Introduction to Symmetry Analysis

Chapter 2 - Dimensional Analysis

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2.1 Introduction

What is a dimension?

Definition 2.1. *A dimension is a measurable property of a physical system that can be varied by a dilational transformation of the units of measurement. The value of each variable of the system is proportional to a power monomial function of the fundamental dimensions.*

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2.2 The Two-Body Problem in a Gravitational Field

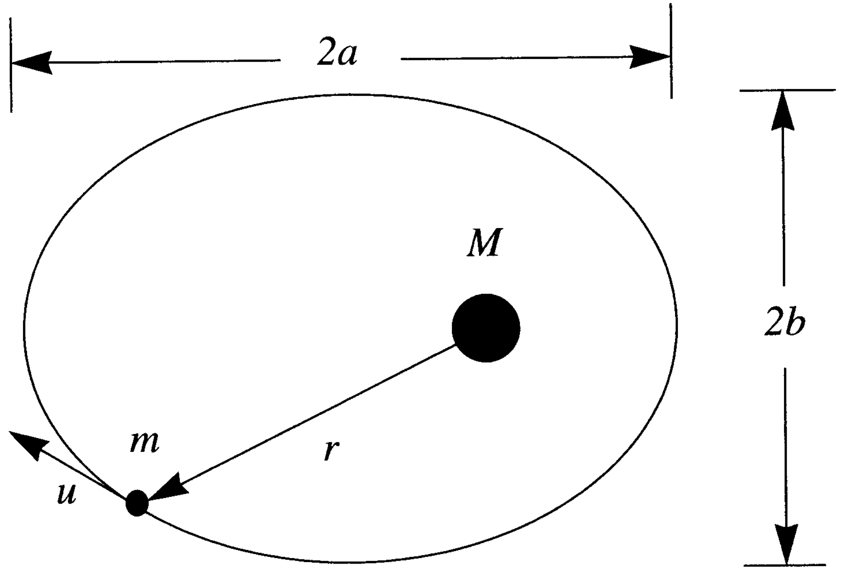

Figure 2.1 Elliptical orbit of a planet about the Sun

Newtonian law of gravitation

$$F = -G\frac{Mm}{r^2}, \qquad (2.1)$$

$$G = 6.670 \times 10^{-11}\ \text{N-m}^2/\text{kg}^2$$

Parameters of the problem

$$\hat{a} = L, \qquad \hat{b} = L, \qquad \hat{M} = M, \qquad \hat{m} = M, \qquad \hat{T} = T, \qquad \hat{G} = \frac{L^3}{MT^2}. \quad (2.3)$$

$M =$ mass, $L =$ length, and $T =$ time

There are six parameters and three fundamental dimensions. So we can expect the solution to depend on three dimensionless numbers

$$\Pi_1 = \frac{m}{M}, \qquad \Pi_2 = \frac{b}{a} \quad (2.4)$$

and

$$\Pi_3 = \frac{GMT^2}{a^3} \quad (2.5)$$

These variables must be related by a dimensionless function of the form

$$\psi = \Psi(\Pi_1, \Pi_2, \Pi_3) \quad (2.6)$$

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or

$$\frac{GMT^2}{(r_{\text{mean}})^3} = F\left(\frac{m}{M}, e\right) \tag{2.7}$$

The mean radius is defined as $r_{\text{mean}} = \sqrt{ab}$

Theory tells us that

$$F\left(\frac{m}{M}, e\right) = 4\pi^2\left(\frac{1}{(1 + m/M)(1 - e^2)^{3/4}}\right) \tag{2.8}$$

Table 2.1. *The planets and their orbits.*

Heavenly body	Mass (Earth masses)	Diameter (Earth diameters)	Mean orbit Radius (10^6 km)	Eccentricity	Orbital period (years)
Sun	332,488.0	109.15	—	—	—
Mercury	0.0543	0.38	57.9	0.2056	0.241
Venus	0.8136	0.967	108.1	0.0068	0.615
Earth	1.0000	1.000	149.5	0.0167	1.000
Mars	0.1069	0.523	227.8	0.0934	1.881
Jupiter	318.35	10.97	777.8	0.0484	11.862
Saturn	95.3	9.03	1426.1	0.0557	29.458
Uranus	14.58	3.72	2869.1	0.0472	84.015
Neptune	17.26	3.38	4495.6	0.0086	164.788
Pluto	<0.1	0.45	5898.9	0.2485	247.697

Table 2.1 The planets and their orbits

The mass of the Earth is 5.975×10^{24} kg and the mean diameter is 12742.46 km

The eccentricity of a planet's orbit is

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}. \quad (2.2)$$

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For all the planets $m/M \ll 1$, and for all but Mercury and Pluto e^2 is very small. In the limits $m/M \to 0$ and $e \to 0$ the right-hand side of (2.8) approaches the finite limit $4\pi^2 = 39.4784$.

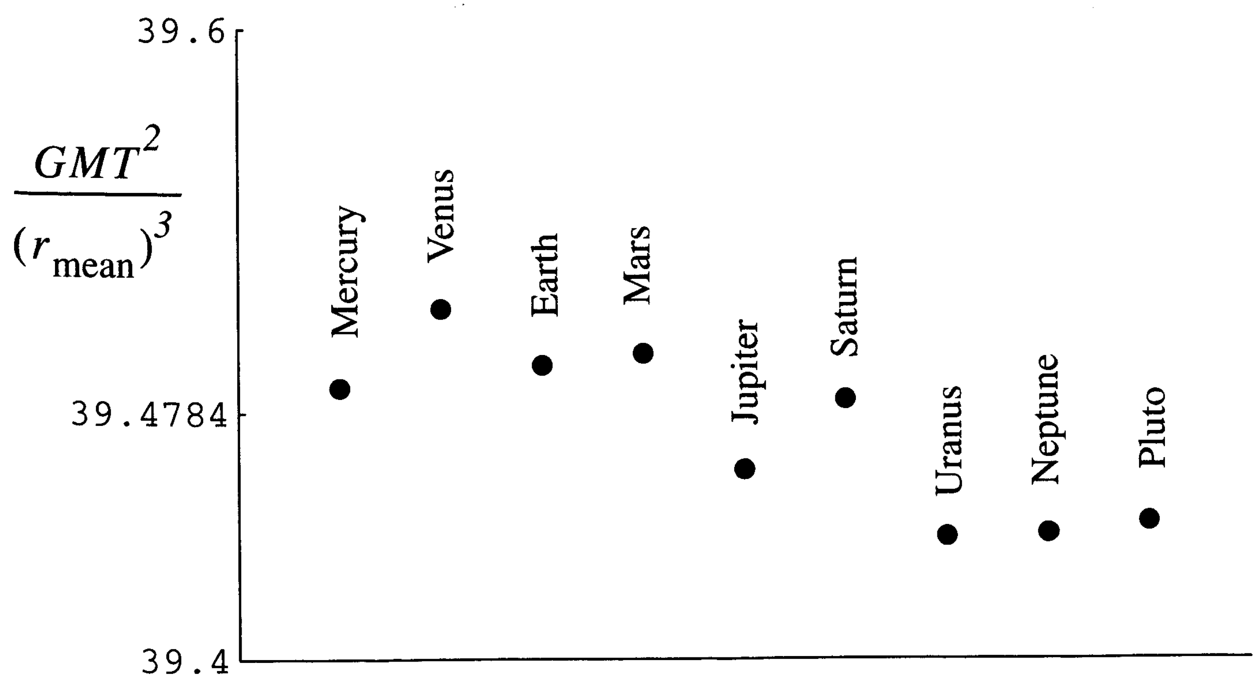

Figure 2.2 *Kepler's third law for the Solar System.*

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2.3 The Drag on a Sphere

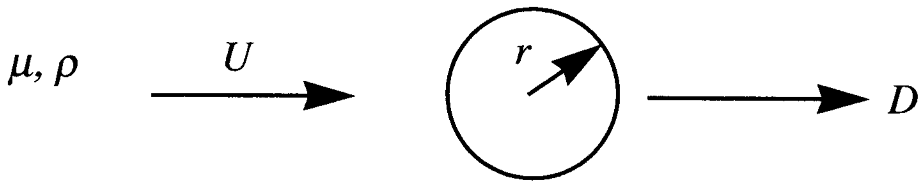

Figure 2.3 Viscous flow past a sphere

The parameters of the problem are related to one another through a function of the form

$$\psi_0 = \Psi_0[D, \mu, \rho, U, r] \tag{2.9}$$

Dimensions of the governing parameters

$$\hat{D} = \frac{ML}{T^2}, \qquad \hat{\mu} = \frac{M}{LT}, \qquad \hat{\rho} = \frac{M}{L^3}, \qquad \hat{U} = \frac{L}{T}, \qquad \hat{r} = L. \tag{2.10}$$

The fact that the parameters have dimensions highly restricts the kind of drag functions that are possible. For example, suppose we guess that the drag law has the form

$$0 = D - (\mu + \rho + U + r), \tag{2.11a}$$

If we introduce the dimensions of each parameter the expression has the form

$$0 = \frac{ML}{T^2} - \left(\frac{M}{LT} + \frac{M}{L^3} + \frac{L}{T} + L\right). \tag{2.11b}$$

Suppose the units of mass are changed from kilograms to grams. Then the number for the drag will increase by a factor of a thousand. But the expression in parentheses will not increase by this factor and the equality will not be satisfied. In effect the drag of the sphere will seem to depend on the choice of units and this is impossible. The conclusion is that (2.11a) can not possibly describe the drag of a sphere.

