# AA 218 – Problem 8.3

Solve the second order ODE

$$yy_{xx} - (y_x)^2 - a^2y^3 = 0$$

Symmetry Groups by inspection

Since x does not appear explicitly the equation is invariant under a simple translation in x.

$$\tilde{x} = x + s$$

$$\tilde{y} = y$$

$$X^a = \frac{\partial}{\partial x}$$

Try a dilation group

$$\tilde{x} = e^a x$$

$$\tilde{y} = e^b y$$

$$e^{2b-2a}yy_{xx} - e^{2b-2a}(y_x)^2 - e^{3b}a^2y^3 = 0$$

$$2b - 2a = 3b$$

$$b = -2a$$

$$\tilde{x} = e^a x$$

$$\tilde{y} = e^{-2a} y$$

$$X^b = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$$

Commutator table

| | $X^a$ | $X^b$ |
|---|---|---|
| $X^a$ | $0$ | $-X^a$ |
| $X^b$ | $X^a$ | $0$ |

$X^a$ is the ideal of the Lie algebra. Use this to achieve the first reduction.

# First reduction

$$y y_{xx} - (y_x)^2 - a^2 y^3 = 0$$

$$X^a = \frac{\partial}{\partial x}$$

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy_x}{0} = \frac{dy_{xx}}{0}$$

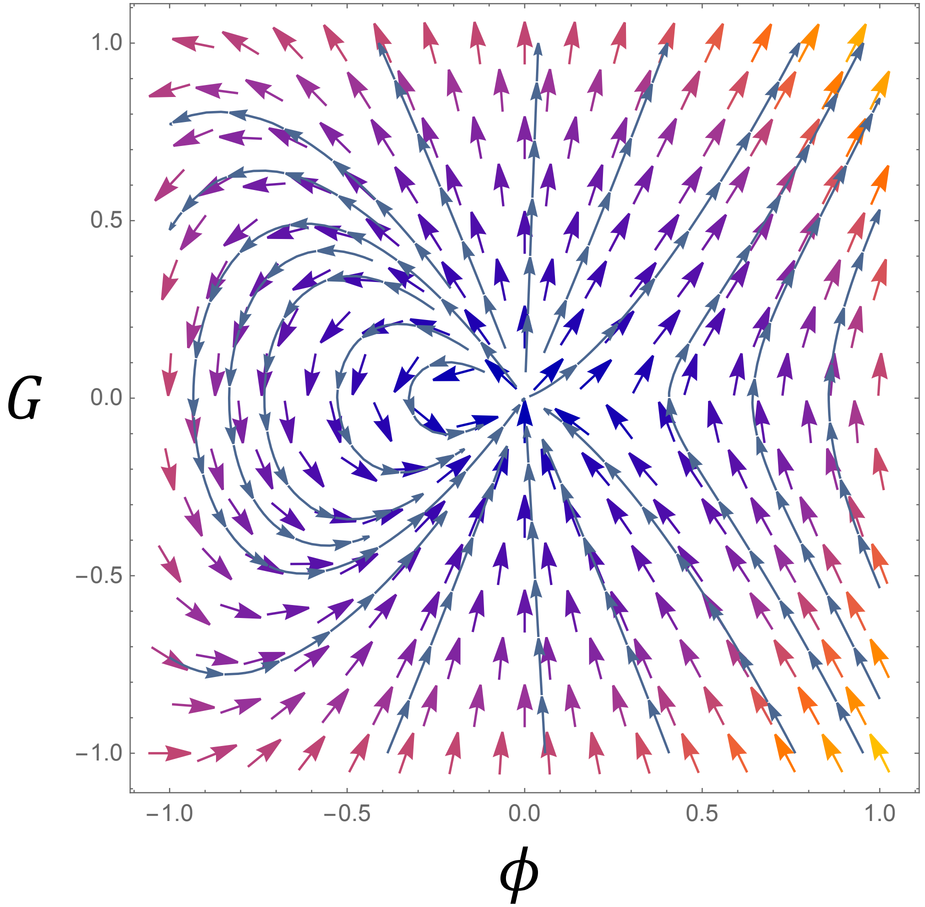
Invariants

$$\phi = y \qquad G = y_x$$

$$\frac{dG}{d\phi} = \frac{y_{xx}}{y_x} = \frac{y_x}{y} + \frac{a^2 y^2}{y_x} = \frac{G^2 + a^2 \phi^3}{\phi G}$$

