NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1256

THE BOUNDARY LAYERS IN FLUIDS WITH LITTLE FRICTION

By H. Blasius

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INTRODUCTION

1. The vortices forming in flowing water behind solid bodies are not represented correctly by the solution of the potential theory nor by Helmholtz's jets. Potential theory is unable to satisfy the condition that the water adheres at the wetted bodies, and its solutions of the fundamental hydrodynamic equations are at variance with the observation that the flow separates from the body at a certain point and sends forth a highly turbulent boundary layer into the free flow. Helmholtz's theory attempts to imitate the latter effect in such a way that it joins two potential flows, jet and still water, nonanalytical along a stream curve. The admissibility of this method is based on the fact that, at zero pressure, which is to prevail at the cited stream curve, the connection of the fluid, and with it the effect of adjacent parts on each other, is canceled. In reality, however, the pressure at these boundaries is definitely not zero, but can even be varied arbitrarily. Besides, Helmholtz's theory with its potential flows does not satisfy the condition of adherence nor explain the origin of the vortices, for in all of these problems, the friction must be taken into account on principle, according to the vortex theorem.

When a cylinder is dipped into flowing water, for example, the flow corresponds, qualitatively, to the known potential, but as the water adheres to the cylinder, a boundary layer forms on the cylinder wall in which the velocity rises from zero at the wall to the value given by the potential flow. In this boundary layer, the friction plays an essential part because of the marked velocity difference; on it also depends the extent of the velocity-decreasing wall effect, which must be conveyed by shearing forces into the fluid, that is, the

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thickening of the boundary layer. That the outer flow separates at a certain place, and that the water, set in violent rotation at the boundary, leads into the open, must be explainable from the processes in the boundary layer.

The exact treatment of this question was undertaken originally by Prandtl (Verhandlungen des intern. Math. Kongress, 1904). This explanation of the separation is repeated below. Since the integration of the hydrodynamic equations with friction is a too difficult problem, he assumed the internal friction as being small, but retained the condition of adherence at the boundary surface. In the present report, several problems, based upon the simplified hydrodynamic equations resulting from Prandtl's article, are worked out. They refer to the formation of boundary layers on solid bodies and the origin of separation of jets from these boundary layers suggested by Prandtl. The writer wishes to thank Prof. L. Prandtl for the suggestion of this article.

2. The constant of the internal friction is assumed small as in Prandtl's report. The boundary layers then become correspondingly thin; the fluid maintains its normal (potential) velocity up to near the boundary surface. Nevertheless, the decrease in velocity to value zero, and, as the calculation will show, the separation in this boundary layer must, naturally, continue, and so the potential flow is not completely regained, even at arbitrarily little friction; rather the separation and the transformation of the flow effected through it behind the body must prevail even at arbitrarily small friction.

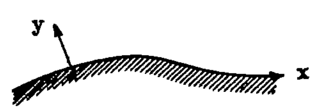

Figure 1

The procedure is limited to two-dimensional flow and coordinates parallel and at right angles to the boundary (arc length and normal distance). In spite of its curvature, the type of the basic equations in the narrow space of the boundary does not differ perceptibly from that for rectangular coordinates. With ϵ as order of magnitude of the boundary-layer thickness

$$\frac{\partial u}{\partial y} \sim \frac{1}{\epsilon}, \quad \frac{\partial^2 u}{\partial y^2} \sim \frac{1}{\epsilon^2}$$

as the velocity u over this distance is to increase from zero to normal values; u, $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, and $\frac{\partial^2 u}{\partial x^2}$ have normal value; from the equation of continuity follows then $\frac{\partial v}{\partial y} \sim 1$, and by integration, $v \sim \epsilon$. The terms in the fundamental equations obtain then the following order of magnitude[1]

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$1 \qquad 1\cdot 1 \qquad \epsilon\cdot\frac{1}{\epsilon} \qquad 1 \qquad 1 \qquad \frac{1}{\epsilon^2}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + k\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\epsilon \qquad 1\cdot \epsilon \qquad \epsilon\cdot 1 \qquad \qquad \epsilon \qquad \frac{1}{\epsilon}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$1 \qquad 1$$

The friction gains influence when it is put at $k \sim \epsilon^2$; this gives the relationship between boundary-layer thickness and smallness of friction constant. In the first equation, the term $\partial^2 u/\partial x^2$ cancels out; in the second equation, only $\partial p/\partial y \sim \epsilon$ or, when allowing for the coordinate curvature, ~ 1 remains[1]. In both cases,

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[1]Allowance for the curvature of the coordinates produces, as is apparent when reforming the differential quotients, only in the second equation a not-to-be-neglected term $\rho u^2/r$ if r is the radius of curvature. This term is of the order of magnitude, unity.

the effect of the pressure on y is to be disregarded since, in the narrow space of the boundary layer, the integration of $\partial p/\partial y$ can, at the most, produce pressure differences of the order of magnitude ϵ^2 or ϵ, or, in other words, pressure and pressure difference $\partial p/\partial x$ are independent of y, hence, are "impressed" by the outer flow on the boundary layer. The velocity of the outer flow next to the boundary layer is denoted by $\bar{u}$ and is to be regarded solely as function of x because the really existing dependence on y, when compared with the substantial variations in the boundary layer itself, can be ignored in the sense of the foregoing omissions; v is accordingly $\sim \epsilon = \sqrt{k}$, hence becomes zero with k. The remaining fundamental equations for the boundary layers are then:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x}\right) + k\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Boundary conditions are

for $y = 0$: $u = 0$ $v = 0$

for $y = \infty$: $u = \bar{u}$

These equations establish, to a certain extent, a basis for a special mechanics of boundary layers, since the outer flow enters only in "impressed" manner.

3. The qualitative explanation for the separation of flow according to Prandtl is as follows: the pressure difference, and with it the acceleration, is, apart from the friction term, constant throughout the boundary layer, but the velocity near the wall is lower. As a result, the velocity here drops sooner below the value

zero for pressure rise than outside, thus giving rise to return flow and jet formation, as indicated by the velocity profiles in the figure below.

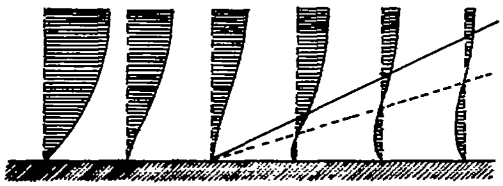


Figure 2

The region of separation itself is therefore characterized by

$$\frac{\partial u}{\partial y} = 0 \quad \text{for} \quad y = 0$$

This explanation does not work like the Helmholtz jet theory with an ad hoc assumption, but only with the concepts forming the basis of the present hydrodynamic equations. The stream line, which bounds the separated part of the flow, departs at a certain angle from the area of separation since the stream function ψ develops around the separation point $[x]$ in the following manner:

$$\psi = c_1 y^3 + c_2 (x - [x]) y^2$$

As a less important effect, it is to be foreseen that, as a consequence of the stagnation of water effected by adhesion, the flow is pushed away from the body. Through this and the reformed flow aft of the body, the flow upstream from the body is, of course, affected also, so that the assumption of potential flow is insufficient for quantitative accuracy of results and must be replaced by experimental recording of the pressure distribution.

I. BOUNDARY LAYER FOR THE STEADY MOTION ON A FLAT PLATE IMMERSED PARALLEL TO THE STREAM LINES

The flow proceeds parallel to the x-axis. The plate starts in the origin of the coordinates and lies on the positive x-axis.