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The drag expression must be invariant under a three parameter dilation group.

$$\tilde{M} = e^{m} M, \qquad \tilde{L} = e^{l} L, \qquad \tilde{T} = e^{t} T, \tag{2.12}$$

We can derive the required drag expression as follows.

Step1

Scale the units of mass using the one-parameter group

$$\tilde{M} = e^{m} M, \qquad \tilde{L} = L, \qquad \tilde{T} = T. \tag{2.13}$$

The effect is to transform the parameters as follows.

$$\tilde{D} = e^{m} D, \qquad \tilde{\mu} = e^{m} \mu, \qquad \tilde{\rho} = e^{m} \rho, \qquad \tilde{U} = U, \qquad \tilde{r} = r. \tag{2.14}$$

The drag expression must be independent of the scaling parameter m and therefore must be of the form.

$$\psi_0 = \Psi_1\left[\frac{D}{\rho}, \frac{\rho}{\mu}, U, r\right] \tag{2.15}$$

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The dimensions of the variables remaining are

$$\frac{\hat{D}}{\hat{\rho}} = \frac{L^4}{T^2}, \qquad \frac{\hat{\rho}}{\hat{\mu}} = \frac{T}{L^2}, \qquad \hat{U} = \frac{L}{T}, \qquad \hat{r} = L. \tag{2.16}$$

Step 2

Let the units of length be scaled according to

$$\tilde{L} = e^l L, \qquad \tilde{T} = T. \tag{2.17}$$

The effect of this group on the new variables is

$$\frac{\tilde{D}}{\tilde{\rho}} = e^{4l}\frac{D}{\rho}, \qquad \frac{\tilde{\rho}}{\tilde{\mu}} = e^{-2l}\frac{\rho}{\mu}, \qquad \tilde{U} = e^l U, \qquad \tilde{r} = e^l r. \tag{2.18}$$

The drag relation must be independent of the scaling parameter *l*.

A functional form that accomplishes this is

$$\psi_0 = \Psi_3\left[\frac{D}{\rho U^2 r^2}, \frac{\rho U^2}{\mu}, \frac{r}{U}\right]. \tag{2.19}$$

2 would make more sense

The dimensions of these variables are

$$\frac{\hat{D}}{\hat{\rho}\hat{U}^2\hat{r}^2} = 1, \qquad \frac{\hat{\rho}\hat{U}^2}{\hat{\mu}} = \frac{1}{T}, \qquad \frac{\hat{r}}{\hat{U}} = T. \tag{2.20}$$

Step 3

Finally scale the units of time

$$\tilde{T} = e^t T. \tag{2.21}$$

The effect of this group on the remaining variables is

$$\frac{\tilde{D}}{\tilde{\rho}\tilde{U}^2\tilde{r}^2} = \frac{D}{\rho U^2 r^2}, \qquad \frac{\tilde{\rho}\tilde{U}^2}{\tilde{\mu}} = e^{-t}\frac{\rho U^2}{\mu}, \qquad \frac{\tilde{r}}{\tilde{U}} = e^t \frac{r}{U}. \tag{2.22}$$

The drag relation must be independent of the scaling parameter t. Finally

$$\boxed{\psi_0 = \Psi[C_D, Re]} \tag{2.23}$$

where

$$C_D = \frac{D}{\frac{1}{2}\rho U^2(\pi r^2)}, \qquad Re = \frac{\rho U(2r)}{\mu}. \tag{2.24}$$

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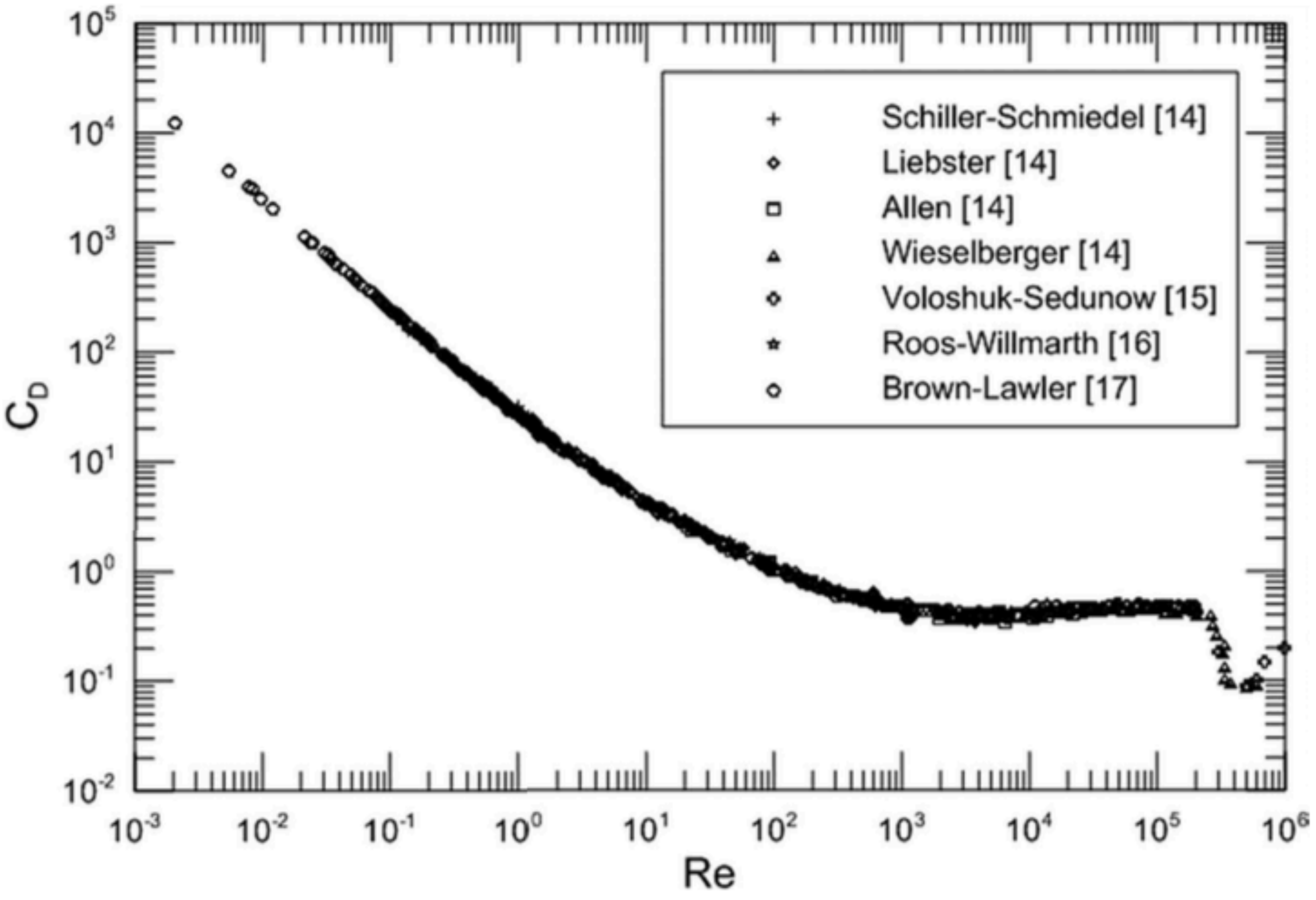

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Measurements of circular cylinder drag versus Reynolds number taken by a variety of investigators.

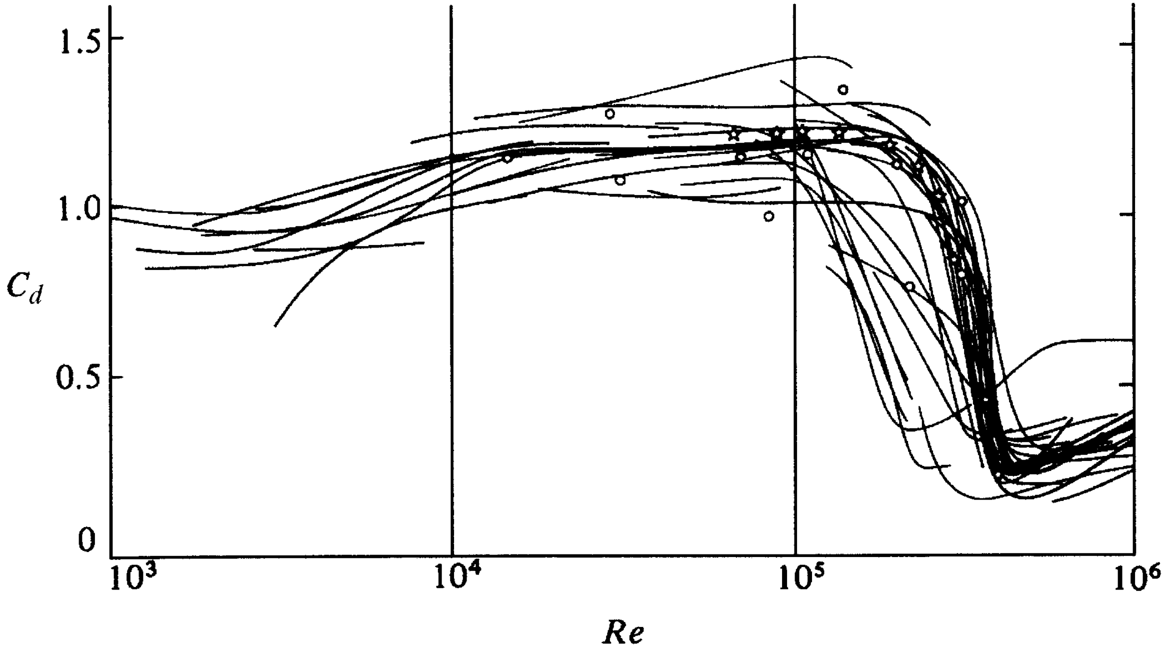

Figure 2.4 Experimental measurements of the drag of a circular cylinder

The data shows a huge amount of scatter - why?

$$\psi = \Psi\left[\frac{D}{\rho U^2 r^2}, \frac{\rho U r}{\mu}, \frac{\upsilon_1}{U}, \frac{\upsilon_2}{U}, \ldots, \frac{\lambda_1}{r}, \frac{\lambda_2}{r}, \ldots\right], \quad (2.31)$$

The drag of a sphere or a cylinder depends on a wide variety of length and velocity scales that we have ignored!

