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Introduction to Symmetry Analysis

Chapter 5 - Introduction to One-Parameter Lie Groups

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Problem 1.5 (i)

Continuity places no restriction on the parameters

$$\frac{\partial \tilde{u}^k}{\partial \tilde{x}^k} = e^{c-b}\frac{\partial u^k}{\partial x^k} = 0$$

$$\frac{\partial u^k}{\partial x^k} = 0$$

$$\frac{\partial \tilde{u}^i}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}^k}\left(\tilde{u}^i\tilde{u}^k + \frac{\tilde{p}}{\varrho}\delta^i_k\right) - \nu\frac{\partial^2 \tilde{u}^i}{\partial \tilde{x}^k \partial \tilde{x}^k} = 0$$

Four parameter group $\tilde{x}^k = e^a x^k$ $\tilde{t} = e^b x^k$ $\tilde{u}^i = e^c u^i$ $\tilde{p} = e^d p$

$$e^{c-b}\frac{\partial u^i}{\partial t} + e^{-a}\frac{\partial}{\partial x^k}\left(e^{2c}\, u^i u^k + e^d\frac{p}{\varrho}\delta^i_k\right) - \nu\, e^{c-2a}\frac{\partial^2 u^i}{\partial x^k \partial x^k} = 0$$

$$e^{c-b}\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x^k}\left(e^{2c-a}\, u^i u^k + e^{d-a}\frac{p}{\varrho}\delta^i_k\right) - \nu\, e^{c-2a}\frac{\partial^2 u^i}{\partial x^k \partial x^k} = 0$$

The Navier-Stokes equations are invariant under the dilation group if and only if

$$e^{c-b} = e^{2c-a} = e^{d-a} = e^{c-2a}$$

$$\frac{x}{t^k}$$

In other words $c - b = 2c - a = d - a = c - 2a$

$b = 2a$ $c = -a$ $d = -2a$

One parameter group $\tilde{x}^k = e^a x^k$ $\tilde{t} = e^{2a} x^k$ $\tilde{u}^i = e^{-a} u^i$ $\tilde{p} = e^{-2a} p$

t

5.1 The Symmetry of Functions

Definition 5.1. A mathematical relationship between variables is said to possess a symmetry property if one can subject the variables to a group of transformations and the resulting expression reads the same in the new variables as the original expression. The relationship is said to be invariant under the transformation group.

Translation along horizontal lines

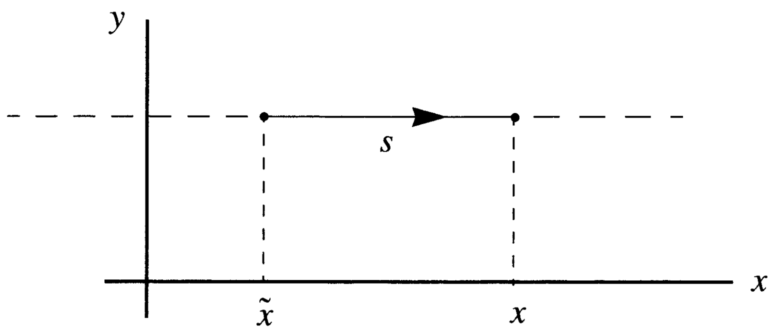

Fig. 5.1. Mapping of points by a translation group.

$$x = \tilde{x} + s,$$
$$y = \tilde{y}$$

$$\left.\begin{aligned} x &= \tilde{x} + s_1 \\ y &= \tilde{y} \\ \\ \tilde{x} &= \tilde{\tilde{x}} + s_2 \\ \tilde{y} &= \tilde{\tilde{y}} \end{aligned}\right\} \rightarrow \begin{cases} x = \tilde{\tilde{x}} + s_3, & s_3 = s_1 + s_2, \\ y = \tilde{\tilde{y}}. \end{cases}$$