In this very elementary case, there is no pressure difference; hence, no separation is expected. However, the calculation is carried out to illustrate the mode of calculation to be used later. The fundamental equations read:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = k\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The equation of continuity is integrated by introducing the stream function ψ:

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

Boundary conditions are:

$$\text{for } y = 0: \quad u = 0, \quad v = 0$$

$$\text{for } y = \infty: \quad u = \bar{u}, \leftarrow \text{constant}$$

1. According to the principle of mechanical similitude, the equations can be simplified when a similitude transformation converting differential equations and boundary conditions are known: multiplying x, y, u, v, ψ by the factors x_o, y_o, u_o, v_o, and ψ_o results in

$$\frac{\rho u_o}{x_o} = \frac{k}{{y_o}^2}; \quad v_o = \frac{u_o y_o}{x_o}; \quad \psi_o = u_o y_o; \quad u_o = \bar{u}$$

as conditions that the problem and its solution are transformed, and that, through the transformation, ρ, k, $\bar{u} = 1$ are created. The four equations still leave a degree of freedom in the choice of the factors x_o, y_o, u_o, v_o, and ψ_o. The last three equations define the factors assumed by u, v, and ψ through the transformation; the first states that the desired solution of the problem transforms in itself, provided only that

$$\frac{\rho\bar{u}}{k}\frac{{y_o}^2}{x_o} = 1$$

or in other words, with consideration of the factors which u, v, and ψ assume, the condition can depend only on

$$\frac{\rho\bar{u}}{k}\frac{y^2}{x}$$

By this argument, the number of independent variables is reduced. Next

$$\xi = 1/2\sqrt{\frac{\rho\bar{u}}{k}} \cdot \frac{y}{\sqrt{x}}$$

$$\psi = \sqrt{\frac{k\bar{u}}{\rho}}\sqrt{x} \cdot \zeta$$

are introduced; ζ is then sole function of ξ and

$$u = 1/2\ \bar{u}\zeta'$$

$$v = 1/2\sqrt{\frac{k\bar{u}}{\rho}}\frac{1}{\sqrt{x}}\left(\xi\zeta' - \zeta\right)$$

Insertion in the differential equation gives

$$\zeta\zeta'' = -\ \zeta'''$$

Boundary conditions:

for $\xi = 0$: $\zeta' = 0$ $\zeta = 0$ from $u = 0$; $v = 0$;

for $\xi = \infty$: $\zeta' = 2$ from $u = \bar{u}$.

2. The integration of these and subsequent equations is effected by expansion in series: expansion in powers for $\xi = 0$, asymptotic approximations for $\xi = \infty$. The boundary conditions at both points being given, one and two integration constants, respectively, occur in the expansions. They are defined by the fact that both expansions must agree, at an arbitrary point in the function value ζ, to the first and second differential quotient. The agreement of all differential quotients is then assured by the differential equation.

3. Solution of the above equation by expansion in powers

$$\zeta\zeta'' = -\zeta'''$$

for $\xi = 0$ with the boundary conditions at this point

$$\zeta' = 0 \qquad \zeta = 0$$

is effected by

$$\zeta = \sum_{n=0}^{\infty} (-1)^n \frac{c_n \alpha^{n+1}}{(3n + 2)!} \xi^{3n+2}$$

which is so chosen that the coefficients c_n to be defined are whole positive numbers, which simplifies calculation. The factor α^{n+1} brings out the nature of entry of the integration constant; c_o, which otherwise would occur as such, can then be put as $c_o = 1$. The recursion formula for c_n reads

$$c_n = \sum_{\nu=0}^{n-1} \binom{3n-1}{3\nu} c_\nu c_{n-1-\nu}$$

The first of the thus computed coefficients are:

$$c_o = 1 \qquad c_1 = 1 \qquad c_2 = 11 \qquad c_3 = 375 \qquad c_4 = 27{,}897$$

$$c_5 = 3{,}817{,}137 \qquad {}^2c_6 = 865{,}874{,}115 \qquad c_7 = 298{,}013{,}289{,}795$$

On account of the convergence, the denominator $(3n + 2)!$ was used in the previous equations; ζ' and ζ'' are easily formed.

[2]The coefficients c_6 and c_7 in the original thesis are incorrect. This error has no effect until the fourth decimal.

4. There is an additive integration constant for ζ in the asymptotic approximation of ξ because

$$\text{for } \xi = \infty: \quad \zeta' = 2$$

hence,

$$\zeta = 2\xi + \text{const.} = 2\eta$$

so that η appears as new coordinate shifted toward ξ.

To compute a first correction ζ_1, put

$$\zeta = 2\eta + \zeta_1$$

which gives

$$2\eta\zeta_1'' = -\zeta_1'''$$

with the squares of the corrections disregarded, hence by integration:

$$\zeta_1 = \gamma \int_\infty^\eta d\eta \int_\infty^\eta e^{-\eta^2} d\eta = \gamma\eta \int_\infty^\eta e^{-\eta^2} d\eta + \frac{\gamma}{2} e^{-\eta^2}$$

$$\zeta_1' = \gamma \int_\infty^\eta e^{-\eta^2} d\eta \qquad \zeta_1'' = \gamma e^{-\eta^2}$$

The general procedure for computing the other terms is such that further minor corrections ζ_n are added and its squares disregarded. The result is a set of linear differential equations for ζ_n, the left, homogeneous side always the same; at the right, the error appears as "impressed force" which the sum of the preceding approximations, inserted in the differential equations, leaves.

5. The object is reached much quicker by the following argument: The differential equation for ζ_1

$$2\eta\zeta_1'' = -\zeta_1'''$$

arises from the original equation

$$\zeta\zeta'' = -\zeta'''$$

when the roughest approximation $\zeta = 2\eta$ is inserted at the left for ζ. Obviously, ζ has the least effect at this point, and the differential equation is then integrated as if ζ were known at this point.

$$\zeta = \int^{\eta} d\eta \int^{\eta} d\eta e^{-\int^{\eta} \zeta d\eta}$$

The three integration constants are contained in the arbitrary low limits. Putting $\zeta = 2\eta$ at the right gives ζ_1 at the left, but putting $\zeta = 2\eta + \zeta_1$ at the right gives

$$\zeta = \int^{\eta} d\eta \int^{\eta} d\eta e^{-\eta^2} e^{-\int^{\eta} \zeta_1 d\eta}$$

or with consideration to the boundary conditions

$$\zeta = 2\eta + \gamma \int_{\infty}^{\eta} d\eta \int_{\infty}^{\eta} d\eta e^{-\eta^2}\left(1 - \int_{\infty}^{\eta} \zeta_1 d\eta\right)$$

$$= 2\eta + \zeta_1 - \gamma \int_{\infty}^{\eta} d\eta \int_{\infty}^{\eta} d\eta e^{-\eta^2} \int_{\infty}^{\eta} \zeta_1 d\eta$$

Hence, the second asymptotic approximation

$$\zeta_2 = -\gamma^2 \int_{\infty}^{\eta} d\eta \int_{\infty}^{\eta} d\eta \cdot e^{-\eta^2} \int_{\infty}^{\eta} d\eta \int_{\infty}^{\eta} d\eta \int_{\infty}^{\eta} e^{-\eta^2} d\eta$$

By partial integration

$$\zeta_2'' = -\frac{\gamma^2}{4}(2\eta^2 + 1)e^{-\eta^2} \int_{\infty}^{\eta} e^{-\eta^2} d\eta - \frac{\gamma^2}{4}\eta e^{-2\eta^2}$$

$$\zeta_2' = \frac{\gamma^2}{4}\eta e^{-\eta^2} \int_{\infty}^{\eta} e^{-\eta^2} d\eta - \frac{\gamma^2}{4}\left\{\int_{\infty}^{\eta} e^{-\eta^2} d\eta\right\}^2 + \frac{\gamma^2}{8} e^{-2\eta^2}$$

$$\zeta_2 = -\frac{3\gamma^2}{8} e^{-\eta^2} \int_{\infty}^{\eta} e^{-\eta^2} d\eta - \frac{\gamma^2}{4}\eta \left\{\int_{\infty}^{\eta} e^{-\eta^2} d\eta\right\}^2 + \frac{\gamma^2}{2} \int_{\infty}^{\eta} e^{-2\eta^2} d\eta$$

6. A general statement about such integrations reads as follows: According to the formula

$$\int_\infty^\eta e^{-\eta^2}\eta^n d\eta = -\frac{1}{2}\eta^{n-1}e^{-\eta^2} + \frac{n-1}{2}\int_\infty^\eta e^{-\eta^2}\eta^{n-2}d\eta$$

to be gained by partial integration, each integral of this form can be reduced to the functions $e^{-\eta^2}$ and $\int_\infty^\eta e^{-\eta^2}d\eta$ multiplied by powers of η. After several such integrals are obtained, the innermost is transformed, if necessary, in the indicated manner. The integral $e^{-\eta^2}$ or $\int_\infty^\eta e^{-\eta^2}d\eta$ multiplied by powers, appears then below the penultimate integral sign. The former gives no new difficulty; the latter can be reduced by partial integration to the two functions $e^{-\eta^2}$ and $\int_\infty^\eta e^{-\eta^2}d\eta$

$$\int_\infty^\eta d\eta \int_\infty^\eta e^{-\eta^2}d\eta = \eta \int_\infty^\eta e^{-\eta^2}d\eta + \frac{1}{2}e^{-\eta^2}$$

$$\int_\infty^\eta \eta d\eta \int_\infty^\eta e^{-\eta^2}d\eta = \frac{1}{2}\eta^2 \int_\infty^\eta e^{-\eta^2}d\eta - \frac{1}{4}\int_\infty^\eta e^{-\eta^2}d\eta + \frac{1}{4}\eta e^{-\eta^2}$$

$$\int_\infty^\eta \eta^2 d\eta \int_\infty^\eta e^{-\eta^2}d\eta = \frac{1}{3}\eta^3 \int_\infty^\eta e^{-\eta^2}d\eta + \frac{1}{6}\eta^2 e^{-\eta^2} + \frac{1}{6}e^{-\eta^2} \quad \text{and so forth.}$$