In the limit of vanishing Reynolds number the drag of a sphere is given by

$$C_D = \frac{24}{Re}. \tag{2.25}$$

If we insert the expressions for the Drag coefficient and Reynolds number into Equation (2.25) the drag law becomes

$$\frac{D}{\mu U r} = 6\pi. \tag{2.26}$$

Note that at low Reynolds number the drag of a sphere is independent of the density of the surrounding fluid. In this limit there is only one dimensionless parameter in the problem proportional to the product C_D x Re.

One might conjecture that the same kind of law applies to the low Reynolds number flow past a circular cylinder. In this case the drag force is replaced by the drag force per unit span with units

$$\hat{D}_{\text{cylinder}} = \frac{M}{T^2} \qquad (2.27)$$

with drag coefficient

$$C_{D_{\text{cylinder}}} = \frac{D_{\text{cylinder}}}{\frac{1}{2}\rho U^2(2r)}. \qquad (2.28)$$

Assume that in the limit of vanishing Reynolds number the drag coefficient of a circular cylinder follows the same law as for the sphere.

$$C_{D_{\text{cylinder}}} = \frac{\psi}{Re}. \qquad (2.29)$$

If we restore the dimensioned variables in (2.29) the result is

$$\frac{D_{\text{cylinder}}}{\mu U} = \psi, \qquad (2.30)$$

This is a completely incorrect result!

2.4 The Drag on a Sphere in High Speed Flow

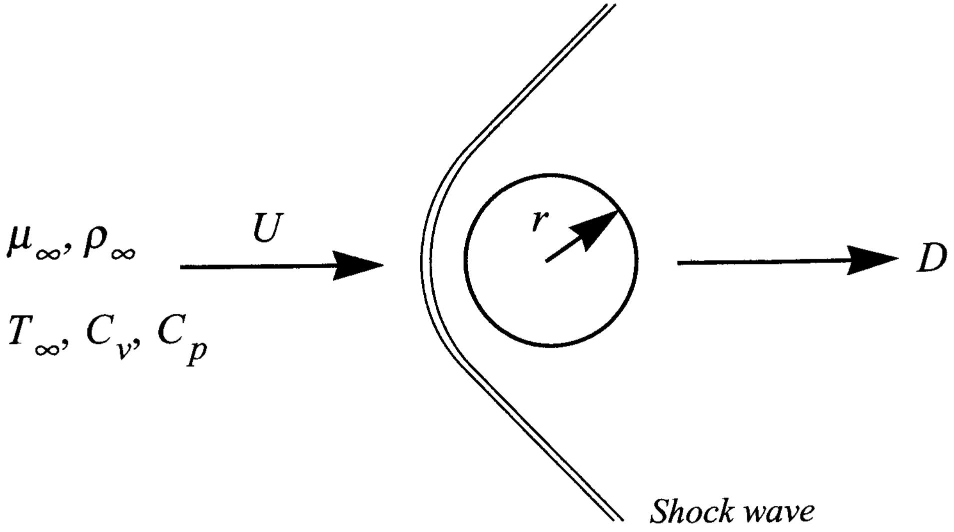

Figure 2.5 High speed flow past a sphere

The dimensions of the new variables are

$$\hat{T}_\infty = \Theta, \qquad \hat{C}_p = \frac{M^2}{L^2\Theta}, \qquad \hat{C}_v = \frac{M^2}{L^2\Theta}. \tag{2.32}$$

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There are now two additional dimensionless variables related to the fact that the sphere motion significantly changes the temperature of the oncoming gas.

$$\Pi_1 = \frac{U^2}{C_v T_\infty}, \qquad \Pi_2 = \frac{C_p}{C_v}. \tag{2.33}$$

The drag relation is now a function of four dimensionless variables.

$$\psi = \Psi[C_D, Re, M_\infty, \gamma] \tag{2.34}$$

The Mach number is used in (2.34).

$$M_\infty = \frac{U}{a_\infty}, \tag{2.35}$$

where

$$a_\infty^2 = \gamma R T_\infty. \tag{2.36}$$

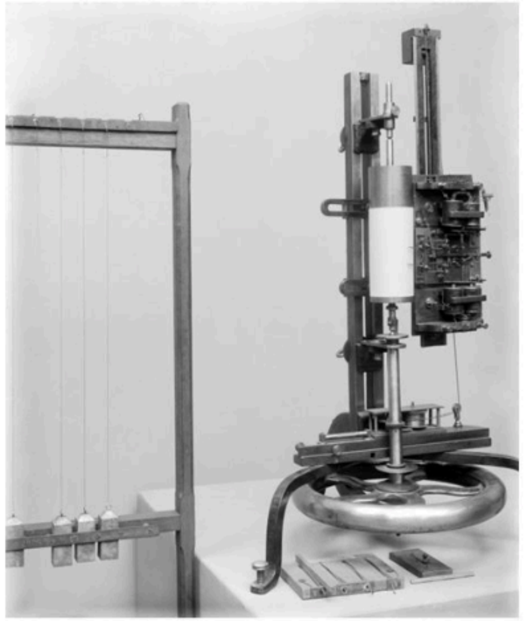
Without loss of generality we can write

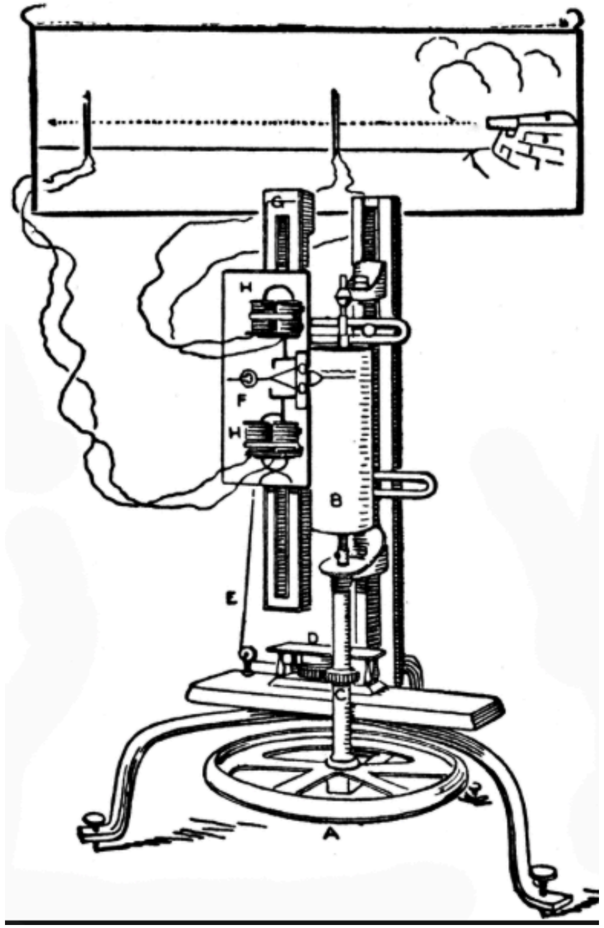
$$C_D = F[Re, M_\infty, \gamma]. \tag{2.37}$$

Miller and Bailey, (JFM 93, 1979) found that the best measurements of sphere drag at supersonic Mach numbers were the cannonball measurements of Francis Bashforth taken for the British Royal Navy.

BASHFORTH'S CHRONOGRAPH, 1864-1873

- Based at the Royal Military Academy, Woolwich, the applied mathematician **Francis Bashforth** devises a **chronograph** sensitive enough to detect small variations in ballistic trajectory.
- He uses the resulting data to analyse the effects of air resistance and atmospheric conditions upon trajectories, and spends the next 20 years preparing comprehensive ready reckoner tables for Royal Navy gunnery officers [see 1886].