A reflection and a translation

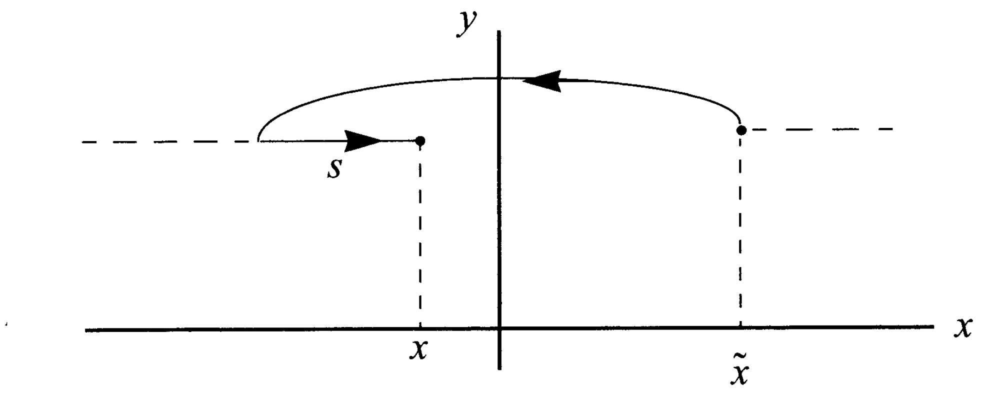

Fig. 5.2. A transformation that is not a group.

$$x = -\tilde{x} + s,$$
$$y = \tilde{y}$$

$$\left.\begin{array}{l} x = -\tilde{x} + s_1 \\ y = \tilde{y} \end{array}\right\} \quad \left.\begin{array}{l} \tilde{x} = -\tilde{\tilde{x}} + s_2 \\ \tilde{y} = \tilde{\tilde{y}} \end{array}\right\} \longrightarrow \left\{\begin{array}{l} x = \tilde{\tilde{x}} + s_1 - s_2, \\ y = \tilde{\tilde{y}}. \end{array}\right.$$

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5.3 One-Parameter Lie Groups

Definition 5.2. *Let the vector $\boldsymbol{x} = (x^1, x^2, \ldots, x^n)$ lie in some continuous open set D on the n-dimensional Euclidean manifold $\mathbb{R}^n$. Define the transformation*

$$T^s : \{x^j = F^j[\tilde{\boldsymbol{x}}, s],\ \ j = 1, \ldots, n\}. \tag{5.5}$$

The functions F^j are infinitely differentiable with respect to the real variables $\boldsymbol{x}$ and are analytic functions of the real continuous parameter s, which lies in an open interval, S.

The transformation T^s is a one-parameter Lie group with respect to the binary operation of composition if and only if:

(i) There is an identity *element $s \rightarrow s_0$ such that $\tilde{\boldsymbol{x}}$ is mapped to itself:*

$$T^{s_0} : \{\tilde{x}^j = F^j[\tilde{\boldsymbol{x}}, s_0],\ \ j = 1, \ldots, n\}. \tag{5.6}$$

Note that the identity element can always be arranged to be zero.

(ii) For every value of s there is an inverse *$s \rightarrow s_{\mathrm{inv}}$ such that $\boldsymbol{x}$ is returned to $\tilde{\boldsymbol{x}}$:*

$$T^{s_{\mathrm{inv}}} : \{\tilde{x}^j = F^j[\boldsymbol{x}, s_{\mathrm{inv}}],\ \ j = 1, \ldots, n\}. \tag{5.7}$$

(iii) The binary operation of composition produces a transformation that is a member of the group $T^{s_1} \cdot T^{s_2} = T^{s_3}$ i.e., the group is closed. Consider two members of the group,

$$T^{s_1} : \{x^j = F^j[\tilde{\boldsymbol{x}}, s_1], \;\; j = 1, \ldots, n\} \tag{5.8}$$

and

$$T^{s_2} : \{\tilde{x}^j = F^j[\tilde{\tilde{\boldsymbol{x}}}, s_2], \;\; j = 1, \ldots, n\}. \tag{5.9}$$

If we compose T^{s_1} and T^{s_2}, the result is

$$T^{s_3} : \{x^j = F^j[\boldsymbol{F}[\tilde{\tilde{\boldsymbol{x}}}, s_2], s_1] = F^j[\tilde{\tilde{\boldsymbol{x}}}, s_3], \;\; j, \ldots, n\}, \tag{5.10}$$

where $s_3 = \phi[s_1, s_2] \in S$. The function ϕ defining the law of composition of T^s is an analytic function of $s_1 \in S$ and $s_2 \in S$ and is commutative ($s_3 = \phi[s_1, s_2] = \phi[s_2, s_1]$); thus Lie groups are Abelian.