If, as above, the integral can be quadratic in $e^{-\eta^2}$, four types must be distinguished:

$$e^{-2\eta^2}, \qquad e^{-\eta^2}\int_\infty^\eta e^{-\eta^2}d\eta, \qquad \left\{\int_\infty^\eta e^{-\eta^2}d\eta\right\}^2, \qquad \int_\infty^\eta e^{-2\eta^2}d\eta$$

multiplied by powers of η. The first and fourth types give nothing new. Partial integration provides for the second the formula

$$\int_\infty^\eta e^{-\eta^2}d\eta \int_\infty^\eta e^{-\eta^2}d\eta = \left\{\int_\infty^\eta e^{-\eta^2}d\eta\right\}^2 - \int_\infty^\eta e^{-\eta^2}d\eta \int_\infty^\eta e^{-\eta^2}d\eta$$

or

$$\int_{\infty}^{\eta} e^{-\eta^2} d\eta \int_{\infty}^{\eta} e^{-\eta^2} d\eta = \frac{1}{2}\left\{\int_{\infty}^{\eta} e^{-\eta^2} d\eta\right\}^2$$

and

$$\int_{\infty}^{\eta} \eta e^{-\eta^2} d\eta \int_{\infty}^{\eta} e^{-\eta^2} d\eta = -\frac{1}{2} e^{-\eta^2} \int_{\infty}^{\eta} e^{-\eta^2} d\eta + \frac{1}{2} \int_{\infty}^{\eta} e^{-2\eta^2} d\eta$$

$$\int_{\infty}^{\eta} \eta^2 e^{-\eta^2} d\eta \int_{\infty}^{\eta} e^{-\eta^2} d\eta = -\frac{1}{2}\eta e^{-\eta^2} \int_{\infty}^{\eta} e^{-\eta^2} d\eta + \frac{1}{4}\left\{\int_{\infty}^{\eta} e^{-\eta^2} d\eta\right\}^2 - \frac{1}{8} e^{-2\eta^2}$$

and so forth.

Likewise for the third type

$$\int_{\infty}^{\eta}\left\{\int_{\infty}^{\eta} e^{-\eta^2} d\eta\right\}^2 d\eta = \eta\left\{\int_{\infty}^{\eta} e^{-\eta^2} d\eta\right\}^2 + e^{-\eta^2} \int_{\infty}^{\eta} e^{-\eta^2} d\eta - \int_{\infty}^{\eta} e^{-2\eta^2} d\eta$$

$$\int_{\infty}^{\eta}\left\{\int_{\infty}^{\eta} e^{-\eta^2} d\eta\right\}^2 \eta d\eta = \frac{1}{2}\eta^2\left\{\int_{\infty}^{\eta} e^{-\eta^2} d\eta\right\}^2 + \frac{1}{2}\eta e^{-\eta^2} \int_{\infty}^{\eta} e^{-\eta^2} d\eta - \frac{1}{4}\left\{\int_{\infty}^{\eta} e^{-\eta^2} d\eta\right\}^2 + \frac{1}{8} e^{-2\eta^2}$$

and so forth.

Since no new types for integrals are introduced by these formulas, the indicated tables of formulas govern all integrals in which $e^{-\eta^2}$ occurs no more than twice. Any number of successive integrations over such functions are possible; the powers of η involved are unrestricted. The formulas for ζ_2 in section 5 were obtained by this method. These integrations will be met again later. With the type of integration results thus known, the calculations can be made by utilizing a formula with indeterminant coefficients.

7. With this differential equation, it is possible also to define the error that afflicts the present solution as a result of the effected

omissions. It is easily verified that 2η, $2\eta + \zeta_1$, $2\eta + \zeta_1 + \zeta_2$ remain below the true value of ζ. An upper limit can also be found by employing the previously given (see Section 5) form, somewhat modified

$$\zeta = 2\eta + \gamma \int_\infty^\eta d\eta \int_\infty^\eta d\eta e^{-\eta^2} e^{-\int_\infty^\eta (\zeta - 2\eta) d\eta}$$

of computing a finer from a rougher approximation: a rather arbitrarily chosen upper limit, such as the first term of the semiconvergent expansion of ζ_1, for instance, is entered for $\zeta - 2\eta$, thus

$$\zeta - 2\eta < \frac{\gamma}{4} \frac{e^{-\eta^2}}{\eta^2}$$

and an asymptotically finer upper limit for ζ'', ζ', ζ is computed from this assumption. It is insured so long as the latter remains below the assumed one. The calculation gives (according to the general formula)

$$\int_\eta^\infty e^{-\eta^2} \frac{d\eta}{\eta^\nu} = \frac{1}{2} \frac{e^{-\eta^2}}{\eta^{\nu+1}} - \frac{\nu + 1}{2} \int_\eta^\infty e^{-\eta^2} \frac{d\eta}{\eta^{\nu+2}}$$

$$\left. < \frac{1}{2} \frac{e^{-\eta^2}}{\eta^{\nu+1}} \right]$$

$$\int_\eta^\infty (\zeta - 2\eta) d\eta < \frac{\gamma}{8} \frac{e^{-\eta^2}}{\eta^3} = \vartheta$$

$$e^{-\int_\infty^\eta (\zeta - 2\eta) d\eta} < e^{\vartheta}$$

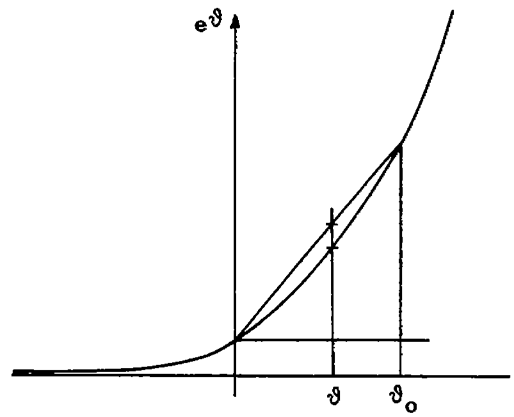


Figure 3

The upper limit for e^{ϑ} is then found according to the above figure as

$$e^{\vartheta} < 1 + \sigma\vartheta \qquad \sigma = \frac{e^{\vartheta_o} - 1}{\vartheta_o}$$

where ϑ_o is the highest existing value of ϑ, hence corresponds to the value of the coordinates for which ζ is to be computed. It results in

$$\zeta < 2\eta + \gamma \int_{\eta}^{\infty} d\eta \int_{\eta}^{\infty} d\eta e^{-\eta^2}\left(1 + \frac{\sigma\gamma}{8} \frac{e^{-\eta^2}}{\eta^3}\right)$$

$$< 2\eta + \zeta_1 + \frac{\sigma\gamma^2}{128} \frac{e^{-2\eta^2}}{\eta^5}$$

similarly for ζ' and ζ'':

$$\zeta' > 2 + \zeta_1 - \frac{\sigma\gamma^2}{32}\frac{e^{-2\eta^2}}{\eta^4}$$

$$\zeta'' < \zeta_1'' + \frac{\sigma\gamma^2}{8}\frac{e^{-2\eta^2}}{\eta^3}$$

A more accurate execution of the integrals affords a more accurate result.