Miller and Bailey, JFM 93, 1979

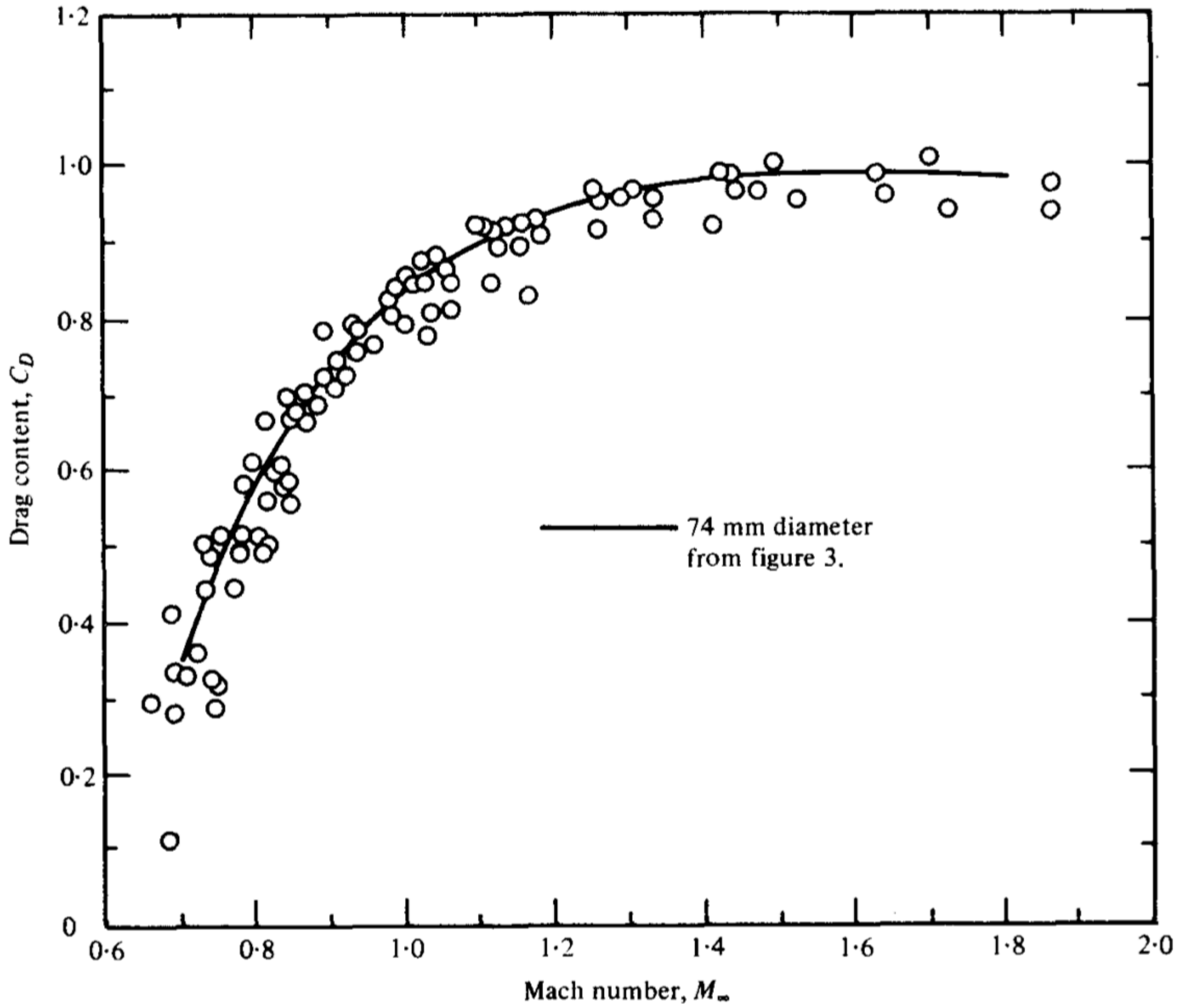


FIGURE 2. Variation of drag with Mach number for 74 mm diameter hollow sphere calculated from Bashforth (1870).

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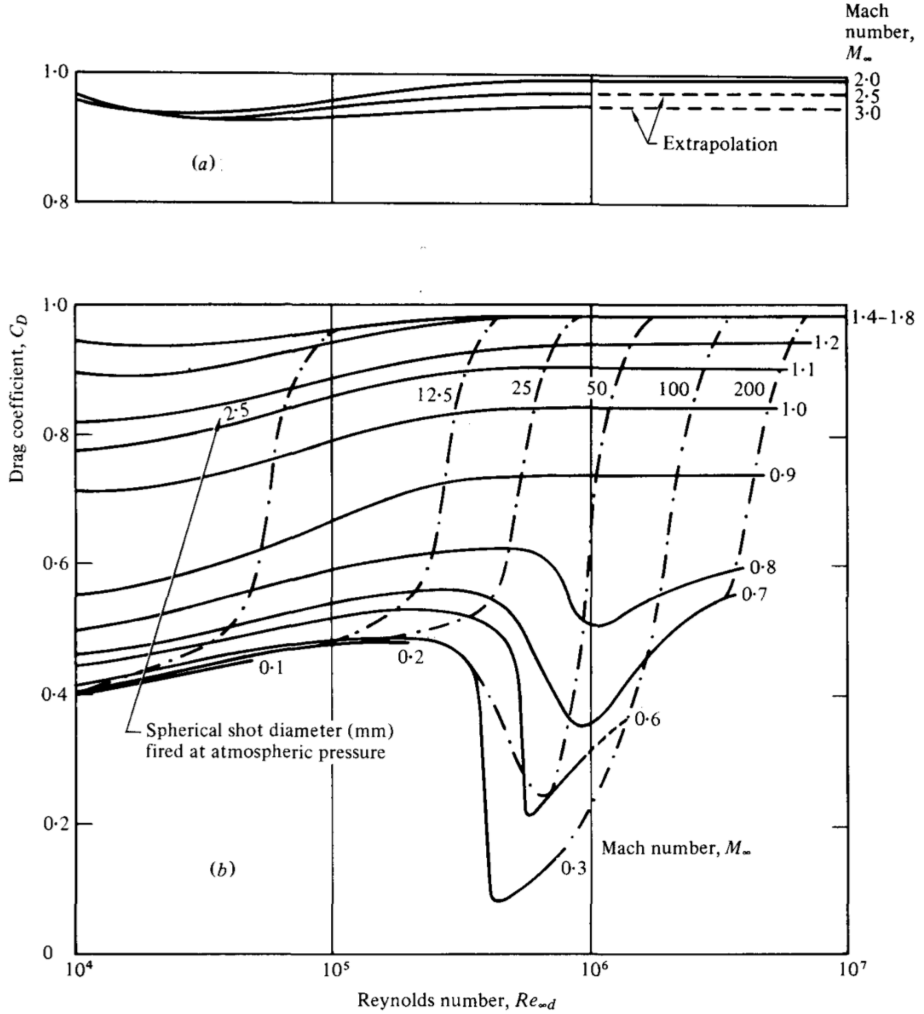

FIGURE 4. Summary of sphere drag measurements at high Reynolds numbers: (a) $2{\cdot}0 \leqslant M_\infty \leqslant 3{\cdot}0$, and (b) $0{\cdot}2 \leqslant M_\infty \leqslant 1{\cdot}8$ (curve is in two parts because C_D reaches a maximum between $M_\infty = 1{\cdot}6$ and $1{\cdot}8$).

As the Mach number increases the drag coefficient tends to become independent of both Reynolds number and Mach number with

$$C_D \approx 1, \tag{2.38}$$

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2.4 The Buckingham Pi Theorem

Dimensional analysis makes use of a simple, purely algorithmic procedure that is extremely general and can be applied to practically any physical problem. The various steps are as follows.

(1) Identify the physical variables relevant to the problem $(a_1, a_2, \ldots, a_\alpha)$.

(2) Determine the fundamental dimensions of each physical variable. The total number of dimensions is $(d_1, d_2, \ldots, d_\beta)$ $(\beta \leq \alpha)$. Each variable is a power monomial function of its dimensions,

$$\hat{a}_i = d_1^{k_1} d_2^{k_2} \cdots d_\beta^{k_\beta}. \tag{2.39}$$

where $k_1, k_2, \ldots, k_\beta$ are usually but not always integers.

(3) *Buckingham's Pi Theorem* – A relationship between physical variables $\psi = f[a_1, a_2, \ldots, a_\alpha]$ must be expressible in a form that is invariant under a β-parameter dilation group applied to the fundamental dimensions:

$$\tilde{d}_1 = e^{\delta_1} d_1, \qquad \tilde{d}_2 = e^{\delta_2} d_2, \ldots, \qquad \tilde{d}_\beta = e^{\delta_\beta} d_\beta. \tag{2.40}$$

(4) The algorithm for accomplishing step 3 is to apply a one-parameter dilation group to each dimension in succession. New variables are created at each step, which are independent of the dimension being varied. This process is continued until all the dimensions are exhausted. In the final result, the physical problem can only depend on dimensionless variables via a function of the form $\psi = \Psi[\Pi_1, \Pi_2, \ldots, \Pi_\gamma]$. Usually $\gamma = \alpha - \beta$. Occasionally the dimensions of the variables are such that two or more dimensions may be eliminated in a single step. In this case the number of dimensionless variables is larger than $\alpha - \beta$. See Exercise 2.9 for an example. This notion can be quantified by forming the $\beta \times \alpha$ matrix of exponents of the dimensions of the physical variables. The actual count of dimensionless variables is α minus the rank of this matrix. If the rank is less than β then two or more dimensions can be combined.

