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Introduction to Symmetry Analysis

Chapter 7 - Differential Functions and Notation

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7.1.1 Superscript Notation for Dependent and Independent Variables

$$\boldsymbol{y} = (y^i), \qquad i = 1, \ldots, m, \tag{7.10}$$

$$\boldsymbol{x} = (x^j), \qquad j = 1, \ldots, n. \tag{7.11}$$

$$\begin{aligned} \tilde{x}^j &= F^j[\boldsymbol{x}, \boldsymbol{y}, s], \qquad j = 1, \ldots, n, \\ \tilde{y}^i &= G^i[\boldsymbol{x}, \boldsymbol{y}, s], \qquad i = 1, \ldots, m. \end{aligned} \tag{7.12}$$

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7.1.2 Subscript Notation for Derivatives

$$y_x \equiv \frac{dy}{dx}, \qquad y_{xx} \equiv \frac{d^2y}{dx^2}, \ldots, \qquad y_{(p-1)x} \equiv \frac{d^{p-1}y}{dx^{p-1}}, \qquad y_{px} \equiv \frac{d^py}{dx^p}. \tag{7.13}$$

$$\tilde{y}_{\tilde{x}} \equiv \frac{d\tilde{y}}{d\tilde{x}}, \qquad \tilde{y}_{\tilde{x}\tilde{x}} \equiv \frac{d^2\tilde{y}}{d\tilde{x}^2}, \ldots, \qquad \tilde{y}_{(p-1)\tilde{x}} \equiv \frac{d^{p-1}\tilde{y}}{d\tilde{x}^{p-1}}, \qquad \tilde{y}_{p\tilde{x}} \equiv \frac{d^p\tilde{y}}{d\tilde{x}^p}. \tag{7.14}$$

$$\begin{aligned} \frac{\partial F}{\partial x} &= F_x, \qquad & \frac{\partial F}{\partial y} &= F_y, \\ \frac{\partial G}{\partial x} &= G_x, \qquad & \frac{\partial G}{\partial y} &= G_y \end{aligned} \tag{7.15}$$

$$\begin{aligned} \frac{\partial^2 F}{\partial x^2} &= F_{xx}, \qquad & \frac{\partial^2 F}{\partial x\,\partial y} &= F_{xy}, \qquad & \frac{\partial^2 F}{\partial y^2} &= F_{yy}, \\ \frac{\partial^2 G}{\partial x^2} &= G_{xx}, \qquad & \frac{\partial^2 G}{\partial x\,\partial y} &= G_{xy}, \qquad & \frac{\partial^2 G}{\partial y^2} &= G_{yy}, \end{aligned} \tag{7.16}$$

$$y_1 \equiv \frac{dy}{dx}, \qquad y_2 \equiv \frac{d^2y}{dx^2}, \ldots, \qquad y_{p-1} \equiv \frac{d^{p-1}y}{dx^{p-1}}, \qquad y_p \equiv \frac{d^p y}{dx^p} \qquad (7.17)$$

$$y^i_j \equiv \frac{\partial y^i}{\partial x^j}, \qquad y^i_{j_1 j_2} \equiv \frac{\partial^2 y^i}{\partial x^{j_1}\, \partial x^{j_2}}, \qquad y^i_{j_1 j_2 j_3} \equiv \frac{\partial^3 y^i}{\partial x^{j_1}\, \partial x^{j_2}\, \partial x^{j_3}}, \ldots,$$

$$y^i_{j_1 j_2 j_3 \cdots j_p} \equiv \frac{\partial^p y^i}{\partial x^{j_1}\, \partial x^{j_2}\, \partial x^{j_3} \cdots \partial x^{j_p}} \qquad (7.18)$$

$$\boldsymbol{y}_1 \equiv (y^i_j) = \left(\frac{\partial y^1}{\partial x^1}, \ldots, \frac{\partial y^1}{\partial x^n}, \frac{\partial y^2}{\partial x^1}, \ldots, \frac{\partial y^2}{\partial x^n}, \ldots\ldots, \frac{\partial y^m}{\partial x^1}, \ldots, \frac{\partial y^m}{\partial x^n}\right) \qquad (7.19)$$

$$\boldsymbol{y}_2 \equiv (y^i_{j_1 j_2}) = (y^1_{11}, y^1_{12}, \ldots, y^1_{nn}, \ldots, y^m_{11}, y^m_{12}, \ldots, y^m_{nn}) \qquad (7.20)$$

$$\boldsymbol{y}_3 \equiv (y^i_{j_1 j_2 j_3}) \qquad (7.21)$$

$$\boldsymbol{y}_p \equiv (y^i_{j_1 j_2 j_3 \cdots j_p}). \qquad (7.22)$$

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7.1.3 Curly-Brace Subscript Notation for Functions That Transform Derivatives