# Phase Portrait

$$\frac{dG}{d\phi} = \frac{G^2 + a^2\phi^3}{\phi G}$$



# Second reduction

$$\frac{dG}{d\phi} = \frac{G^2 + a^2\phi^3}{\phi G} = \frac{B}{A}$$

$$\xi = -2\phi \qquad \eta = -3G$$

Use the second group

$$\tilde{x} = e^a x \qquad \tilde{\phi} = e^{-2a}\phi$$

$$\tilde{y} = e^{-2a} y \quad \rightarrow \quad \tilde{G} = e^{-3a} G$$

$$\tilde{y}_{\tilde{x}} = e^{-3a} y_x \qquad \tilde{G}_{\tilde{\phi}} = e^{-a} G_\phi$$

## Integrating factor

$$M = \frac{1}{A\eta - B\xi} = \frac{1}{-3\phi G^2 + 2\phi(G^2 + a^2\phi^3)} = \frac{1}{2a^2\phi^4 - \phi G^2}$$

$$d\psi = \frac{a^2\phi^3 + G^2}{2a^2\phi^4 - \phi G^2} d\phi - \frac{\phi G}{2a^2\phi^4 - \phi G^2} dG$$

$$d\psi = \frac{a^2\phi^3 + G^2}{2a^2\phi^4 - \phi G^2} d\phi - \frac{G}{2a^2\phi^3 - G^2} dG$$

$$\psi = \frac{1}{2} ln(2a^2\phi^3 - G^2) + f(\phi)$$

$$\psi = ln\left(\frac{1}{\phi}(2a^2\phi^3 - G^2)^{1/2}\right)$$

$$\psi_\phi = \frac{3a^2\phi^2}{2a^2\phi^3 - G^2} + f'(\phi) = \frac{a^2\phi^3 + G^2}{2a^2\phi^4 - G^2}$$