It is important to recognize that the dimensionless parameters generated by the algorithm just described are not unique. For example in the case of the sphere we could have wound up with the following, equally correct, result.

$$\phi = \Phi[C_D Re, Re] = \Phi\left[\frac{D}{\mu U r}, \frac{\rho U r}{\mu}\right] \tag{2.41}$$

In this form the drag law has a finite value in the limit of vanishing Reynolds number.

$$\lim_{Re \to 0} \Phi\left[\frac{D}{\mu U r}, \frac{\rho U r}{\mu}\right] = 6\pi. \tag{2.42}$$

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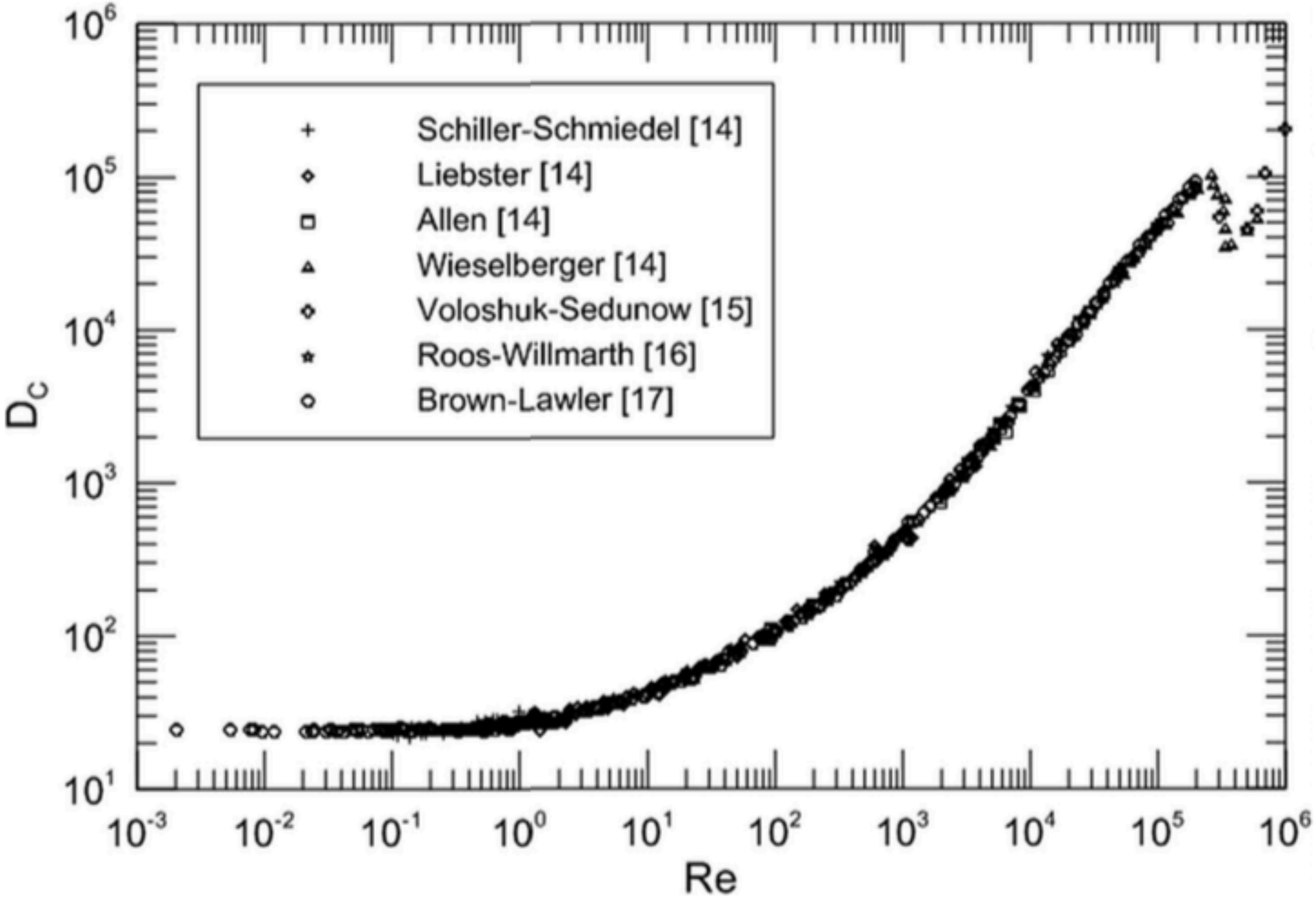

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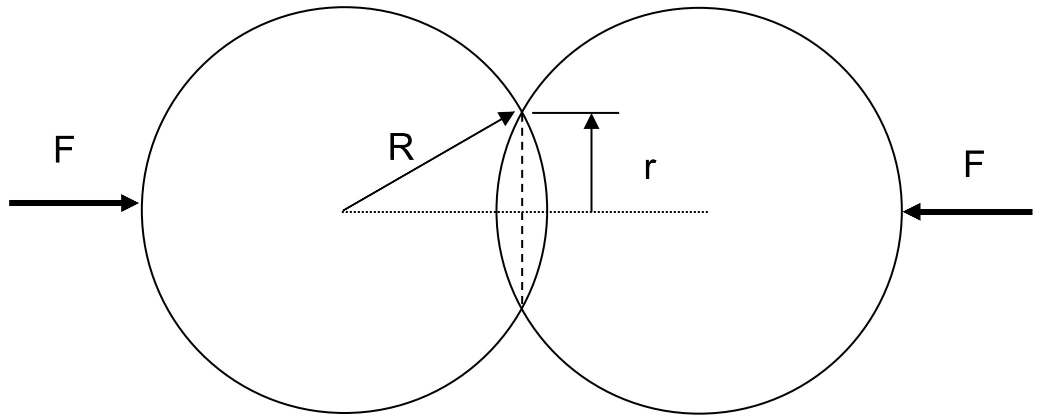
Sometimes two dimensions drop out in a single step

Elastic spheres pressed together



Parameters

$$\hat{R} = L$$

$$\hat{r} = L$$

$$\hat{F} = ML / T^2$$

$$\hat{E} = M / LT^2$$ Young's modulus

Let the units of mass be scaled according to

$$\tilde{M} = e^m M \qquad \tilde{L} = L \qquad \tilde{T} = T$$

The effect of this group on the parameters is

$$\tilde{R} = R \qquad \tilde{r} = r \qquad \tilde{F} = e^m F \qquad \tilde{E} = e^m E$$

The force relation must be independent of the scaling parameter *m*.

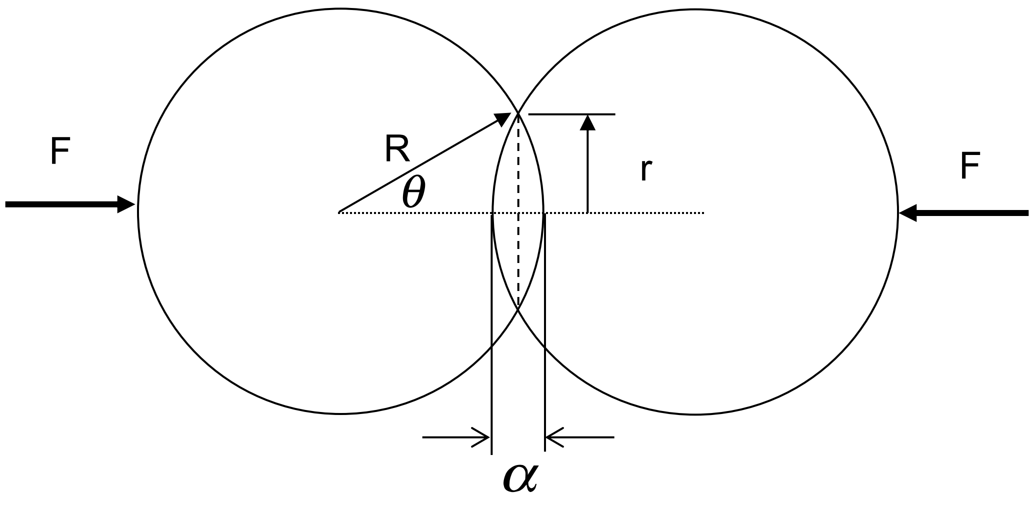
A functional form that accomplishes this is

$$\psi_0 = \Psi_1\left[R, r, \frac{F}{E}\right]$$

Note that both mass and time have been eliminated. Eliminate length

$$\psi_0 = \Psi_2\left[\frac{r}{R}, \frac{F}{Er^2}\right]$$

Exact solution



$$\cos[\theta]=1-\frac{\alpha}{2R}\cong 1-\frac{\theta^2}{2}$$

$$\theta=\left(\frac{\alpha}{R}\right)^{1/2}$$

$$\frac{r}{R}=\sin[\theta]\cong\left(\frac{\alpha}{R}\right)^{1/2}$$

$$\frac{\alpha}{R}=\left(\frac{r}{R}\right)^{2}$$

$$\alpha^3=\frac{9}{2R}\left(\left(\frac{1-\sigma^2}{E}\right)F\right)^2$$

$$\frac{\alpha}{R}=\left(\frac{9}{2}\right)^{1/3}\left(\left(\frac{1-\sigma^2}{E}\right)F\right)^{2/3}\frac{1}{R^{4/3}}$$

$$\frac{r}{R}=\left(\frac{9}{2}\right)^{1/6}\left(\left(\frac{1-\sigma^2}{E}\right)\frac{F}{r^2}\right)^{1/3}\left(\frac{r}{R}\right)^{2/3}$$

$$\left(\frac{r}{R}\right)^{1/3}=\left(\frac{9}{2}\right)^{1/6}\left(\left(\frac{1-\sigma^2}{E}\right)\frac{F}{r^2}\right)^{1/3}$$

$$\boxed{\frac{r}{R}=\frac{3}{\sqrt{2}}\left(\left(\frac{1-\sigma^2}{E}\right)\frac{F}{r^2}\right)}$$

Poisson's ratio

$$\sigma=\frac{E}{2G}-1$$

G – shear modulus

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2.6 Concluding Remarks

2.7 Exercises

2.1 Under the influence of surface tension, a liquid rises to a height H in a glass tube of diameter D (Figure 2.8). How does H depend on the parameters of the problem?