8. The connection of the two developments and the determination of the integration constants (α, γ, and $\eta - \xi$) is as follows: To separate the integration constant α,

$$Z = \frac{1}{\sqrt[3]{\alpha}}\zeta, \qquad X = \sqrt[3]{\alpha}\,\xi$$

is introduced in the power development (3), which results in

$$Z = \frac{1}{\sqrt[3]{\alpha}}\zeta = \sum_{n=0}^{\infty}(-1)^n\frac{c_n}{(3n+2)!}X^{3n+2}$$

$$\frac{dZ}{dX} = \frac{1}{\alpha^{2/3}}\zeta' = \sum_{n=0}^{\infty}(-1)^n\frac{c_n}{(3n+1)!}X^{3n+1}$$

$$\frac{d^2Z}{dX^2} = \frac{1}{\alpha}\zeta'' = \sum_{n=0}^{\infty}(-1)^n\frac{c_n}{(3n)!}X^{3n}$$

The displacement of η relative to ξ is expressed by introducing the integration constant β

$$\sqrt[3]{\alpha}\cdot\eta = X - \beta$$

The formulas are completed by inserting

$$\zeta = 2\eta + \zeta_1 + \zeta_2$$

$$\zeta' = 2 + \zeta_1' + \zeta_2'$$

$$\zeta'' = \zeta_1'' + \zeta_2''$$

from the asymptotic approximation (4) and (5). A graph is made for Z and its differential quotients from which the following values are quoted:

X =	0	0.8	1.0	1.2	1.4	1.9	2.0	2.05	2.1
Z =	0	0.317	0.492	0.701	0.938	1.63	1.79	1.8561	1.94
$\frac{dZ}{dX}$ =	0	.784	.961	1.121	1.257	1.50	1.53	1.5479	1.56
$\frac{d^2Z}{dX^2}$ =	1	—	—	—	.639	.34	.28	.2582	.23

The terms of the power series are computed up to $\frac{c_7 X^{23}}{23!}$; further terms are extrapolated, in part, from the difference series of the logarithms of the coefficients. The location of the asymptotes is already quite apparent in figure 4.

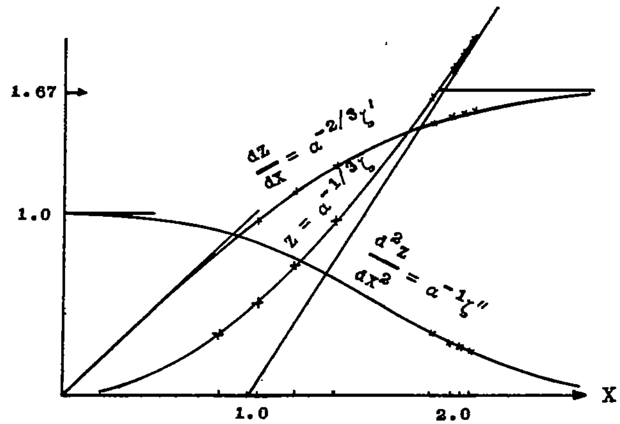


Figure 4

Since owing to $\zeta = 2\eta$ asymptotic, $Z = \frac{1}{\alpha^{2/3}}(X - \beta)$, rough approximation values can already be read for α and β: $\alpha = 1.30$, $\beta = 0.96$, with $X = 2.05$ as connecting coordinate for $\eta = 1.00$. The corresponding values of $\frac{d^2Z}{dX^2}$ and ζ'' give for γ: $\gamma = 0.92$. The calculation is more rigorous when α, γ, and the connecting coordinate related to $\eta = 1$ are varied by minor corrections and these then computed from linear equations. To judge the accuracy, it is stated that our calculation for X = 2.05 gave

$$Z = 1.8561, \qquad \frac{dZ}{dX} = 1.5479, \qquad \frac{d^2Z}{dX^2} = 0.2582, \qquad \frac{d^3Z}{dX^3} = -0.479$$

where the fourth decimal is no longer certain. Asymptotic approximation for $\eta = 1$ gives (using Markoff's $\int_t^\infty e^{-t^2}dt$)

$$\zeta = 2 + 0.04454 \cdot \gamma + \begin{Bmatrix} 0.00012 \\ 0.00106 \end{Bmatrix} \gamma^2$$

$$\zeta' = 2 - 0.13940 \cdot \gamma - \begin{Bmatrix} 0.00076 \\ 0.00423 \end{Bmatrix} \gamma^2$$

$$\zeta'' = 0.36788 \cdot \gamma + \begin{Bmatrix} 0.00462 \\ 0.01692 \end{Bmatrix} \gamma^2$$

the top numerals in the $\{\}$ stemming from ζ_2, the bottom numerals from the upper limit defined in (7). (The latter is, as stated before, rather rough.) The "temporary assumption" about the upper limit of ζ gives: $\zeta < 2 + 0.092\ \gamma$. The upper limits are therefore guaranteed (reference 7). Hence, the result

$\alpha = 1.3266, \qquad X = 2.0494, \qquad \gamma = 0.9227, \qquad (\beta = 0.9508)$

It can be safely assumed that α ranges between 1.326 and 1.327.

9. From it, it can be computed, for example, what drag a plate of width b and length l is subjected to when dipped parallel to the flow lines into a flow moving at velocity $\bar{u}$. The drag per unit of surface is

$$X_y = k\frac{\partial u}{\partial y} = k\frac{1}{2}\bar{u}\zeta''\frac{1}{2}\sqrt{\frac{\rho\bar{u}}{k}}\,\frac{1}{\sqrt{x}}$$

$$= \frac{\alpha}{4}\sqrt{k\rho\bar{u}^3}\,\frac{1}{\sqrt{x}}$$

Integration over the plate gives

$$b \cdot \int_0^l X_y\, dx = \frac{\alpha}{2} b \sqrt{k\rho l\bar{u}^3}$$

hence, when the water flows at both sides of the plate

$$\text{drag} = 1.327 \cdot b \sqrt{k\rho l\bar{u}^3}$$

II. CALCULATION OF REGION OF SEPARATION BEHIND A BODY DIPPED INTO A UNIFORM FLOW

1. The following problem is treated: In an otherwise parallel flow, a cylindrical body is immersed symmetrically to the direction of flow. The boundary-layer coordinates are computed from the point of division of the flow. The quantity $\overline{u}$ is expanded as function of x in a power series. For the integration of the fundamental equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \overline{u}\frac{\partial \overline{u}}{\partial x} + \frac{k}{\rho}\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\overline{u} = \sum_{l=0}^{\infty} q_l x^{2l+1}$$

the formula

$$\psi = \sum_{l=0}^{\infty} \chi_l(y) x^{2l+1}$$

is used, with due regards to the symmetrical conditions for the stream function ψ; u and v are obtained then by differentiation.

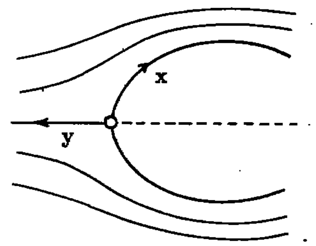


Figure 5

Consistent with the general boundary conditions, the functions $\chi_l(y)$ must then satisfy the boundary conditions

$$\chi_l' = 0 \qquad \chi_l = 0 \qquad \text{for } y = 0$$

$$\chi_l' = q_l \qquad \text{for } y = \infty$$

hence

$$\chi_l = q_l y + r_l$$

r_l is the constant of integration. From insertion in the first fundamental equation, the differential equations for χ are obtained as:

$$\sum_{\lambda=0}^{l}(2\lambda + 1)(\chi_\lambda'\chi_{l-\lambda}' - \chi_\lambda\chi_{l-\lambda}'') = \sum_{\lambda=0}^{l}(2\lambda + 1)q_\lambda q_{l-\lambda} + \frac{k}{\rho}\chi_l'''$$

which for $l = 0$ is quadratic, for $l > 0$ linear in the χ_l function to be defined. This equation can, like the preceding problem, be solved by expanding $y = 0$ in powers, for $y = \infty$ approximating asymptotically and joining both. Subsequently, it is shown that the asymptotic approximation can be omitted, since the power series already identifies the asymptote and therefore the integration constant with sufficient accuracy. The calculation is restricted to χ_0 and χ_1, that is, the first and third powers of x. Because, since the corresponding coefficients q_o and q_1 in $\overline{u}$ already indicate a first increasing, then decreasing velocity – the case, in which presumably separation occurs, is characterized by $q_o > 0$, $q_1 < 0$ – the type of pressure distribution required in the introduction (3) is already supplied by the first two powers; hence, it is to be expected that χ_o and χ_1, even though not quantitatively exact, already represent the effect of the separation. In one of the problems treated in similar manner later on the next approximation was also computed; and it substantiated the admissibility of the limitation to the first two powers of x.

