2/4} D_\alpha[\zeta_a]. \tag{9.128}$$

The second matching condition in (9.124) is automatically satisfied if (9.125) is satisfied. The first relation in (9.125) becomes

$$s_a\left(\left.\frac{dM[-\frac{\alpha}{2},\frac{1}{2},s]}{ds}\right|_{s=s_a} - M\left[-\frac{\alpha}{2},\frac{1}{2},s_a\right]\right) + \frac{1+\alpha}{2}M\left[-\frac{\alpha}{2},\frac{1}{2},s_a\right] = 0, \tag{9.129}$$

where $s_a = \zeta_a^2/(2(1+\varepsilon))$. Using the identity

$$s\frac{dM[a,b,s]}{ds} + (b-a-s)M[a,b,s] = (b-a)M[a-1,b,s], \tag{9.130}$$

Equation (9.129) becomes

$$M\left[-\frac{\alpha}{2}-1, \frac{1}{2}, \frac{\zeta_a^2}{2(1+\varepsilon)}\right] = 0. \qquad (9.131)$$

Given the diffusivity ratio $1+\varepsilon$, the matching condition (9.131) provides a relation between α and ζ_a. The second condition in (9.125) yields

$$\zeta_a \left.\frac{dD_\alpha}{d\zeta}\right|_{\zeta=\zeta_a} + \left(1+\alpha-\frac{\zeta_\alpha^2}{2}\right) D_\alpha[\zeta_a] = 0. \qquad (9.132)$$

Now use the following identities for parabolic cylinder functions:

$$\begin{aligned} &\frac{dD_\alpha}{d\zeta} + D_{\alpha+1} - \frac{\zeta}{2} D_\alpha = 0, \\ &D_{\alpha+2} - \zeta D_{\alpha+1} + (\alpha+1) D_\alpha = 0. \end{aligned} \qquad (9.133)$$

Using (9.133) to simplify (9.132) gives

$$D_{\alpha+2}[\zeta_a] = 0, \qquad (9.134)$$

or, in terms of hypergeometric functions,

$$\frac{M\left[-\frac{\alpha}{2}-1, \frac{1}{2}, \frac{\zeta_a^2}{2}\right]}{\Gamma\left[-\frac{1}{2}-\frac{\alpha}{2}\right]} - \sqrt{2}\zeta_a \frac{M\left[-\frac{\alpha}{2}-\frac{1}{2}, \frac{3}{2}, \frac{\zeta_a^2}{2}\right]}{\Gamma\left[-1-\frac{\alpha}{2}\right]} = 0. \qquad (9.135)$$

Equations (9.131) and (9.135) provide two equations in two unknowns, allowing one to solve for α and ζ_a given ε. The results are plotted in Figure 9.6.

The remarkable feature of this problem, the feature that makes it an important problem, is that when full numerical solutions are carried out for a general initial condition, it is found that after an early period of non-self-similar development the solution approaches the one similarity solution that exists for a

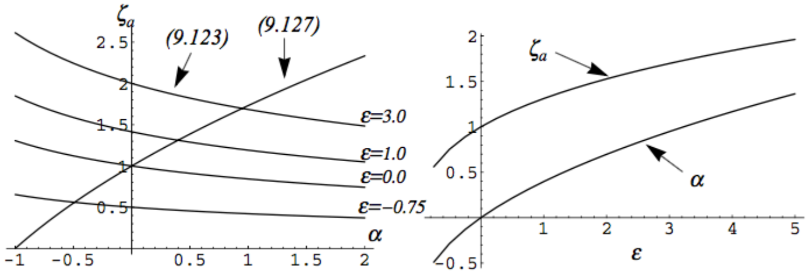

Fig. 9.6. Solution parameters of the modified diffusion problem.

given choice of the material constant ε (see Barenblatt [9.3], Figures 3.3 and 3.4, and the discussion in Sections 3.2.4 and 3.2.5). By one means or another, symmetry finds a way! The group that leaves the asymptotic solution invariant is the same dilation group of the heat equation we encountered in Section 9.4, Equation (9.83),

$$\tilde{x} = e^{a}x, \qquad \tilde{t} = e^{2a}t + (e^{2a} - 1)t_0, \qquad \tilde{p} = e^{-(1+\alpha)a}p, \qquad (9.136)$$

where the constant α is determined from the solution of the nonlinear eigenvalue problem (9.131) and (9.135).