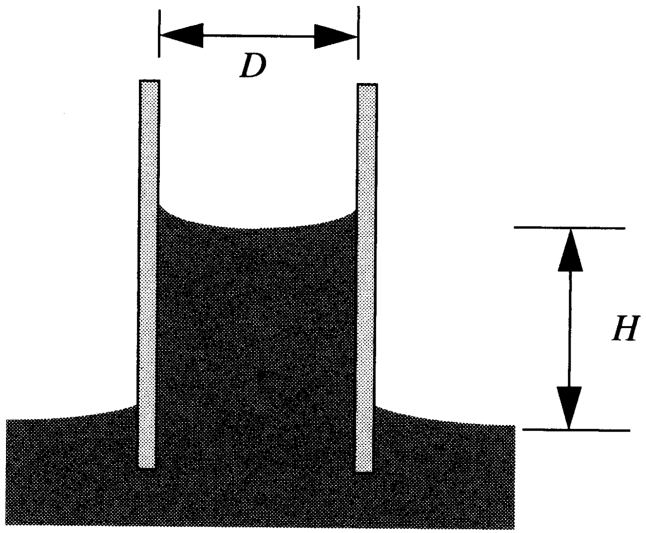

Fig. 2.8.

2.2 Estimate the time of oscillation of a small drop of liquid under its own surface tension.

2.3 When a drop of water strikes a surface at sufficiently low speed, surface tension keeps it round, so it makes a circular spot. As the impact speed is increased, dynamic forces overcome the smoothing effect of surface tension, and the drop becomes unstable and forms a spiky shape as shown in Figure 2.9. (Thanks to Milton Van Dyke for this problem [2.8].) How does the speed at which the impact becomes unstable depend on the properties of the drop? Retain only the essential properties, so that your result involves only a single unknown constant that could be determined from an experiment. Thus you may wish to assume that viscosity is negligible, the properties of the surrounding air are unimportant, etc. See if your result makes sense. For example, does the critical speed depend on the surface tension in the way you would expect?

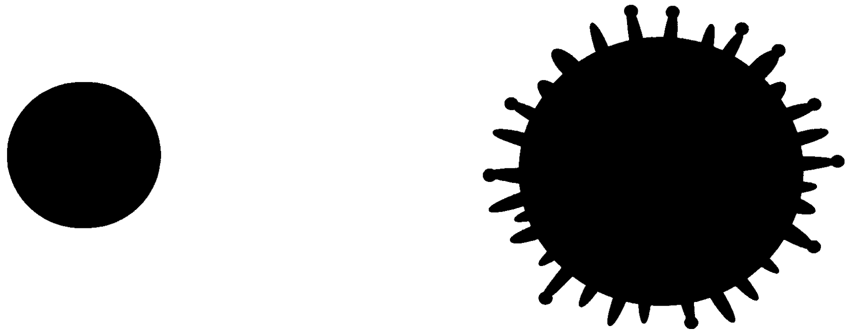

Fig. 2.9.

2.4 Estimate the velocity of fall of a small heavy sphere in a viscous fluid of lower density than the sphere under the influence of gravity. Compare your result with the exact solution. How long does it take the sphere to reach its terminal velocity when dropped from rest?

2.5 Liquid in an open container flows through a long horizontal pipe into a second container as shown in Figure 2.10. How does the time for the liquid level to reach equilibrium depend on the parameters of the problem?

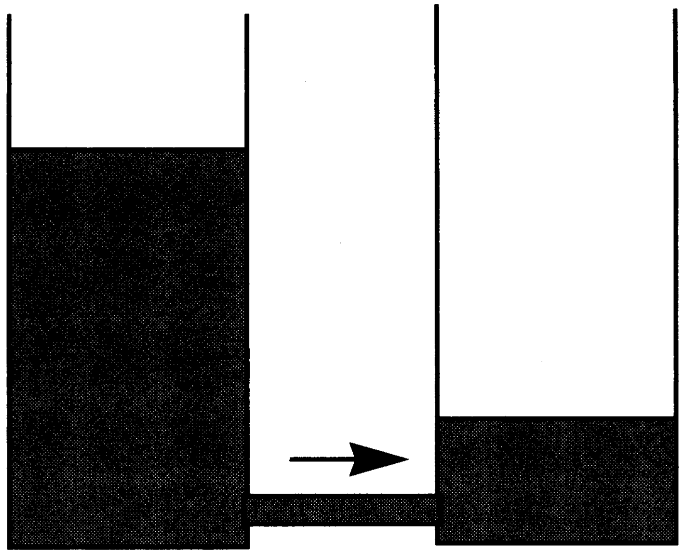

Fig. 2.10.

2.6 Use dimensional analysis to find how the rowing speed depends on the number of oarsmen for racing shells. This problem is discussed by McMahon [2.9] and Barenblatt [2.4]. Use the following asssumptions.

Table 2.2. *Rowing times for 1-, 2-, 4-, and 8-man shells from three previous Olympics. The distance traveled in each case is 2000 m.*

	Time (s)			
Olympics	1 oarsman	2 oarsmen	4 oarsmen	8 oarsmen
Atlanta	404.85	376.98	356.93	342.74
Barcelona	411.40	377.32	355.04	329.53
Seoul	409.86	381.13	363.11	—

(i) The boats are geometrically similar.
(ii) The boat weight W per oarsman is constant.
(iii) Each oarsman contributes the same power, P.
(iv) The only hindering force is skin friction, and the friction coefficient is constant over the wetted area. The friction coefficient is defined as $c_f = \tau_{\text{wall}}/(\frac{1}{2}\rho U^2)$, where τ_{wall} is the wall shear stress.

Hint. Find how the volume of the displaced water varies with the number of oarsman and the length of the boat. Equate the expenditure of energy on skin friction to the power supplied by the oarsman. Data for men's rowing over a 2-km course from three recent Olympic summer games are presented in Table 2.2. Plot the data in logarithmic coordinates and compare with your prediction. Notice that in the context of this problem the number of oarsmen is a fundamental dimension.

2.7 Critique the assumptions in Exercise 2.6 – particularly (i), which seems to suggest that the shells get wider as they get longer to accommodate more rowers.

(i) How does the problem work out if the width of the shell is assumed to be constant?

(ii) Suppose the drag is primarily due to the generation of waves and skin friction can be neglected. How will the speed depend on the number of oarsman? Do these results shake your confidence in the solution developed in Exercise 2.6?

(iii) Work the case where the race is carried out by fleas on a lake of honey.

2.8 What is the speed of the wave in a row of falling dominos on a table? Add whatever simplifying assumptions you feel are reasonable, such as perfectly rigid dominos, constant coefficient of friction between the dominos and the table, etc. This problem is the subject of a pair of journal papers by Stronge [2.10] and Stronge and Shu [2.11] as well as a note in the *SIAM Review* Problems and Solutions. The problem was proposed by Daykin [2.12] and solved by McLachlan et al. [2.13].

2.9 Show that if two equal-size elastic spheres are pressed together, the radius of the circle of contact varies as the one-third power of the force between them. How does it vary with the radius of the spheres?

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2.10 One of the well-known observations in blood flow is that the viscous shear stress at the wall of an artery is approximately independent of the diameter of the artery. Consider a bifurcation where the flow in one large artery splits into two smaller adjoining arteries of equal size. How are the diameters of the smaller arteries related to the diameter of the large artery?

2.11 Use dimensional analysis to deduce how the weight a man can lift depends on his own weight. Assume that the strength of a muscle varies as its cross-sectional area. See if your result correlates the data in Table 2.3, taken from the 1969 *World Almanac* for the 1968 Senior National AAU weightlifting championships. How much did the heavyweight lifter weigh?