(iv) The group is associative*:* $(T^{s_1} \cdot T^{s_2}) \cdot T^{s_3} = T^{s_1} \cdot (T^{s_2} \cdot T^{s_3})$*. Consider three elements of the group,*

$$x^j = F^j[\tilde{\mathbf{x}}, s_1], \qquad \tilde{x}^j = F^j[\tilde{\tilde{\mathbf{x}}}, s_2], \qquad \tilde{\tilde{x}}^j = F^j[\tilde{\tilde{\tilde{\mathbf{x}}}}, s_3]. \quad (5.11)$$

Composing the first two and then the third leads to

$$\begin{array}{ccc} x^j = F^j[\tilde{\mathbf{x}}, s_1] & \tilde{x}^j = F^j[\tilde{\tilde{\mathbf{x}}}, s_2] & \tilde{\tilde{x}}^j = F^j[\tilde{\tilde{\tilde{\mathbf{x}}}}, s_3] \\ & x^j = F^j[\tilde{\tilde{\mathbf{x}}}, \phi[s_1, s_2]] & \\ & & x^j = F^j[\tilde{\tilde{\tilde{\mathbf{x}}}}, \varphi[s_1, s_2, s_3]] \end{array} \quad (5.12)$$

Composing the second and third and then the first leads to the same result

$$\begin{array}{ccc} x^j = F^j[\tilde{\mathbf{x}}, s_1] & \tilde{x}^j = F^j[\tilde{\tilde{\mathbf{x}}}, s_2] & \tilde{\tilde{x}}^j = F^j[\tilde{\tilde{\tilde{\mathbf{x}}}}, s_3] \\ & & \tilde{x}^j = F^j[\tilde{\tilde{\tilde{\mathbf{x}}}}, \phi[s_2, s_3]] \\ & x^j = F^j[\tilde{\tilde{\tilde{\mathbf{x}}}}, \varphi[s_1, s_2, s_3]] & \end{array} \quad (5.13)$$

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5.4 Invariant functions

Example 5.1 Invariance of a parabola under dilation. Transform

$$\Psi[x, y] = y/x^2 \tag{5.14}$$

$$T^{\mathrm{dil}} : \{x = s\tilde{x},\ \ y = s^n\tilde{y},\ \ s > 0\}. \tag{5.15}$$

$$\Psi[x, y] = y/x^2 = e^{s(n-2)}(\tilde{y}/\tilde{x}^2). \tag{5.17}$$

Require $n = 2$ for invariance

$$\Psi[x, y] = y/x^2 = \tilde{y}/\tilde{x}^2 = \Psi[\tilde{x}, \tilde{y}]. \tag{5.18}$$

Definition 5.3. *A function* $\Psi[\boldsymbol{x}]$ *is said to be invariant under the Lie group* $T^s : \{x^j = F^j[\tilde{\boldsymbol{x}}, s], \;\; j = 1, \ldots, n\}$ *if and only if*

$$\Psi[\boldsymbol{x}] = \Psi[\boldsymbol{F}[\tilde{\boldsymbol{x}}, s]] = \Psi[\tilde{\boldsymbol{x}}]. \tag{5.19}$$

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5.5 Infinitesimal form of a Lie group

Expand the group

$$\tilde{x}^j = F^j[\boldsymbol{x}, s], \tag{5.20}$$

in a Taylor series about the identity element $s_0 = 0$.

$$\tilde{x}^j = x^j + s\left[\frac{\partial F^j}{\partial s}\right]_{s=0} + O(s^2) + \cdots, \qquad j = 1, \ldots, n. \tag{5.22}$$

The infinitesimals of the group are

$$\boxed{\xi^j[\boldsymbol{x}] = \left[\frac{\partial}{\partial s} F^j[\boldsymbol{x}, s]\right]_{s=0}, \qquad j = 1, \ldots, n.} \tag{5.23}$$

The vector ξ^j is called the vector field of the group.