This is an example of a wide class of important problems in filtration, and further examples can be found in the work of Baikov, Gladkov, and Wiltshire [9.4] and Baikov [9.5].

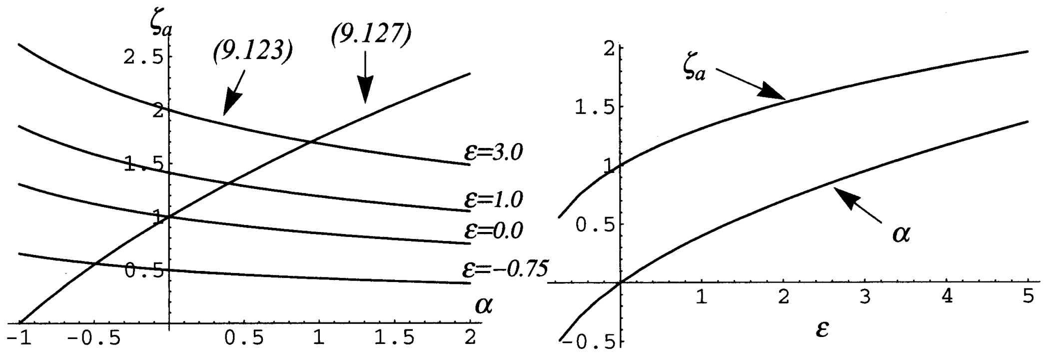

Fig. 9.6. Solution parameters of the modified diffusion problem.

$$\zeta_a = \frac{x_a}{(2\kappa(t + t_0))^{1/2}}.$$

Example computation from Barenblatt
Scaling, self-similarity and intermediate asymptotics

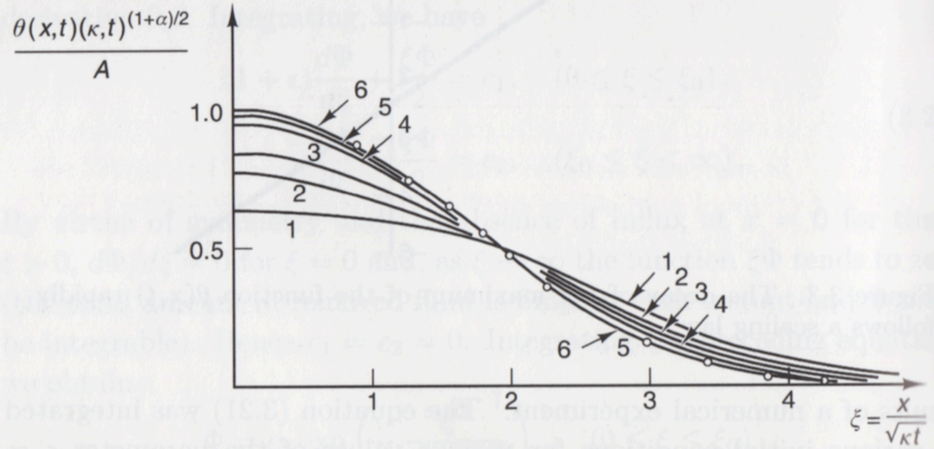

Figure 3.4. Transition to self-similar intermediate asymptotics of the solution to the non-self-similar problem (3.21) with $\epsilon = 1$ and initial data $u(x,0) = 10$ $(0 \le x \le 0.1)$, $u(x,0) \equiv 0$ $(x > 0.1)$. Curves 1–6 correspond respectively to $t = 0.001, 0.002, 0.003, 0.015, 0.040$, and finally 0.225 and all greater values. The open circles are the values of the function determined by solving the nonlinear eigenvalue problem.