$$\begin{aligned} \tilde{x} &= F[x, y, s], \\ \tilde{y} &= G[x, y, s], \\ \tilde{y}_{\tilde{x}} &= G_{\{1\}}[x, y, y_x, s]. \end{aligned} \tag{7.23}$$

$$\begin{aligned} \tilde{x} &= x + s\xi[x, y], \\ \tilde{y} &= y + s\eta[x, y], \\ \tilde{y}_{\tilde{x}} &= y_x + s\eta_{\{1\}}[x, y, y_x]. \end{aligned} \tag{7.24}$$

$$
\begin{aligned}
\tilde{x} &= x + s\xi[x, y], \\
\tilde{y} &= y + s\eta[x, y], \\
\tilde{y}_{\tilde{x}} &= y_x + s\eta_{\{1\}}[x, y, y_x], \\
\tilde{y}_{\tilde{x}\tilde{x}} &= y_{xx} + s\eta_{\{2\}}[x, y, y_x, y_{xx}].
\end{aligned}
\tag{7.25}
$$

$$
\begin{aligned}
\tilde{x} &= x + s\xi[x, y], \\
\tilde{y} &= y + s\eta[x, y], \\
\tilde{y}_{\tilde{x}} &= y_x + s\eta_{\{x\}}[x, y, y_x], \\
\tilde{y}_{\tilde{x}\tilde{x}} &= y_{xx} + s\eta_{\{xx\}}[x, y, y_x, y_{xx}].
\end{aligned}
\tag{7.26}
$$

$$\tilde{y}^i_{j_1 j_2} = G^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, s], \tag{7.27}$$

$$\tilde{y}^i_{j_1 j_2} = y^i_{j_1 j_2} + s\eta^i_{\{j_1 j_2\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2], \tag{7.28}$$

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7.1.4 The Total Differentiation Operator

Definition 7.1. *The total differentiation operator with respect to the jth independent variable is.*

$$\frac{D}{Dx^j} = D_j = \frac{\partial}{\partial x^j} + y^i_j \frac{\partial}{\partial y^i} + y^i_{j_1 j} \frac{\partial}{\partial y^i_{j_1}} + \cdots + y^i_{j_1 j_2 \cdots j_p j} \frac{\partial}{\partial y^i_{j_1 j_2 \cdots j_p}}. \qquad (7.29)$$

$$D = \frac{D}{Dx} = \frac{\partial}{\partial x} + y_x \frac{\partial}{\partial y} + y_{xx} \frac{\partial}{\partial y_x} + \cdots + y_{(p+1)x} \frac{\partial}{\partial y_{px}}. \qquad (7.30)$$

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7.1.5 Definition of a Differential Function

Definition 7.2. *Let z denote the infinite sequence of variables and derivatives,*

$$z = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots), \tag{7.31}$$

and let $\langle z \rangle$ denote any finite subsequence of z. A differential function $\Psi[\langle z \rangle]$ *is a locally analytic function of $\langle z \rangle$ (i.e., expandable in a Taylor series about some point $\langle z_0 \rangle$). The space of differential functions is denoted by $\mathcal{A}$.*

7.1.6 Total Differentiation of Differential Functions

Definition 7.3. *The total differentiation operator acting in the space of differential functions is the infinite operator*

$$D_j = \frac{\partial}{\partial x^j} + y^i_j \frac{\partial}{\partial y^i} + y^i_{j_1 j} \frac{\partial}{\partial y^i_{j_1}} + y^i_{j_1 j_2 j} \frac{\partial}{\partial y^i_{j_1 j_2}} + \cdots. \qquad (7.32)$$

$$\Psi[\langle z \rangle] = \Psi[x, y, y_x, y_{xx}, y_{xxx}, \ldots, y_{px}]. \qquad (7.33)$$

$$D\Psi = \frac{\partial \Psi}{\partial x} + y_x \frac{\partial \Psi}{\partial y} + y_{xx} \frac{\partial \Psi}{\partial y_x} + y_{xxx} \frac{\partial \Psi}{\partial y_{xx}} + \cdots + y_{(p+1)x} \frac{\partial \Psi}{\partial y_{px}}. \qquad (7.34)$$

$$\Psi[\langle z\rangle] = \Psi[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \ldots, \boldsymbol{y}_p]. \tag{7.35}$$

$$D_j \Psi = \frac{\partial \Psi}{\partial x^j} + y^i_j \frac{\partial \Psi}{\partial y^i} + y^i_{j_1 j} \frac{\partial \Psi}{\partial y^i_{j_1}} + \cdots + y^i_{j_1 \cdots j_p j} \frac{\partial \Psi}{\partial y^i_{j_1 \cdots j_p}}. \tag{7.36}$$