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5.6 Lie series

Substitute $\tilde{x}^j = F^j[\boldsymbol{x}, s]$ into $\Psi[\tilde{\boldsymbol{x}}]$

$$\Psi[\tilde{\boldsymbol{x}}] = \Psi[\boldsymbol{F}[\boldsymbol{x}, s]]. \tag{5.24}$$

Now expand (5.24) in a Taylor series about the identity element $s = 0$:

$$\Psi[\tilde{\boldsymbol{x}}] = \Psi[\boldsymbol{x}] + s\left[\frac{\partial \Psi}{\partial s}\right]_{s=0} + \frac{s^2}{2!}\left[\frac{\partial^2 \Psi}{\partial s^2}\right]_{s=0} + \frac{s^3}{3!}\left[\frac{\partial^3 \Psi}{\partial s^3}\right]_{s=0} + \cdots. \tag{5.25}$$

Using the chain rule

$$\left[\frac{\partial \Psi}{\partial s}\right]_{s=0} = \frac{\partial \Psi}{\partial F^j}\left[\frac{\partial F^j}{\partial s}\right]_{s=0} = \xi^j \frac{\partial \Psi}{\partial F^j} = \xi^j \frac{\partial \Psi}{\partial x_j} \tag{5.26}$$

the expansion (5.25) becomes the *Lie series* representation of the function Ψ:

$$\Psi[\tilde{\boldsymbol{x}}] = \Psi[\boldsymbol{x}] + s\left(\xi^j \frac{\partial \Psi}{\partial x^j}\right) + \frac{s^2}{2!}\xi^j \frac{\partial}{\partial x^j}\left(\xi^{j_1} \frac{\partial \Psi}{\partial x^{j_1}}\right) + \frac{s^3}{3!}\xi^j \frac{\partial}{\partial x^j}\left(\xi^{j_1} \frac{\partial}{\partial x^{j_1}}\left(\xi^{j_2} \frac{\partial \Psi}{\partial x^{j_2}}\right)\right) + \cdots. \tag{5.27}$$

where $j_1, j_2 \ldots$ are dummy indices that are summed from 1 to ~~4~~ n.

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Theorem 5.1. *The analytic function* $\Psi[\boldsymbol{x}]$ *is invariant under the Lie group* $T^s : \{\tilde{x}^j = F^j[\boldsymbol{x}, s],\ \ j, \ldots, n\}$ *or, equivalently, the infinitesimal group* $\xi^j[\boldsymbol{x}]$, $j, \ldots, n$, *if and only if* $\Psi[\boldsymbol{x}]$ *satisfies the condition*

$$\boxed{\xi^j[\boldsymbol{x}]\frac{\partial \Psi}{\partial x^j} = 0.} \tag{5.28}$$

The operator

$$\boxed{X \equiv \xi^j[\boldsymbol{x}]\frac{\partial}{\partial x^j}} \tag{5.29}$$

is called the *group operator.*

The Lie series can be written concisely using the group operator. Any analytic function can be expanded as

$$\boxed{\Psi[\tilde{\boldsymbol{x}}] = \Psi[\boldsymbol{x}] + s(X\Psi) + \frac{s^2}{2!}X(X\Psi) + \frac{s^3}{3!}X(X(X\Psi)) + \cdots.} \tag{5.30}$$

The Lie series can formally be written as the exponential map.