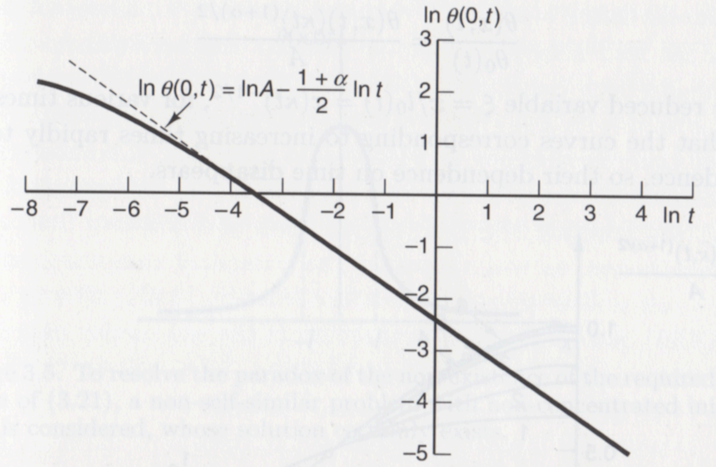

Figure 3.3. The decay of the maximum of the function $\theta(x,t)$ rapidly follows a scaling law.

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9.6 Nonclassical Symmetries

There is a nonlinear alternative to the Lie algorithm that, in several applications, has led to the identification of new point symmetries of differential equations that do not correspond to classical Lie symmetries. The basic idea is to replace the requirement that the differential equation be invariant under a certain symmetry with a somewhat less restrictive requirement that the equation admit a symmetry over a limited set of solutions of the equation.

In the Lie procedure one solves the pth order extended invariance condition

$$X_{\{p\}}\Psi^i = 0, \qquad i = 1, \ldots, m \tag{9.137}$$

for the unknown infinitesimals $(\xi^i[\boldsymbol{x}, \boldsymbol{y}], \eta^i[\boldsymbol{x}, \boldsymbol{y}])$ subject to the constraint imposed by the requirement that $\boldsymbol{y}[\boldsymbol{x}]$ is a solution of the original system of equations. Namely.

$$\Psi^i = 0. \tag{9.138}$$

Equation (9.138) is used to replace derivatives of the y^i that appear in (9.137). When (9.137) is parsed, the result is the set of *linear* determining PDEs for the infinitesimals and the solution of this system is the set of classical point symmetries of the system of equations.

Alternatively, one can search for symmetries that are valid only over some set of invariant solutions of the system of equations. This is accomplished by adding to (9.137) an additional constraint in the form of the invariance condition on a solution. Let the invariant solution be expressed in the form

$$\Omega^i[\boldsymbol{x}, \boldsymbol{y}] = y^i - \Phi^i[\boldsymbol{x}] = 0. \tag{9.139}$$

The invariance condition is

$$X\Omega^i = \eta^i - \xi^j y^i_j = 0, \qquad i = 1, \ldots, m \tag{9.140}$$

Recalling the definition of the characteristic function, $\mu^i = \eta^i - \xi^j y^i_j$, introduced in Section 9.3, the new condition is simply $\mu^i = 0$. The difficulty with this approach is that when (9.140) is is used to replace m of the y^i_j in (9.137) and the equation is parsed, the result is a set of *nonlinear* determining PDEs for the infinitesimals.

Some simplification is possible. Note that the extended operator that we are dealing with is one where $\mu^i = 0$ in (9.86). The implication of this is that if (ξ^j, η^i) satisfy (9.86) then $(f\xi^j, f\eta^i)$ also satisfy (9.86) where $f[\boldsymbol{x}, \boldsymbol{y}]$ is any scalar function. This permits one of the ξ^j to be set to unity without loss of generality. Nevertheless, solving the determining equations in this case is much more difficult and only a few examples are known. Note that the set of solutions that admit the symmetry X is determined once X has been identified.

9.6.1 A Non-classical Point Group of the Heat Equation

To illustrate these ideas let's look again at the heat equation

$$u_t - u_{xx} = 0. \tag{9.141}$$

The invariance condition (9.58) after replacing u_{xx} by u_t is given in equation (9.65) and repeated here for convenience.

$$\begin{aligned}\eta_{\{t\}} - \eta_{\{xx\}} = (\eta_t - \eta_{xx}) &+ 2u_x u_{xt}(\tau_u) + 2u_x u_t(\xi_u + \tau_{xu}) \\ &+ u_x^2(2\xi_{xu} - \eta_{uu}) + u_x^2 u_t(\tau_{uu}) + u_x^3(\xi_{uu}) + 2u_{xt}(\tau_x) \\ &+ u_t(\tau_{xx} + 2\xi_x - \tau_t) + u_x(\xi_{xx} - \xi_t - 2\eta_{xu}) = 0\end{aligned} \tag{9.142}$$

In the usual approach, with the replacement complete, all other derivatives of $u[x, t]$ are not restricted in any way and in order for (9.142) to be satisfied, the coefficients of various products of derivatives of u are set to zero forming the linear determining equations for the unknown infinitesimals $\xi[x, t, u]$, $\tau[x, t, u]$ and $\eta[x, t, u]$.