2. The equations for χ_o and χ_1 are

$$\chi_o'^2 - \chi_o\chi_o'' = q_o^2 + \frac{k}{\rho}\chi_o'''$$

$$\chi_o'\chi_1' - \chi_o\chi_1'' + 3(\chi_1'\chi_o' - \chi_1\chi_o'') = 4q_oq_1 + \frac{k}{\rho}\chi_1'''$$

The manner of entry of q_o, q_1, k, ρ can be established by mechanical similarity. Here also, the first two terms indicate universal significance in some respects. Hence, writing

$$\overline{u} = q_o x \pm q_1 x^3 \qquad \psi = \chi_o x \pm \chi_1 x^3$$

and introducing the following quantities

$$\xi = \sqrt{\frac{q_1}{q_o}}\, x \qquad \eta = \sqrt{\frac{\rho q_o}{2k}}\, y \qquad \zeta_o = \sqrt{\frac{2\rho}{kq_o}}\, \chi_o \qquad \zeta_1 = \sqrt{\frac{2\rho q_o}{kq_1^2}}\, \chi_1$$

for x, y, χ_o, χ_1 gives

$$\overline{u} = \sqrt{\frac{q_o^3}{q_1}}(\xi \pm \xi^3)$$

$$\psi = \sqrt{\frac{kq_o^2}{2\rho q_1}}(\zeta_o \xi \pm \zeta_1 \xi^3)$$

$$u = \frac{1}{2}\sqrt{\frac{q_o^3}{q_1}}(\zeta_o' \xi \pm \zeta_1' \xi^3) \quad \text{etc.}$$

ζ_o and ζ_1 satisfy, as functions of η, the differential equations

$$\zeta_o'^2 - \zeta_o \zeta_o'' = 4 + \zeta_o'''$$

$$4\zeta_o' \zeta_1' - 3\zeta_o'' \zeta_1 - \zeta_o \zeta_1'' = 16 + \zeta_1'''$$

Boundary conditions

for $\eta = 0$:	$\zeta_o = 0$	$\zeta_o' = 0$	$\zeta_1 = 0$	$\zeta_1' = 0$
for $\eta = \infty$:		$\zeta_o' = 2$		$\zeta_1' = 2$

3. For ζ_o the power series

$$\zeta = \sum_{\mu=2}^{\infty} \frac{\alpha^{\mu+1} b_\mu}{\mu!} \eta^\mu$$

is entered.

Insertion in the differential equation gives:

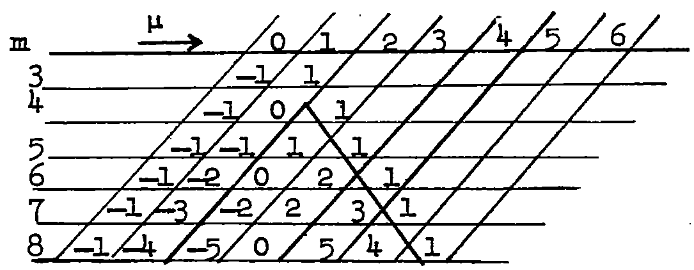
b_2 arbitrarily = 1, since α already is integration constant.

$\alpha^4 \beta_3 = -4$; since, in the formula of the integration constant α, no allowance was made for the homogeneity of the equation for ζ_o, α appears again in this equation.

$b_4 = 0$; the curvature of the velocity profile does not change, at first, since the friction in its effect is two terms ahead of the inertia; starting from $m = 5$, it is

$$b_m = \sum_{\mu=2}^{m-3} \left[\binom{m-3}{\mu-1} - \binom{m-3}{\mu} \right] b_\mu b_{m-1-\mu}$$

The coefficients of these recurrence formulas can, like all numbers combined this way from binomial coefficients, be computed from a diagram similar to Pascal's triangle, whose start is the following:



and in which each term is the sum of those above it. Only the framed-in portion, consistent with the foregoing limits of sums, is counted.

The first 13 coefficients are

$$b_2 = 1 \qquad b_3 = -\frac{4}{\alpha^4} \qquad b_4 = 0$$

$$b_5 = 1 \qquad b_6 = 2b_3 \qquad b_7 = 2b_3^{\,2}$$

$$b_8 = -1 \qquad b_9 = -4b_3 \qquad b_{10} = -16b_3^{\,2}$$

$$b_{11} = 27 - 16b_3^{\,3} \qquad b_{12} = 181b_3 \qquad b_{13} = 840b_3^{\,2}$$

4. Besides α, two more integration constants due to the asymptotic approximation are involved, which, as in the preceding problem, should join the computed power series. For the present

purposes (calculation of point of separation), it is, however, sufficient to know α, and, as stated before, it will be seen that α can already be computed with sufficient accuracy by means of the power series.

Put $Z_o = \frac{1}{\alpha}\zeta_o$, $H = \alpha\eta$ and plot $\frac{dZ_o}{dH}$ as function of H from the power series. $\frac{dZ_o}{dH}$ itself is still dependent on $b_3 = -\frac{4}{\alpha_4}$ and shall, for the correct value of α, approach the asymptote $\frac{dZ_o}{dH} = \frac{2}{\alpha^2}$. For other values of α, it approaches no asymptote at all, as a result of which as fig. 6 shows, the method for defining α is very sensitive. The value $\alpha = 1.515$ is obtained; the last cipher is no longer certain.

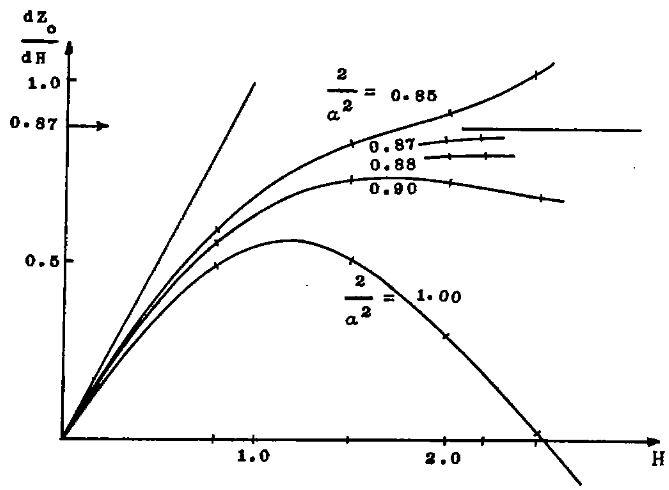


Figure 6

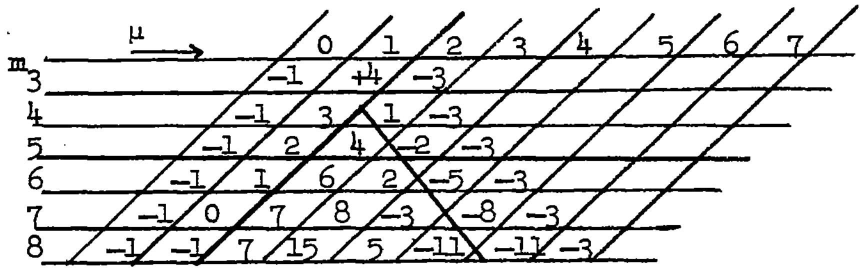
5. The calculation of ζ_1 by the above linear equation and the boundary conditions is effected in similar manner: power formula

$$\zeta_1 = \delta \sum_{\mu=2}^{\infty} \frac{c_\mu}{\mu!}\eta^\mu$$

$c_2 = 1$, since δ already is integration constant; $\delta c_3 = -16$; $c_4 = 0$; and for $m \geqq 5$:

$$c_m = \sum_{\mu=2}^{m-3} \left[-3\binom{m-3}{\mu-2} + 4\binom{m-3}{\mu-1} - \binom{m-3}{\mu} \right] \alpha^{\mu+1} b_\mu c_{m-1-\mu}$$

Here also the coefficients in these formulas can be computed from a diagram whose first line ($m = 3$) consists of the numbers -1, $+4$, -3, while the others follow by addition:



The first coefficients are

$$c_2 = 1; \quad \delta c_3 = -16; \quad c_4 = 0; \quad c_5 = 4\alpha^3; \quad c_6 = 6\alpha^3 c_3 - 8;$$

$$c_7 = -32c_3; \quad c_8 = 17\alpha^6; \quad c_9 = 30\alpha^6 c_3 - 224\alpha^3;$$

$$c_{10} = -576\alpha^3 c_3 - 256; \quad c_{11} = 2048c_3 + 294\alpha^9;$$

$$c_{12} = 783\alpha^9 c_3 - 5092\alpha^6; \quad c_{13} = -17392\alpha^6 c_3 + 59648\alpha^3;$$

$$c_{14} = 221952\alpha^3 c_3 - 315\alpha^{12} - 136192;$$

$$c_{15} = -11025\alpha^{12} c_3 - 1024000c_3 - 54864\alpha^9;$$

$$c_{16} = 174168\alpha^9 c_3 - 221296\alpha^6.$$

6. The asymptotic approach is again disregarded, the integration constant δ being defined by the condition that ζ_1' must have the asymptote $\zeta_1' = 2$. Figure 7 shows the terms of the power series

for ζ_1', those free from c_3 and those multiplied by c_3, as curves A and B, that is

$$\zeta_1' = \delta \cdot A - 16 \cdot B$$

after which ζ_1' is plotted for different values of δ.