$$\Psi[\tilde{\boldsymbol{x}}] = e^{sX}\Psi[\boldsymbol{x}]. \tag{5.31}$$

5.7 Solving the characteristic equation $X\Psi[x]=0$

As discussed in Chapter 2, the linear first-order PDE $\xi^j[\boldsymbol{x}](\partial\Psi/\partial x^j) = 0$ has an associated system of $n-1$ characteristic first-order ODEs of the form

$$\boxed{\frac{dx^1}{\xi^1[\boldsymbol{x}]} = \frac{dx^2}{\xi^2[\boldsymbol{x}]} = \frac{dx^3}{\xi^3[\boldsymbol{x}]} = \cdots = \frac{dx^n}{\xi^n[\boldsymbol{x}]}} \tag{5.32}$$

with integrals

$$\psi^i = \Psi^i[\boldsymbol{x}], \qquad i = 1, \ldots, n-1, \tag{5.33}$$

aligned along solutions of the system of n ODEs

$$\boxed{\frac{dx^j}{ds} = \xi^j[\boldsymbol{x}], \qquad j = 1, \ldots, n,} \tag{5.34}$$

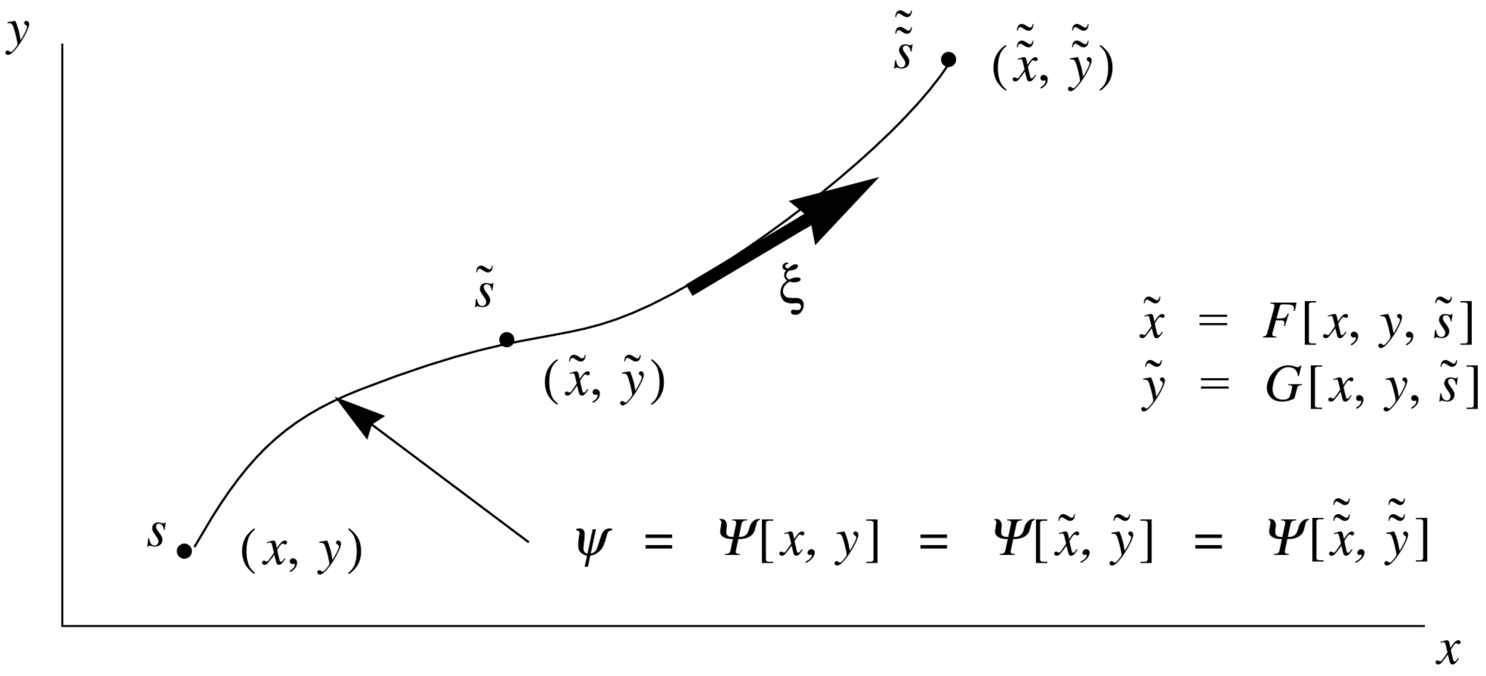

Fig. 5.3. Mapping of points along a single characteristic curve.