$$f'(\phi) = \frac{-2a^2\phi^3 + G^2}{2a^2\phi^4 - \phi G^2} = -\frac{1}{\phi}$$

$$f(\phi) = -ln(\phi)$$

# Integrate once to get to the general solution

$$\psi = ln\left(\frac{1}{\phi}(2a^2\phi^3 - G^2)^{1/2}\right)$$

Let

$$\psi = ln\left(C_1^{1/2}\right)$$

$$C_1 = \frac{1}{\phi}(2a^2\phi^3 - G^2)^{1/2}$$

$$C_1^{1/2} = 2a^2\phi - \frac{G^2}{\phi^2}$$

$$G = \pm(2a^2\phi^3 - C_1)^{1/2}$$

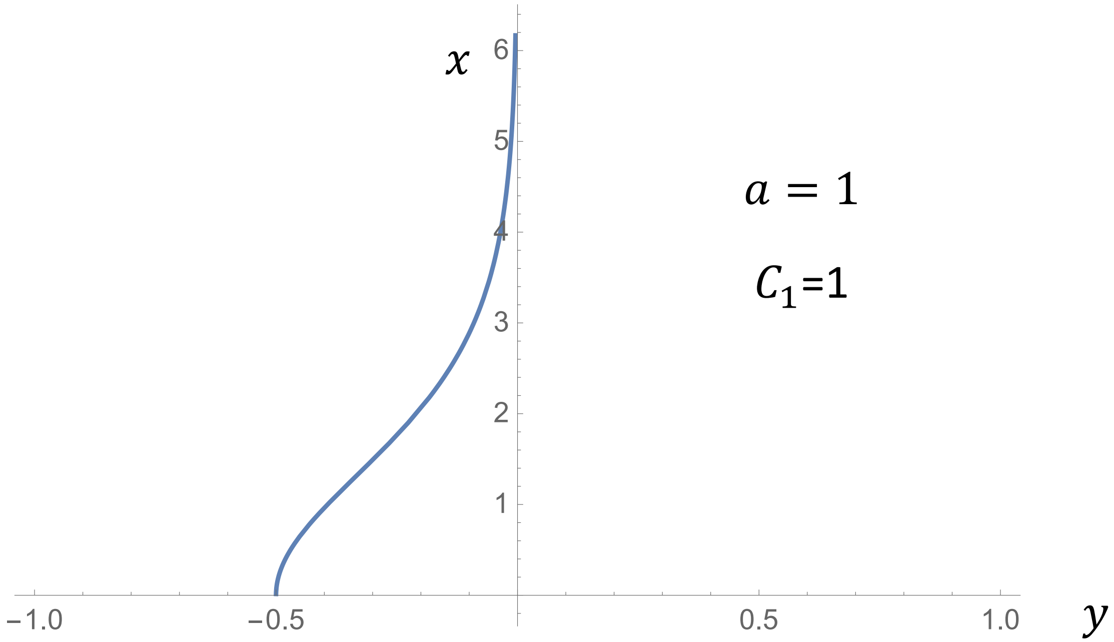
$$y_x = \pm(2a^2y^3 - C_1)^{1/2}$$

$$dx = \frac{dy}{\pm(2a^2y^3 - C_1)^{1/2}}$$

$$x = \int \frac{dy}{\pm(2a^2y^3 - C_1)^{1/2}} + C_2$$

## Mathematica will integrate the solution

$$x = \int \frac{dy}{\pm(2a^2y^3 - C_1y^2)^{1/2}} + C_2 = \pm 2y \frac{\sqrt{-C_1 + 2a^2y}}{\sqrt{C_1}\sqrt{y^2(-C_1 + 2a^2y)}} ArcTanh\left(\frac{\sqrt{-C_1 + 2a^2y}}{\sqrt{C_1}}\right)$$




$a = 1$

$C_1$=1

# Search for an invariant solution under the dilation group

$$\tilde{x} = e^{a}x$$

$$\tilde{y} = e^{-2a}y$$

$$X^b = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$$

Assume an invariant solution of the form

$$\Psi = y - f(x) = 0$$

$$X^b\Psi = -x\frac{df}{dx} - 2y = 0$$

$$x\frac{df}{dx} = -2f$$

$$f = \frac{C_3}{x^2}$$

Substitute $f$ into the original equation to determine the constant

$$yy_{xx} - (y_x)^2 - a^2y^3 =$$

$$\frac{C_3}{x^2}\left(6\frac{C_3}{x^4}\right) - 4\frac{C_3}{x^6} - a^2\left(\frac{C_3}{x^2}\right)^3 =$$

$$\frac{C_3}{x^2}\left(6\frac{C_3}{x^4}\right) - 4\frac{C_3^2}{x^6} - a^2\left(\frac{C_3}{x^2}\right)^3 =$$

$$2\frac{C_3^2}{x^6} - a^2\left(\frac{C_3}{x^2}\right)^3 = 0$$

$$C_3 = \frac{2}{a^2}$$

Invariant solution

$$y = \frac{2}{a^2x^2} \qquad y_x = -2^{1/2}ay^{3/2}$$

$$G = \mp 2^{1/2}a\phi^{3/2}$$

Express the invariant solution in terms of $\phi$ and $G$

$$y = \frac{2}{a^2x^2}$$

$$\frac{1}{x} = \pm\left(\frac{a^2y}{2}\right)^{1/2}$$

$$y_x = -\frac{4}{a^2x^3}$$

$$\frac{1}{x} = -\left(\frac{a^2y_x}{4}\right)^{1/3}$$

$$-\left(\frac{a^2y_x}{4}\right)^{1/3} = \pm\left(\frac{a^2y}{2}\right)^{1/2}$$

$$y_x = \mp\frac{4}{a^2}\left(\frac{a^2y}{2}\right)^{3/2} = \mp 2^{1/2}ay^{3/2}$$