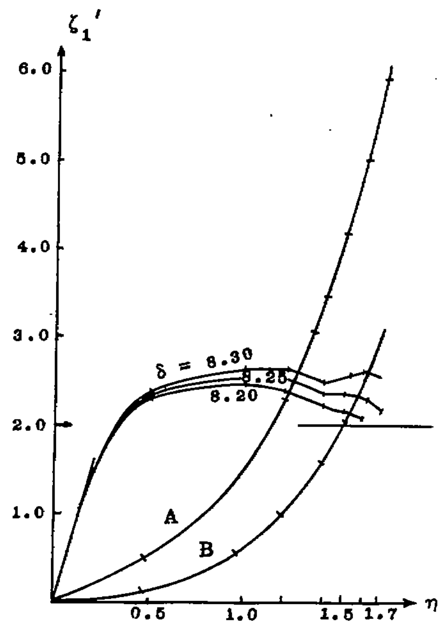


Figure 7

This curve indicates that the convergence of the series is rather poor in spite of the great number of computed coefficients c, even at $\eta = 1.6$. In any event, the terms indicate, when identical powers of α are combined, a satisfactory variation so that the series are still practicable. The correct value of δ ranges between 8.20 and 8.30. The curve rises, at first, very quickly and approaches its asymptote from above. This marked influence on u near $\eta = 0$ compared to $\eta = \infty$ permits u in the case of separation to change signs at the boundary before it does on the outside.

7. Proceeding to the calculation of the point of separation, it will be remembered from (1) that, quantitatively, the results are not exact, since only the first and third powers of x were taken into consideration. The point of separation $[\xi]$ is defined by

$$0 = \frac{\partial u}{\partial y} = \sqrt{\frac{\rho q_o^4}{8kq_1}} \left(\zeta_o'' [\xi] \pm \zeta_1'' [\xi]^3\right) \qquad \text{for} \quad \eta = 0$$

or by (3) and (5)

$$\alpha^3 \pm \delta [\xi]^2 = 0$$

By (4) and (6), respectively, $\alpha = 1.515$, $\delta = 8.25$. Thus, in the case of the lower prefix, the only one of interest, the coordinate of the point of separation is

$$[\xi] = 0.65$$

hence

$$[x] = 0.65 \sqrt{\frac{q_o}{q_1}}$$

with

$$\bar{u} = q_o x - q_1 x^3$$

The maximum of the velocity (minimum pressure) lies therefore at

$$x = 0.577 \sqrt{\frac{q_o}{q_1}}$$

while zero velocity in the outside flow would not be reached till $x = 1 \cdot \sqrt{\frac{q_o}{q_1}}$. Accordingly, the point of separation is 12 percent of the total boundary-layer length behind the pressure maximum. The obtained figures are independent of friction constant, density, and a proportional increase of all velocities.

According to Prandtl's diagram (section 3 of Introduction) the stream line $\psi = 0$ diverges from the boundary at a certain angle, which is computed as follows: In the vicinity of the point of separation, the development of the expression for ψ given in (2) reads

$$\psi = \sqrt{\frac{kq_o^2}{2\rho q_1}} \frac{1}{3!}((\zeta_o''' [\xi] - \zeta_1''' [\xi]^3)\eta^3 + 3(\zeta_o'' - 3\zeta_1'' [\xi]^2)(\xi - [\xi])\eta^2)$$

$\psi = 0$ gives for the divergent-stream line

$$\frac{\eta}{\xi - [\xi]} = 3\frac{3\delta [\xi]^2 - \alpha^3}{16 [\xi]^3 - 4 [\xi]} = 11.5$$

or in the not-reduced coordinates

$$\frac{y}{x - [x]} = 11.5 \sqrt{\frac{2kq_1}{\rho q_o^2}}$$

These formulas are characterized by considerable uncertainty because only two terms of the development of ψ were computed and the higher differential quotients, which represent more subtle processes, are always less accurately computed than the former.

III. FORMATION OF THE BOUNDARY LAYER AND OF THE ZONE OF SEPARATION AT SUDDEN START OF MOTION FROM REST

1. The two preceding problems treated stationary flows. The problem of the growth of the boundary layer is now treated. Assume that a cylinder of arbitrary cross section is suddenly set in motion in a fluid at rest and from $t = 0$ is permanently maintained at constant velocity. At first, the state of potential flow is reached under the single action of the pressure distribution. The thickness of the boundary layer is zero to begin with, so far as the sudden velocity distribution can be obtained at all. The boundary layer develops in the first place under the effect of friction, then through the convective terms. The result is that, after a certain time, the separation starts at the rear of the body and, from there, progresses gradually. Since the fundamental equations refer only to thin boundary layers, they, naturally, represent only the start of the separation process, just as the previous problems dealt with the boundary layer only as far as the zone of separation.

2. The equations involved here are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \bar{u}\frac{\partial \bar{u}}{\partial x} + \kappa\frac{\partial^2 u}{\partial y^2}$$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

κ substitutes for $\frac{k}{\rho}$.

The potential flow which is set up first gives the boundary value $\bar{u}$ as function of x. Since the process for $t = 0$ is singular, the type of development is, for the time being, still unknown; it must be established by successive approximation. The principal influence on the changes has (at small t) the friction, hence, for the first approximation u_o

$$\frac{\partial u_o}{\partial t} = \kappa\frac{\partial^2 u_o}{\partial y^2}$$

The integral of this equation

$$u_o = \frac{2\bar{u}}{\sqrt{\pi}}\int_o^{\eta} e^{-\eta^2}d\eta$$

$$\eta = \frac{y}{2\sqrt{\kappa t}}$$

satisfies the conditions of supplying a vanishing boundary layer for $t = 0$ and of joining the outside flow $u_o = \bar{u}$ for $y = \infty$. The subsequent approximation is obtained by inserting u_o in the convective terms, while time and friction terms obtain $u = u_o + u_1$. The resultant equation for u_1 reads

$$\frac{\partial u_1}{\partial t} = \kappa\frac{\partial^2 u_1}{\partial y^2} + \bar{u}\frac{\partial \bar{u}}{\partial x} \quad \text{(function of } \eta)$$

According to mechanical similarity, this equation is satisfied by the formula

$$u_1 = t\bar{u}\frac{\partial\bar{u}}{\partial x} f(\eta)$$

which is also not contradictory to the boundary condition $u_1 = 0$ for $y = 0$ and $y = \infty$.

After further considerations, which in particular refer to the insertion of x, the quantity u is represented in an expansion in powers of t, the coefficients of which are functions of η, that is, still contain t. These functions are also still dependent on x, but this time x enters the differential equations only as parameter.

3. The formula for ψ is accordingly

$$\psi = 2\sqrt{\kappa t} \sum_{\nu=0}^{\infty} t^{\nu}\chi_{\nu}(x\eta)$$

$$\eta = \frac{y}{2\sqrt{\kappa t}}$$

$$u = \sum_{\nu=0}^{\infty} t^{\nu}\frac{\partial\chi_{\nu}}{\partial\eta}$$

and hence the differential equations for χ

$$\frac{\partial^3\chi_{\mu}}{\partial\eta^3} + 2\eta\frac{\partial^2\chi_{\mu}}{\partial\eta^2} - 4\mu\frac{\partial\chi_{\mu}}{\partial\eta} = 4\sum_{\nu=0}^{\mu-1}\left(\frac{\partial\chi_{\mu-1-\nu}}{\partial\eta}\frac{\partial^2\chi_{\nu}}{\partial x\partial\eta} - \frac{\partial\chi_{\nu}}{\partial x}\frac{\partial^2\chi_{\mu-1-\nu}}{\partial\eta^2}\right)$$

for $\mu = 1$, the right-hand side contains $-4\bar{u}\frac{\partial\bar{u}}{\partial x}$.