Instead we now apply the condition

$$\eta[x, t, u] - \xi[x, t, u]u_x - \tau[x, t, u]u_t = 0. \tag{9.143}$$

Without loss of generality let $\tau = 1$ and make the replacement,

$$u_t = \eta[x, t, u] - \xi[x, t, u]u_x \tag{9.144}$$

in (9.142). The result is

$$\eta_{\{t\}} - \eta_{\{xx\}} = (\eta_t - \eta_{xx}) + 2u_x(\eta - \xi u_x)\xi_u + u_x^2(2\xi_{xu} - \eta_{uu}) + u_x^3(\xi_{uu}) + (\eta - \xi u_x)2\xi_x + u_x(\xi_{xx} - \xi_t - 2\eta_{xu}) = 0 \tag{9.145}$$

or, with some rearrangement

$$\begin{aligned} \eta_{\{t\}} - \eta_{\{xx\}} = {} & (\eta_t - \eta_{xx} + 2\eta\xi_x) \\ & + u_x(2\eta\xi_u - 2\xi\xi_x + \xi_{xx} - \xi_t - 2\eta_{xu}) \\ & + u_x^2(2\xi_{xu} - \eta_{uu} - 2\xi\xi_u) + u_x^3(\xi_{uu}) = 0 \end{aligned} \tag{9.146}$$

In order for (9.146) to be satisfied for arbitrary derivatives of u the coefficients in parentheses must be zero and so the determining equations in this case are

$$\begin{aligned} \eta_t - \eta_{xx} + 2\eta\xi_x &= 0, \\ 2\eta\xi_u - 2\xi\xi_x + \xi_{xx} - \xi_t - 2\eta_{xu} &= 0, \\ 2\xi_{xu} - \eta_{uu} - 2\xi\xi_u &= 0, \\ \xi_{uu} &= 0. \end{aligned} \tag{9.147}$$

The nonlinearity of the system (9.147) precludes any sort of elementary approach to a solution including the power series method used in the linear case. The only reasonable way to make progress is to look for simplifying assumptions that lead to interesting solutions. Let

$$\begin{aligned} \eta &= 0 \\ \xi_u &= \xi_t = 0 \end{aligned} \tag{9.148}$$

The determining equations reduce to

$$\xi_{xx} - 2\xi\xi_x = 0. \tag{9.149}$$

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with the solution

$$\xi = -\sqrt{C_2}\tanh[\sqrt{C_2}x + C_1\sqrt{C_2}]. \tag{9.150}$$

Using this procedure we find that the heat equation admits the nonclassical point symmetry

$$X = -\sqrt{C_2}\tanh[\sqrt{C_2}x + C_1\sqrt{C_2}]\frac{\partial}{\partial x} + \frac{\partial}{\partial t}. \tag{9.151}$$

A couple of questions remain. What sort of solution does this symmetry generate? And, what symmetry arises if we assume $\xi = 1$ instead of $\tau = 1$? These are left as exercises for the reader. Nonclassical symmetries are the subject of considerable research and the reader is referred to the treatments in Hydon [9.8] and Baumann [9.9] as well as the papers of Bluman and Cole [9.10] and more recently Clarkson [9.11]. The package **IntroToSymmetry.m** can aid in the search for nonclassical symmetries and several examples are included on the CD.

Note that

$$\xi = -\sqrt{C_2}\coth\left[\sqrt{C_2}x + C_1\sqrt{C_2}\right]$$

Also satisfies

$$\xi_{xx} - 2\xi\xi_x = 0$$

What solutions do these groups correspond to?