5.9 Multi-parameter groups

The projective group in n dimensions

$$T^{\text{projn}}:\left\{\tilde{x}^j=\frac{x^j+a_j+b_{jk}x^k}{1+c_kx^k},\ \ j=1,2,\ldots,n\right\},\qquad (5.53)$$

Let

$$a_j\Rightarrow a_js,\qquad b_{jk}\Rightarrow b_{jk}s,\qquad c_k\Rightarrow c_ks.\qquad (5.54)$$

$$\tilde{x}^j=\frac{x^j+(a_j+b_{jk}x^k)s}{1+(c_kx^k)s},\qquad j=1,2,\ldots,n.\qquad (5.55)$$

Now assume s is infinitsimally small.

Expand and retain only the lowest order term.

$$\tilde{x}^j=x^j+(a_j+b_{jk}x^k-c_kx^kx^j)s,\qquad j=1,2,\ldots,n.\qquad (5.57)$$

The infinitesimals of the n-dimensional projective group are

$$\xi^j(\boldsymbol{x}) = a_j + b_{jk}x^k - c_k x^k x^j, \quad j = 1, 2, \ldots, n. \tag{5.58}$$

The corresponding group operators are

$$X^{a_j} = \frac{\partial}{\partial x^j}, \qquad X^{b_{jk}} = x^k \frac{\partial}{\partial x^j}, \qquad X^{c_k} = x^k x^j \frac{\partial}{\partial x^j}. \tag{5.59}$$

5.9.1 The Commutator

The operators of the group considered above have the interesting and useful property that they form a closed set with respect to commutation. The *commutator* of two group operators X^a and X^b is the operator generated as follows:

$$\{X^a, X^b\} = X^a(X^b) - X^b(X^a). \tag{5.60}$$

Let

$$X^a = \alpha^j[\boldsymbol{x}]\frac{\partial}{\partial x^j}, \qquad X^b = \beta^j[\boldsymbol{x}]\frac{\partial}{\partial x^j}. \tag{5.61}$$

The commutator is

$$\begin{aligned}\{X^a, X^b\} &= \alpha^j[\boldsymbol{x}]\frac{\partial}{\partial x^j}\left(\beta^k[\boldsymbol{x}]\frac{\partial}{\partial x^k}\right) - \beta^k[\boldsymbol{x}]\frac{\partial}{\partial x^k}\left(\alpha^j[\boldsymbol{x}]\frac{\partial}{\partial x^j}\right)\\ &= \left(\alpha^j\frac{\partial \beta^k}{\partial x^j}\right)\frac{\partial}{\partial x^k} - \left(\beta^k\frac{\partial \alpha^j}{\partial x^k}\right)\frac{\partial}{\partial x^j}.\end{aligned} \tag{5.62}$$

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5.10 Lie Algebras

Definition 5.2. *The infinitesimal generators* X^k, $k = 1, \ldots, r$, *of the* r, *parameter Lie group* $T^{a_1,\ldots,a_r} : \{\tilde{x}^j = F^j[x^1, \ldots, x^n; a_1, \ldots, a_r],\ j = 1, \ldots, n\}$ *form an* r*-dimensional Lie algebra* Λ^r *with the following properties. Let* $X^a, X^b, X^c \in \Lambda^r$, *and let* α, β *be real constants. The null algebra is* Λ^0.