As before, the calculation is limited to the first two terms, that is

$$\chi_o = \bar{u}\zeta_o(\eta) \qquad \chi_1 = \bar{u}\frac{\partial\bar{u}}{\partial x}\zeta_1(\eta)$$

hence

$$u = \bar{u}\zeta_o' + t\bar{u}\frac{\partial\bar{u}}{\partial x}\zeta_1'$$

The equations for ζ_o and ζ_1 read then

$$\zeta_o''' + 2\eta\zeta_o'' = 0$$

$$\zeta_1''' + 2\eta\zeta_1'' - 4\zeta_1' = 4(\zeta_o'^2 - \zeta_o\zeta_o'' - 1)$$

Boundary conditions

$$\text{for } \eta = 0: \quad \zeta_o = 0 \quad \zeta_o' = 0$$

$$\zeta_1 = 0 \quad \zeta_1' = 0$$

$$\text{for } \eta = \infty: \quad \zeta_o' = 1 \quad \zeta_1' = 0$$

4. The solutions of the above differential equations, which are to be used in the subsequent problem, are obtained by quadrature when the homogeneous equations are integrated. The latter integrals were obtained by the following consideration: The homogeneous parts of the equations stem from the time and friction term which together form the heat conduction equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial y^2}$$

Of this equation integrals of the form

$$u_n = t^n f_n(\eta)$$

$$\eta = \frac{y}{2\sqrt{kt}}$$

exist, according to similarity considerations, whereby f_n satisfies the differential equation

$$f_n'' + 2\eta f_n' - 4nf_n = 0$$

which the above form possesses.

Thus, for example (see above)

$$u_o = f_o = \frac{-2}{\sqrt{\pi}} \int_{\infty}^{\eta} e^{-\eta^2} d\eta$$

For $\eta = 0$

$$u_o = 1 \quad \text{when} \quad t > 0$$

$$u_o = 0 \quad \text{when} \quad t < 0$$

For $\eta = 0$, hence, for $y = 0$, u_n is proportional to t^n, hence must be representable by superposition of solutions u_o in the following form

$$u_n = n \int_o^{\infty} u_o\left(\frac{y}{2\sqrt{k(t - t_o)}}\right) \cdot t_o^{n-1} dt_o$$

$$= n \int_o^{t} u_o\left(\frac{y}{2\sqrt{k(t - t_o)}}\right) \cdot t_o^{n-1} dt_o$$

since for $y = 0$ it is

$$= n \int_o^{t} t_o^{n-1} dt_o = t^n$$

For the evaluation of this integral, put $t - t_o = \tau$

$$u_n = -n \int_t^{o} u_o\left(\frac{y}{2\sqrt{k\tau}}\right) \cdot (t - \tau)^{n-1} d\tau$$

insert herein

$$\frac{y}{2\sqrt{kt}} = \eta; \qquad \frac{y}{2\sqrt{k\tau}} = \zeta;$$

$$t = \frac{1}{4k}\frac{y^2}{\eta^2}; \qquad \tau = \frac{1}{4k}\frac{y^2}{\zeta^2};$$

$$d\tau = -\frac{1}{2k}\frac{y^2}{\zeta^3}d\zeta;$$

and finally obtain

$$u_n = \frac{4n}{\sqrt{\pi}}t^n\eta^{2n}\int_\infty^\eta \frac{1}{\zeta^3}\left(\frac{1}{\eta^2} - \frac{1}{\zeta^2}\right)^{n-1}d\zeta \int_\infty^\zeta e^{-\vartheta^2}d\vartheta$$

Calculation of this integral by the binomial theorem and the previously cited method of partial integration finally gives

$$f_n = \sum_{\mu=0}^{n}\frac{2^\mu\binom{n}{\mu}}{(2\mu - 1)\ldots3\cdot1}\eta^{2\mu}\int_\infty^\eta e^{-\eta^2}d\eta$$

$$+ \sum_{\nu=1}^{n}\left\{\sum_{\mu=\nu}^{n}(-1)^{\mu+\nu}\frac{2^{\nu-1}\binom{n}{\mu}}{(2\mu - 1)\ldots(2\mu - 2\nu + 1)}\right\}\eta^{2\nu-1}e^{-\eta^2}$$

The other integral is algebraic and equal to the above factor of $\int_\infty^\eta e^{-\eta^2}d\eta$

$$\dot{f}_n = \sum_{\mu=0}^{n}\frac{2^\mu\binom{n}{\mu}}{(2\mu - 1)\ldots3\cdot1}\eta^{2\mu}$$

5. Quantity ζ_o is determined as follows: With the boundary conditions taken into consideration

$$\zeta_0' = 1 + \frac{2}{\sqrt{\pi}} \int_\infty^\eta e^{-\eta^2} d\eta$$

whence by integration

$$\zeta_0 = -\frac{1}{\sqrt{\pi}} + \eta + \frac{2}{\sqrt{\pi}}\left(\eta \int_\infty^\eta e^{-\eta^2} d\eta + \frac{1}{2} e^{-\eta^2}\right)$$

while utilizing

$$\zeta_0'' = \frac{2}{\sqrt{\pi}} e^{-\eta^2}$$

The second differential equation (of the second order for ζ_1') assumes then the form

$$\zeta_1'' + 2\eta\zeta_1'' - 4\zeta_1' = \frac{16}{\sqrt{\pi}} \int_\infty^\eta e^{-\eta^2} d\eta - \frac{8}{\sqrt{\pi}} \eta e^{-\eta^2} + \frac{16}{\pi}\left[\left\{\int_\infty^\eta e^{-\eta^2} d\eta\right\}^2 - \eta e^{-\eta^2} \int_\infty^\eta e^{-\eta^2} d\eta - \frac{1}{2} e^{-2\eta^2} + \frac{1}{2} e^{-\eta^2}\right]$$

The integral of the homogeneous equation for ζ_1' is by (4)

$$f_1 = \alpha(2\eta^2 + 1) + \beta\left[\eta e^{-\eta^2} + (2\eta^2 + 1)\int_\infty^\eta e^{-\eta^2} d\eta\right]$$

The integral of the nonhomogeneous equation would then be obtainable by quadratures. But it is also true that, by twice differentiating, the differential equation finally becomes

$$\zeta'''' + 2\eta\zeta''' = \text{function of } \eta$$

which is easier to integrate as an equation of essentially first order. Hence

$$\zeta'''' = e^{-\eta^2}\int_\infty^\eta e^{\eta^2}\left[\text{function of } \eta\right]d\eta$$

Since the impressed force of the differential equation contains $e^{-\eta^2}$ in each term after twice differentiating, e^{η^2} cancels out, and ζ'''' and then ζ can be integrated, because the functions behind the integrals contain, at the most, $e^{-\eta^2}$ twice, and in addition, powers of η, and must be integrated several times, which can be accomplished by the methods discussed previously (I,6). The result of the rather voluminous calculation reads

$$\zeta_1' = \frac{6}{\pi}\eta e^{-\eta^2}\int_\infty^\eta e^{-\eta^2}d\eta + \frac{2}{\pi}(2\eta^2 - 1)\left\{\int_\infty^\eta e^{-\eta^2}d\eta\right\}^2 + \frac{2}{\pi}e^{-2\eta^2}$$

$$+ \frac{1}{\sqrt{\pi}}\eta e^{-\eta^2} - \frac{4}{\sqrt{\pi}}\int_\infty^\eta e^{-\eta^2}d\eta - \frac{4}{3\pi}e^{-\eta^2}$$

$$+ \alpha(2\eta^2 + 1) + \beta\left[\eta e^{-\eta^2} + (2\eta^2 + 1)\int_\infty^\eta e^{-\eta^2}d\eta\right]$$

$$\zeta_1'' = -\frac{2}{\pi}(2\eta^2 - 1)e^{-\eta^2}\int_\infty^\eta e^{-\eta^2}d\eta + \frac{8}{\pi}\eta\left\{\int_\infty^\eta e^{-\eta^2}d\eta\right\}^2 - \frac{2}{\pi}\eta e^{-2\eta^2}$$

$$- \frac{1}{\sqrt{\pi}}(2\eta^2 + 3)e^{-\eta^2} + \frac{8}{3\pi}\eta e^{-\eta^2}$$

$$+ 4\alpha\eta + \beta\left[2e^{-\eta^2} + 4\eta\int_\infty^\eta e^{-\eta^2}d\eta\right]$$

The reason that the equation here could be integrated in closed form, despite its affinity with the previously stationary problem, is due to the fact that $\frac{\partial u}{\partial t}$ is simpler than $u\frac{\partial u}{\partial x}$, although both have, according to the order of the differential quotient, "heat conduction character."