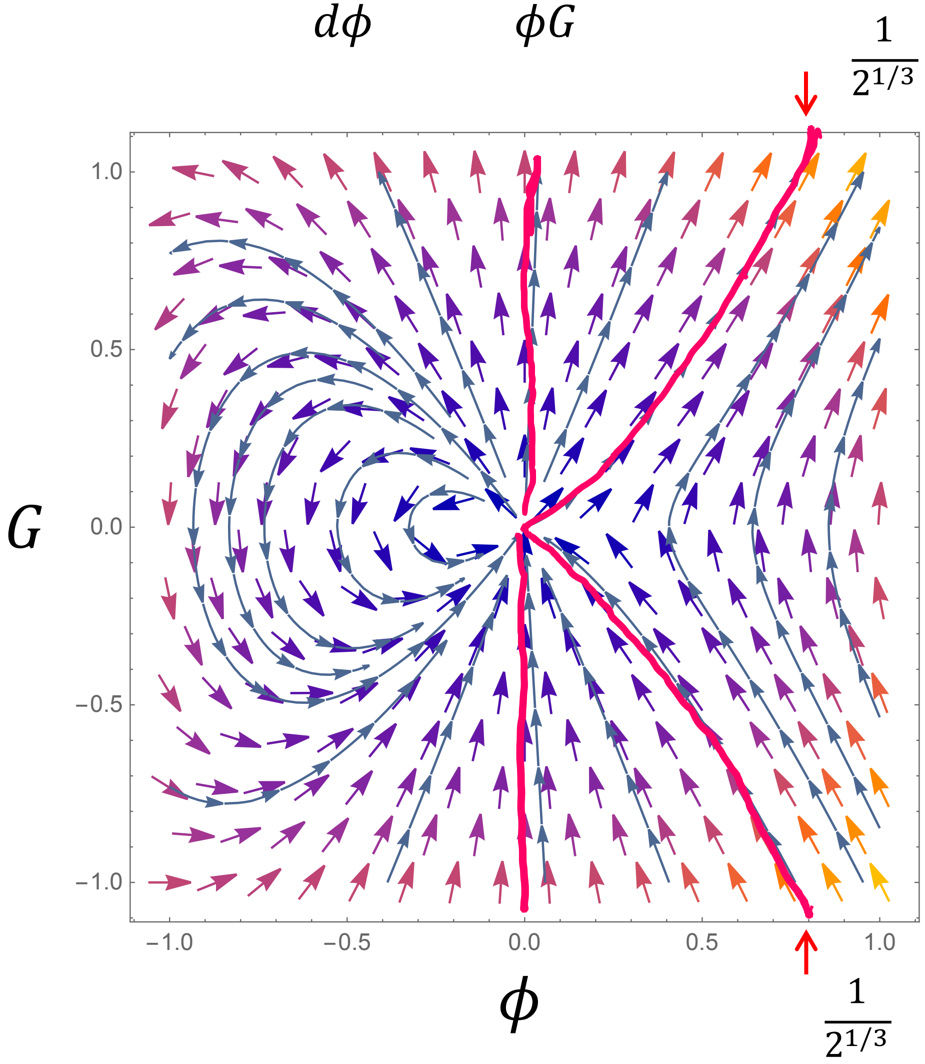
$$\frac{dG}{d\phi} = \frac{G^2 + a^2\phi^3}{\phi G}$$

$$-2^{\frac{1}{2}}\frac{3}{2}a\phi^{\frac{1}{2}} = -\frac{2a^2\phi^3 + a^2\phi^3}{2^{1/2}a\phi^{5/2}}$$

$$-2^{\frac{1}{2}}\frac{3}{2}a\phi^{\frac{1}{2}} = -\frac{3a\phi^{1/2}}{2^{1/2}}$$

# Phase Portrait

$$\frac{dG}{d\phi} = \frac{G^2 + a^2\phi^3}{\phi G}$$




$a = 1$

Invariant solution

$$G = \mp 2^{1/2} a \phi^{3/2}$$

What about the translation group?
Is there an invariant curve?

$$\Psi = y - f(x) = 0$$

$$X^a = \frac{\partial}{\partial x}$$

$$X^a \Psi = -\frac{df}{dx} = 0$$

$$f = C_4$$

Substitute $f$ into the original equation to determine the constant

$$y y_{xx} - (y_x)^2 - a^2 y^3 = 0$$
$$0 - 0 - a^2 (C_4)^3 = 0$$
$$C_4 = 0$$

Invariant solution

$$y = 0$$

or

$$\phi = 0$$