(i) *The Lie algebra* Λ^r *is an* r*-dimensional vector space spanned by the basis set of infinitesimal generators* X^k, $k = 1, \ldots, r$. *Thus*

$$\alpha X^a + \beta X^b = Y, \quad \text{where } Y \in \Lambda^r; \qquad X^a + X^b = X^b + X^a. \tag{5.65}$$

(ii) *The commutator is antisymmetric:*

$$\{X^a, X^b\} = -\{X^b, X^a\}. \tag{5.66}$$

(iii) *The commutator of any two infinitesimal generators of an* r*-parameter Lie group is also an infinitesimal generator that belongs to* Λ^r:

$$\{X^a, X^b\} = \beta_k^{ab} X^k \qquad (\text{sum over } k = 1, \ldots, r). \tag{5.67}$$

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The coefficients β_k^{ab} are the structure constants of the Lie algebra Λ^r. Note that

$$\beta_k^{ab} = -\beta_k^{ba}. \tag{5.68}$$

(iv) *The group operators satisfy the associative rule with respect to addition:*

$$X^a + (X^b + X^c) = (X^a + X^b) + X^c. \tag{5.69}$$

(v) *The group operators satisfy the Jacobi identity,*

$$\{\{X^a, X^b\}, X^c\} + \{\{X^c, X^a\}, X^b\} + \{\{X^b, X^c\}, X^a\} = 0. \tag{5.70}$$

(vi) *It follows from the Jacobi identity that the structure constants defined by the commutation relations (5.67) satisfy*

$$\beta_j^{ab}\beta_k^{jc} + \beta_j^{ca}\beta_k^{jb} + \beta_j^{bc}\beta_k^{ja} = 0 \qquad (\text{sum over } j = 1, \ldots, r). \tag{5.71}$$

(vii) *The commutator may be expanded as*

$$\{\alpha X^a + \beta X^b, X^c\} = \alpha\{X^a, X^c\} + \beta\{X^b, X^c\}. \tag{5.72}$$

Projective group in two dimensions

For $n = 2$ the group (5.53) becomes

$$T^{\text{proj2}} : \left\{ \tilde{x} = \frac{x + a_3 x + a_4 y + a_5}{1 + a_1 x + a_2 y},\ \tilde{y} = \frac{y + a_6 x + a_7 y + a_8}{1 + a_1 x + a_2 y} \right\}. \quad (5.73)$$

$$\begin{aligned} \tilde{x} &= x + (-a_1 x^2 - a_2 xy + a_3 x + a_4 y + a_5)s, \\ \tilde{y} &= y + (-a_1 xy - a_2 y^2 + a_6 x + a_7 y + a_8)s \end{aligned} \quad (5.74)$$

Corresponding group operators

$$\begin{gathered} X^1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \qquad X^2 = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}, \qquad X^3 = x \frac{\partial}{\partial x}, \\ X^4 = y \frac{\partial}{\partial x}, \qquad X^5 = \frac{\partial}{\partial x}, \\ X^6 = x \frac{\partial}{\partial y}, \qquad X^7 = y \frac{\partial}{\partial y}, \qquad X^8 = \frac{\partial}{\partial y}. \end{gathered} \quad (5.75)$$

Table 5.1. *Commutator table of the two-dimensional projective group.*

	X^1	X^2	X^3	X^4	X^5	X^6	X^7	X^8
X^1	0	0	$-X^1$	$-X^2$	$-2X^3 - X^7$	0	0	$-X^6$
X^2	0	0	0	0	$-X^4$	$-X^1$	$-X^2$	$-X^3 - 2X^7$
X^3	X^1	0	0	$-X^4$	$-X^5$	X^6	0	0
X^4	X^2	0	X^4	0	0	$X^7 - X^3$	$-X^4$	$-X^5$
X^5	$2X^3 + X^7$	X^4	X^5	0	0	X^8	0	0
X^6	0	X^1	$-X^6$	$X^3 - X^7$	$-X^8$	0	X^6	0
X^7	0	X^2	0	X^4	0	$-X^6$	0	$-X^8$
X^8	X^6	$X^3 + 2X^7$	0	X^5	0	0	X^8	0

5.14 Exercises

5.1 (1) Show by composition that each of the following transformations is a Lie group:

(i) A projective group

$$\tilde{x} = \frac{x}{1 - sy}, \qquad \tilde{y} = \frac{y}{1 - sy}. \tag{5.97}$$