Table 2.3. *Total weight lifted for different classes.*

Class	Body weight (pounds)	Lifted weight (pounds)
Bantam	123.5	740
Featherweight	132.25	795
Lightweight	148.75	820
Light-heavy	181.75	1025
Middle-heavy	198.25	1055
Heavyweight	?	1280

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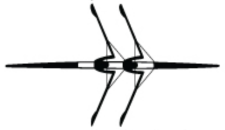
2.12 There is continuing interest in pushing measurements of circular cylinder drag to the highest possible Reynolds numbers. One scheme that has been proposed is to tow a submerged, high-aspect-ratio cylinder behind two nuclear-powered aircraft carriers pulling lines attached to each end of the cylinder. The kinematic viscosity of water is small, the cylinder diameter can be made quite large, and thus high Reynolds numbers ought to be achievable. Assuming only cylinders of a given aspect ratio, say $L/r = 60$, are used, how does the required towing force vary with the Reynolds number based on cylinder diameter? What force would be required to reach a Reynolds number that exceeds the highest available data ($Re = 10^8$, $C_d = 0.6$)? The maximum towing force available is about 10^8 N.

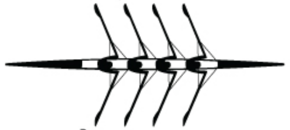
Speed of racing shells

Sculling Boats

Single (1x): Approximately 26 feet long, 11 inches wide, and about 30 lbs. Rowed by one person using two oars.

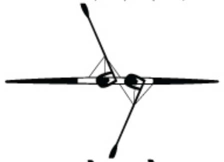


Double (2x): Approximately 32 feet long, 13 inches wide, and about 60 lbs. Rowed by two people using two oars each.

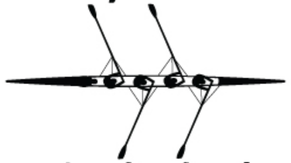


Quad(4x): Approximately 42 feet long and about 115 lbs. Rowed by four people using two oars each.

Sweep Boats

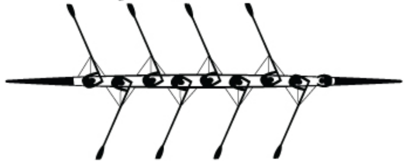


Pair (2-): Approximately 32 feet long, 13 inches wide and about 60 lbs. Rowed by two people using one oar each.

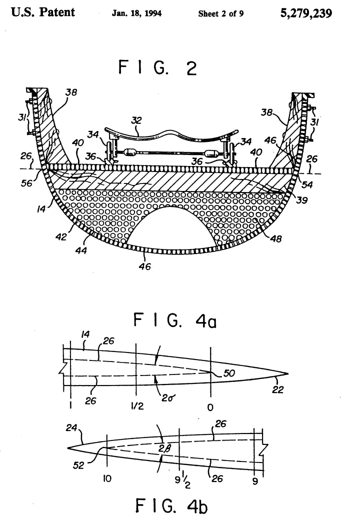


Four (4+): Approximately 42 feet long, 21 inches wide and about 112 lbs. Rowed by four people using one oar each. Coxswain can be in the bow or the stern.

A **Straight Four (4-)** is a similar boat but without a coxswain.



Eight (8): Approximately 60 feet long, 26 inches wide and about 210 lbs. Rowed by eight people using one oar each. Coxswain sits in the stern.



Speed of racing shells - dimensional analysis

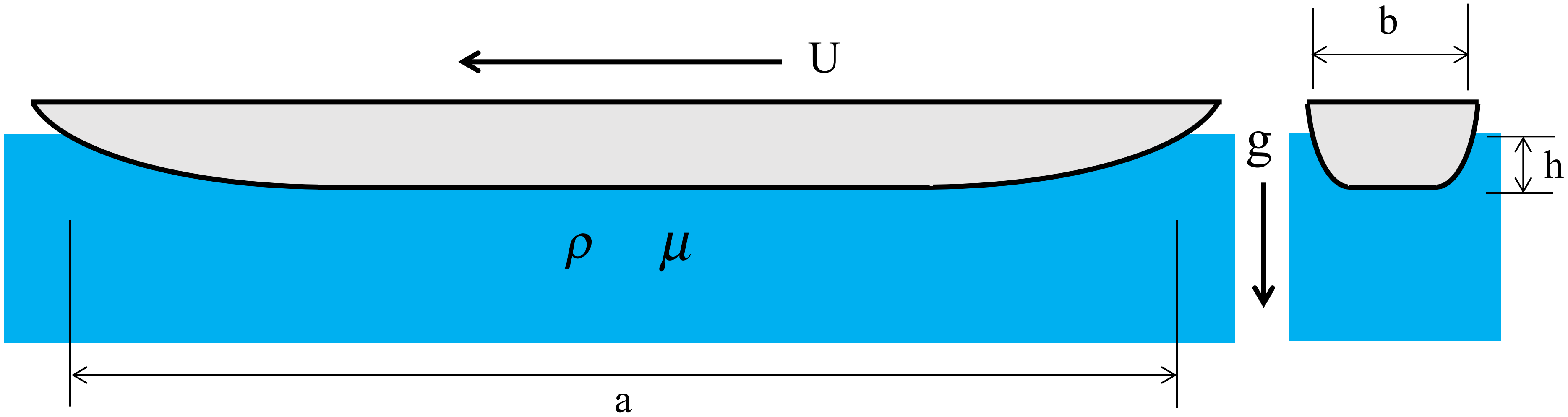
Fundamental dimensions

Mass
Length
Time
Oarsman

N – number of oarsmen

W – weight per oarsmen

P – power per oarsmen



Dimensions of the governing parameters

Fundamental dimensions

M	Mass
L	Length
T	Time
O	Oarsmen

Number of oarsman - $\hat{N} = O$

Weight per oarsman - $\hat{W} = ML/OT^2$

Power per oarsman - $\hat{P} = ML^2/OT^3$

Acceleration of gravity - $\hat{g} = L/T^2$

Boat velocity - $\hat{U} = L/T$

Water density - $\hat{\rho} = M/L^3$

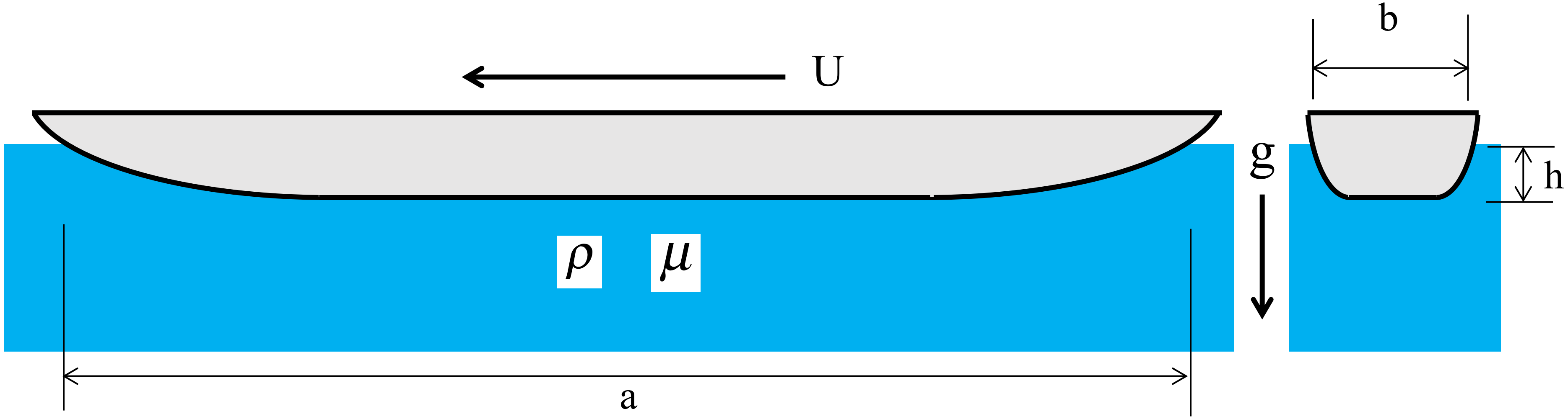
Water viscosity - $\hat{\mu} = M/LT$

Boat length - $\hat{a} = L$

Boat width - $\hat{b} = L$

9 parameters,
4 fundamental dimensions
Expect 5 dimensionless variables

Why is h not a governing parameter ?



Step 1 – Use the number of oarsman to eliminate O, reduce from 9 to 8 parameters.

$$\widehat{NW} = ML/T^2$$

$$\widehat{NP} = ML^2/T^3$$

$$\hat{g} = L/T^2$$

$$\hat{U} = L/T$$

$$\hat{\rho} = M/L^3$$

$$\hat{\mu} = M/LT$$

$$\hat{a} = L$$

$$\hat{b} = L$$

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Step 2 – Use the density to eliminate M, reduce from 8 to 7 parameters.

$$\frac{\widehat{NW}}{\rho} = L^4 / T^2$$

$$\frac{\widehat{NP}}{\rho} = L^5 / T^3$$

$$\hat{g} = L / T^2$$

$$\hat{U} = L / T$$

$$\frac{\hat{\mu}}{\rho} = L^2 / T$$

$$\hat{a} = L$$

$$\hat{b} = L$$

Step 3 – Use the boat velocity to eliminate T, reduce from 7 to 6 parameters.

$$\widehat{\frac{NW}{\rho U^2}} = L^2$$

$$\widehat{\frac{NP}{\rho U^3}} = L^2$$

$$\widehat{\frac{g}{U^2}} = 1/L$$

$$\widehat{\frac{\mu}{\rho U}} = L$$

$$\hat{a} = L$$

$$\hat{b} = L$$

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Step 4 – Use the boat length to eliminate L, reduce from 6 to 5 dimensionless parameters.

$$\widehat{\frac{NW}{\rho U^2 a^2}} = 1$$

$$\widehat{\frac{NP}{\rho U^3 a^2}} = 1$$

$$\widehat{\frac{ga}{U^2}} = 1$$

$$\widehat{\frac{\mu}{\rho U a}} = 1$$

$$\frac{\hat{b}}{a} = 1$$

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There must exist a dimensionless function of the form

$$\psi = \Psi\left(\frac{NW}{\rho U^2 a^2}, \frac{NP}{\rho U^3 a^2}, \frac{\rho U a}{\mu}, \frac{U}{\sqrt{ga}}, \frac{b}{a}\right)$$

Weight number — Power number — Reynolds number — Froude number — Aspect ratio

that governs the problem.