Characteristic equations

$$\frac{dx}{-\sqrt{C_2}\,\mathrm{Tanh}\left[\sqrt{C_2}x+C_1\sqrt{C_2}\right]}=\frac{dt}{1}=\frac{du}{0}$$

or

$$\frac{dx}{-\sqrt{C_2}\,\mathrm{Coth}\left[\sqrt{C_2}x+C_1\sqrt{C_2}\right]}=\frac{dt}{1}=\frac{du}{0}$$

Similarity variables (invariants)

$$\theta=e^{C_2 t}Sinh[\sqrt{C_2}x+C_1\sqrt{C_2}]$$

$$U=u$$

or

$$\theta=e^{C_2 t}\,\mathrm{Cosh}[\sqrt{C_2}x+C_1\sqrt{C_2}]$$

$$U=u$$

What solution does this group correspond to?

Let $u = U[\theta]$

Substitute into the heat equation

$$u_t = U'[\theta]\theta_t$$
$$u_x = U'[\theta]\theta_x$$
$$u_{xx} = U''[\theta](\theta_x)^2 + U'[\theta]\theta_{xx}$$
$$u_t - u_{xx} = U'[\theta](\theta_t - \theta_{xx}) - U''[\theta](\theta_x)^2 = 0$$
$$\theta_t - \theta_{xx} = 0 \Rightarrow U''[\theta] = 0$$
$$U[\theta] = A\theta + B$$

$$\theta = e^{C_2 t} Sinh[\sqrt{C_2}x + C_1\sqrt{C_2}]$$
$$U = u$$
$$or$$
$$\theta = e^{C_2 t}\,\mathrm{Cosh}[\sqrt{C_2}x + C_1\sqrt{C_2}]$$
$$U = u$$

Solutions

$$u = Ae^{C_2 t}\,\mathrm{Sinh}[\sqrt{C_2}x + C_1\sqrt{C_2}] + B$$
$$u = Ae^{C_2 t}\,\mathrm{Cosh}[\sqrt{C_2}x + C_1\sqrt{C_2}] + B$$

Let’s look for a solution of the heat equation using separation of variables

$$u_t = u_{xx}$$

Let $u = g[t]f[x]$

$$u_t = g_t f$$

$$u_x = g f_x$$

$$u_{xx} = g f_{xx}$$

Substitute

$$g_t f = g f_{xx}$$

This equality implies

$$\frac{g_t}{g} = \frac{f_{xx}}{f} = \lambda$$

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The solution of the heat equation derived by separation of variables matches the solution generated from non-classical symmetries.

$$\frac{g_t}{g} = \frac{f_{xx}}{f} = \lambda$$

$$g = Ae^{\lambda t}$$

$$f = \mathrm{Sinh}\left[\sqrt{\lambda}x + a\right]$$

$$f = \mathrm{Cosh}\left[\sqrt{\lambda}x + a\right]$$

$$u = Ae^{\lambda t}\,\mathrm{Sinh}\left[\sqrt{\lambda}x + a\right] + B$$

$$u = Ae^{\lambda t}\,\mathrm{Cosh}\left[\sqrt{\lambda}x + a\right] + B$$

Let

$$\lambda = C_2$$

$$a = C_1\sqrt{C_2}$$

$$u = Ae^{C_2 t}\,\mathrm{Sinh}\left[\sqrt{C_2}x + C_1\sqrt{C_2}\right] + B$$

$$u = Ae^{C_2 t}\,\mathrm{Cosh}\left[\sqrt{C_2}x + C_1\sqrt{C_2}\right] + B$$

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9.2 The elliptic equation below has been used by Mahalingham [9.6] to model the effects of streamwise and cross-stream diffusion in a planar, low-speed, nonpremixed jet flame called a Burke–Schumann flame:

$$\phi_{xx} + \phi_{yy} - c\phi_y = 0. \tag{9.152}$$

Using hand calculations only, find the determining equations and solve for the infinitesimal groups of (9.152). Let

$$\phi[x, y] = e^{(c/2)y} f[x, y], \tag{9.153}$$

and convert the equation to the symmetric form

$$f_{xx} + f_{yy} - \frac{c^2}{4} f = 0. \tag{9.154}$$

Use the package **`IntroToSymmetry.m`** to find the infinitesimals. Compare your results with Reference [9.6].