(ii) A hyperbolic group

$$\tilde{x} = x + s, \qquad \tilde{y} = \frac{xy}{x + s}. \tag{5.98}$$

(iii) An arbitrary translation

$$\tilde{x} = x, \qquad \tilde{y} = y + sf[x], \qquad f[x] \text{ arbitrary}. \tag{5.99}$$

(iv) A helical transformation

$$\begin{aligned} \tilde{x} &= x\cos[s] - y\sin[s], \qquad \tilde{y} = x\sin[s] + y\cos[s], \\ \tilde{z} &= z + ms. \end{aligned} \tag{5.100}$$

(2) Determine the infinitesimal transformation for each case, and then reconstruct the global transformation by series summation.

(3) Set up the characteristic equations and determine the integral invariants of the group for each case. Find these invariants by elimination of the group parameter.

5.2 Is the transformation

$$\tilde{x} = x - sy, \qquad \tilde{y} = y + sx \tag{5.101}$$

a Lie group?

5.3 The Lorentz transformation of the position and time of a particle moving at speed u is

$$\tilde{x} = \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}}, \qquad \tilde{t} = \frac{t - \frac{ux}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}}, \tag{5.102}$$

where c is the speed of light. Show that the transformation is a group with respect to the parameter u. Is it also a group with respect to c? Let $u = -\tanh[a]$. Show that the transformation becomes

$$\tilde{x} = x\cosh[a] + t\sinh[a], \qquad \tilde{y} = x\sinh[a] + t\cosh[a]. \tag{5.103}$$

The Lorentz transformation is a kind of "hyperbolic rotation." Determine the infinitesimal transformation, and compare with an ordinary rotation.

5.4 Carefully work out the steps leading from (5.25) to (5.27) for a Lie group in two variables.

5.5 Consider the transformation

$$\tilde{y} = \frac{a + (1+b)y}{1 + c + dy}. \tag{5.104}$$

Work out the infinitesimal form of the group, and characterize the Lie algebra. Identify the group parameters.

5.6 The following autonomous system of ODEs comes up in the context of a problem involving laminar flame propagation:

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= y - \frac{1}{4}x + \frac{1}{4}x^2. \end{aligned} \tag{5.105}$$

Draw the phase portrait of the system (5.108). Use the Lie series expansion to develop a fourth-order accurate method for solving the equations numerically. Compare your scheme to a standard fourth-order Runge–Kutta method. Solve numerically for $y[x]$ subject to the boundary conditions $y[0]=0$, $y[1]=0$. Use the phase portrait to suggest how to carry out the integration.

5.7 Sum the Lie series to determine the finite transformation corresponding to the group operator,

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - \left(\frac{y^2}{4} + \frac{x}{2}\right)z\frac{\partial}{\partial z}. \tag{5.106}$$

Solve the characteristic equations to determine the two invariants of the group, and show that they are invariant under the finite transformation.

5.8 The equations governing inviscid compressible flow of a general fluid (see Chapter 12) are invariant under an eleven-parameter group with operators

$$X^1 = \frac{\partial}{\partial t}, \qquad X^2 = \frac{\partial}{\partial x}, \qquad X^3 = \frac{\partial}{\partial y}, \qquad X^4 = \frac{\partial}{\partial z},$$

$$X^5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v},$$

$$X^6 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w},$$

$$X^7 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + w\frac{\partial}{\partial u} - u\frac{\partial}{\partial w},$$

$$X^8 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \qquad X^9 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial u}, \qquad X^{10} = t\frac{\partial}{\partial z} + \frac{\partial}{\partial u},$$

$$X^{11} = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \tag{5.107}$$

where t, x, y, z are time and the spatial coordinates, and u, v, w are the velocity components in the corresponding directions. First, see how many finite groups you can work out by inspection. Identify the nature of each group (translation, rotation, dilation, etc.). Sum the Lie series to work out the finite rotation groups. Note that if you want to generate the full form of the three-parameter 3-D rotation group, you will need to sum the Lie series using a three-term group operator with three independent small parameters. Work out the 11×11 commutator table for this group, and identify any subalgebras. Identify any solvable subalgebras.