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Since all racing shells are assumed to be similar in shape use the wetted area as the length scale instead of the boat length.

$$A_w = (abh)^{2/3}$$

$$\psi = \Psi\left(\frac{NW}{\rho U^2 A_w}, \frac{NP}{\rho U^3 A_w}, \frac{\rho U A_w^{1/2}}{\mu}, \frac{U}{g^{1/2} A_w^{1/4}}, \frac{b}{a}\right)$$

An equally valid alternative function is.

$$\phi = \Phi\left(\frac{NW}{\rho g A_w^{3/2}}, \frac{NP}{\rho U^3 A_w}, R_e, F_r, \frac{b}{a}\right)$$

where the Froude number has been combined with the weight number.

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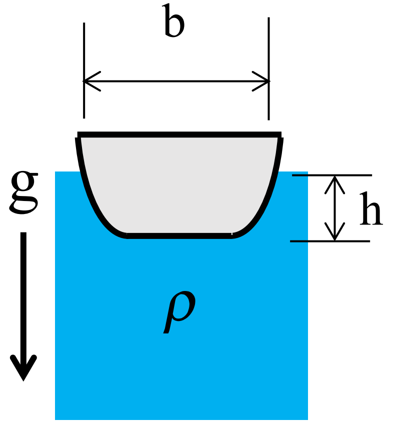
Use Archimedes' principle

$$NW = C_1 \rho g (abh) = C_1 \rho g A_w^{3/2}$$



$$\phi = \Phi\left(C_1, \frac{NP}{\rho U^3 A_w}, \frac{\rho U A_w^{1/2}}{\mu}, \frac{U}{g^{1/2} A_w^{1/4}}, \frac{b}{a}\right)$$

Equate power to drag $\quad NP = \tau_w A_w U \quad$ $\tau_w =$ wall shear stress

$$\frac{NP}{\rho U^3 A_w} = \frac{C_f}{2} \qquad C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

$$\phi = \Phi\left(C_1, \frac{C_f}{2}, R_e, F_r, \frac{b}{a}\right)$$

$$C_1 = \frac{NW}{\rho g A_w^{3/2}} \qquad \frac{C_f}{2} = \frac{NP}{\rho U^3 A_w}$$

Eliminate the wetted area

$$NP = \frac{C_f}{2} \rho U^3 \left(\frac{N^{2/3} W^{2/3}}{\rho^{2/3} g^{2/3} C_1^{2/3}} \right)$$

Velocity is proportional to the 1/9th power of the number of oarsman.

$$\frac{2^{1/3} C_1^{2/9}}{C_f^{1/3}} = \frac{\rho^{1/9} W^{2/9} U}{P^{1/3} g^{2/9} N^{1/9}}$$

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More generally the velocity is proportional to the 1/9th power of the number of oarsman times a function of the Reynolds number, Froude number and aspect ratio.

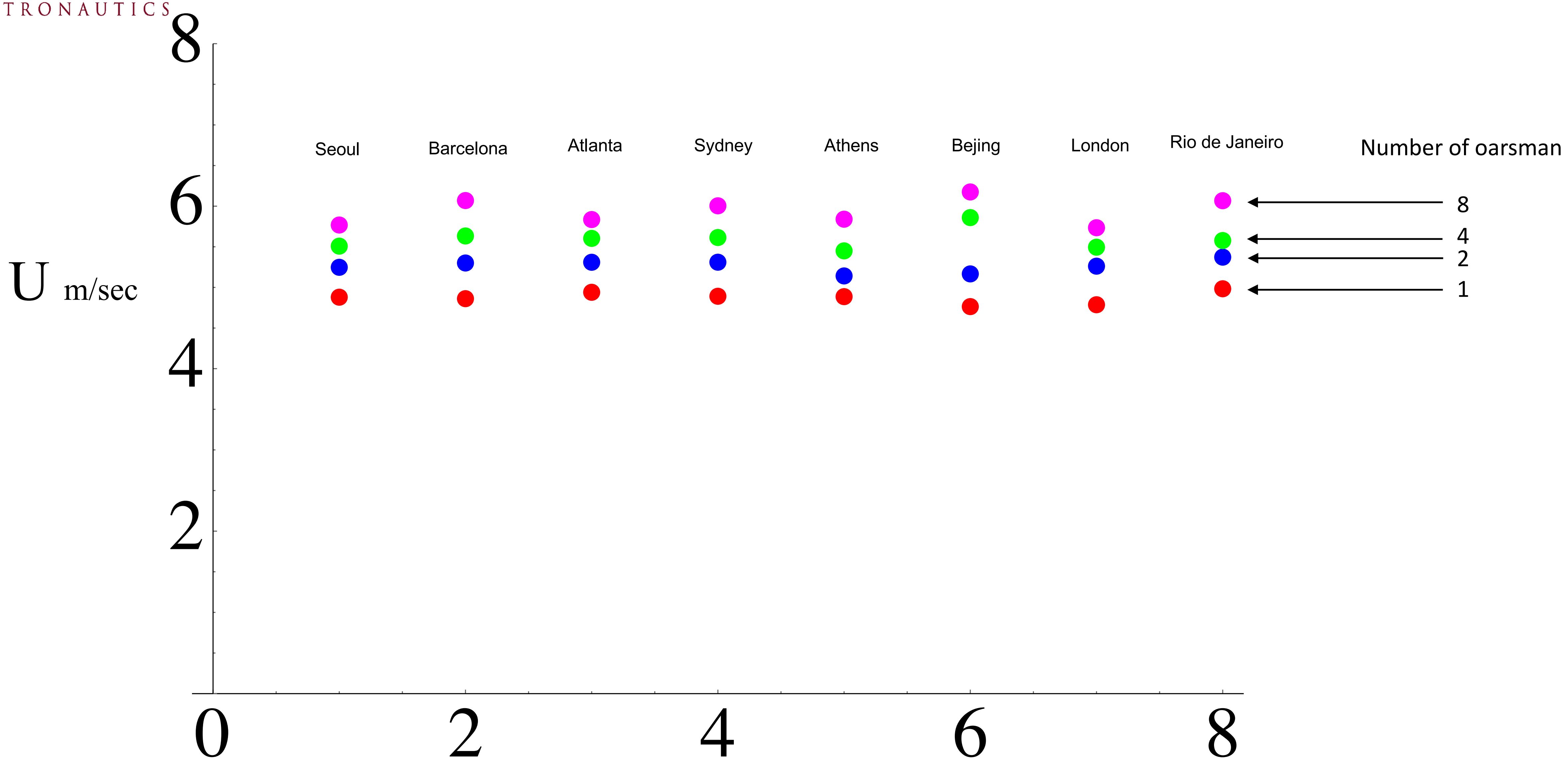
$$\theta = \Theta\left(\frac{\rho^{1/9}W^{2/9}U}{P^{1/3}g^{2/9}N^{1/9}}, \frac{\rho U A_w^{1/2}}{\mu}, \frac{U}{g^{1/2}A_w^{1/4}}, \frac{b}{a}\right)$$

$$\frac{\rho^{1/9}W^{2/9}U}{P^{1/3}g^{2/9}N^{1/9}} = f\left(\frac{\rho U A_w^{1/2}}{\mu}, \frac{U}{g^{1/2}A_w^{1/4}}, \frac{b}{a}\right)$$

$$U = \left(\frac{P^{1/3}g^{2/9}}{\rho^{1/9}W^{2/9}}\right)N^{1/9} f\left(R_e, F_r, \frac{b}{a}\right)$$

Missing from all this is the effect of the aerodynamic drag of the rowers on boat speed.

I collected data from the last eight olympics



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The 1/9th power law scaling seems to work pretty well.

For an 8 man shell, the wetted area is about $20 \text{ x } (2/3) = 13\ m^2$ and the Froude number would be about

$\text{Fr} = 6/(9.8^{1/2}\text{x}13.0^{1/4})$=1.01.

Assuming a drag coefficient of 1 and a frontal area normal to the boat direction of 1 m^2, the aerodynamic drag at 6 $^m/_{sec}$ would be about

$\text{D} = \text{Cd x } (1/2) \text{ x } (1) \text{ x } 6^2 \text{ x } (1) = 18 \text{ N}.$

The force generated by the oarsmen to push the shell through the water is about

$\text{T} = 8 \text{ x } 540\ /6 = 720 \text{ N}.$

$$\frac{\rho^{1/9}W^{2/9}U}{P^{1/3}g^{2/9}N^{1/9}}$$

$\rho = 1000\ kg/m^3$

$P = 540\ J/sec$

$\frac{W}{g} = 102\ kg$