The determination of α and β from the limiting conditions $\zeta_1' = 0$ for $\eta = 0$ and $\eta = \infty$ gives

$$\alpha = 0 \qquad \beta = \frac{3}{\sqrt{\pi}} + \frac{4}{3\pi^{3/2}}.$$

6. For computing the zone of separation, there is

$$0 = \frac{\partial u}{\partial y} = \frac{1}{2\sqrt{\kappa t}}\left(\bar{u}\zeta_0'' + [t]\,\bar{u}\frac{\partial \bar{u}}{\partial x}\zeta_1''\right) \qquad \text{for} \quad \eta = 0$$

Then

$$\zeta_0'' = \frac{2}{\sqrt{\pi}} \qquad \zeta_1'' = \frac{2}{\sqrt{\pi}} + \frac{8}{3\pi^{3/2}}$$

The condition for the time of separation $[t]$ is

$$1 + \left(1 + \frac{4}{3\pi}\right)[t]\frac{\partial \bar{u}}{\partial x} = 0$$

hence, $\frac{\partial \bar{u}}{\partial x}$ must be negative. The separation occurs first where $\frac{\partial \bar{u}}{\partial x}$ has the greatest magnitude. The result applies to cylinders of any cross section; $\bar{u}$ is the corresponding potential flow.

IV. DEVELOPMENT OF ZONE OF SEPARATION FROM REST

AT UNIFORMLY ACCELERATED MOTION

1. Against the physical principles of the foregoing problem, the objection may be raised that the sudden shock might be accompanied

by an interruption of the fluid. Hence, let the solution of the problem assume that, starting from the time $t = 0$, the immersed body is subjected to constant acceleration.

In that case

$$\bar{u} = tw(x)$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} = w + t^2 w\frac{\partial w}{\partial x}$$

2. From considerations similar to those made before, the solution of the differential equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \kappa\frac{\partial^2 u}{\partial y^2}$$

is based on the formula

$$\psi = 2\sqrt{\kappa t} \cdot \sum_{\nu=0}^{\infty} t^{2\nu+1}\chi_{2\nu+1}(x\eta)$$

$$u = \sum_{\nu=0}^{\infty} t^{2\nu+1}\frac{\partial \chi_{2\nu+1}}{\partial \eta}$$

$$\eta = \frac{y}{2\sqrt{\kappa t}}$$

Insertion in the basic equation gives

$$\frac{\partial^3 \chi_{2\lambda+1}}{\partial \eta^3} + 2\eta\frac{\partial^2 \chi_{2\lambda+1}}{\partial \eta^2} - 4(2\lambda + 1)\frac{\partial \chi_{2\lambda+1}}{\partial \eta}$$

$$= 4\sum_{\mu=0}^{\lambda-1}\left[\frac{\partial \chi_{2\mu+1}}{\partial \eta}\frac{\partial^2 \chi_{2\lambda-2\mu-1}}{\partial x \partial y} - \frac{\partial \chi_{2\lambda-2\mu-1}}{\partial x}\frac{\partial^2 \chi_{2\mu+1}}{\partial \eta^2}\right]$$

for $\lambda = 0$, the right-hand side contains $-4w$, for $\lambda = 1$, $-4w\frac{\partial w}{\partial x}$.

The calculation of the state is again limited to the first two terms, while it should be noted that through those two terms, the two terms of the pressure $w + t^2 w\frac{\partial w}{\partial x}$ are also taken into consideration. The impressed force of the next equations contains only earlier development coefficients. For the final equation, however, which supplies the zone of separation, the coefficient of the next term is computed also. For χ_1 and χ_3 the relationship of x can be introduced in the following manner:

$$\chi_1 = w\zeta_1(\eta), \qquad \chi_3 = w\frac{\partial w}{\partial x}\zeta_3(\eta)$$

The differential equations for ζ are then:

$$\frac{\partial^3 \zeta_1}{\partial \eta^3} + 2\eta\frac{\partial^2 \zeta_1}{\partial \eta^2} - 4\frac{\partial \zeta_1}{\partial \eta} = -4$$

$$\frac{\partial^3 \zeta_3}{\partial \eta^3} + 2\eta\frac{\partial^2 \zeta_3}{\partial \eta^2} - 12\frac{\partial \zeta_3}{\partial \eta} = -4 + 4\left[\left(\frac{\partial \zeta_1}{\partial \eta}\right)^2 - \zeta_1\frac{\partial^2 \zeta_1}{\partial \eta^2}\right]$$

Boundary conditions:

$$\text{for } \eta = 0: \quad \left.\begin{aligned} \zeta_1 &= 0, & \frac{\partial \zeta_1}{\partial \eta} &= 0 \\ \zeta_3 &= 0, & \frac{\partial \zeta_3}{\partial \eta} &= 0 \end{aligned}\right\} \text{ from } \begin{aligned} u &= 0 \\ v &= 0 \end{aligned}$$

$$\text{for } \eta = \infty \quad \frac{\partial \zeta_1}{\partial \eta} = 1, \quad \frac{\partial \zeta_3}{\partial \eta} = 0 \quad \text{from } u = tw$$

3. According to the general solutions of the present type of differential equations discussed in III (4), $\frac{\partial \zeta_1}{\partial \eta}$ can be written forthwith, since the nonhomogeneous term -4 is disposed of by $\frac{\partial \zeta_1}{\partial \eta} = 1$; ζ_1 is obtained by integration by the repeatedly cited method (16)

$$\frac{\partial^2 \zeta_1}{\partial \eta^2} = \frac{4}{\sqrt{\pi}} \left[e^{-\eta^2} + 2\eta \int_\infty^\eta e^{-\eta^2} d\eta \right]$$

$$\frac{\partial \zeta_1}{\partial \eta} = 1 + \frac{2}{\sqrt{\pi}} \left[\eta e^{-\eta^2} + (1 + 2\eta^2) \int_\infty^\eta e^{-\eta^2} d\eta \right]$$

$$\zeta_1 = \eta + \frac{2}{3\sqrt{\pi}} \left[-1 + (1 + \eta^2) e^{-\eta^2} \right.$$

$$\left. + (3\eta + 2\eta^3) \int_\infty^\eta e^{-\eta^2} d\eta \right]$$

These functions are quantitatively plotted in figure 8 and given in a table (see Section 6 following).

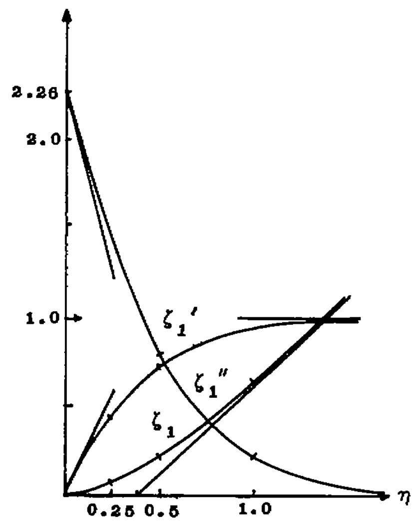


Figure 8

The impressed force on the right-hand side of the second equation is then

$$\frac{16}{\sqrt{\pi}}\int_{\infty}^{\eta} e^{-\eta^2}d\eta + \frac{16}{3\pi}\left[4\eta\int_{\infty}^{\eta} e^{-\eta^2}d\eta + 2e^{-\eta^2}\right]$$

$$+ \frac{16}{3\pi}\left[(-2 + \eta^2)e^{-2\eta^2} + (-4\eta + 4\eta^3)e^{-\eta^2}\int_{\infty}^{\eta} e^{-\eta^2}d\eta\right.$$

$$\left. + (3 + 4\eta^4)\left\{\int_{\infty}^{\eta} e^{-\eta^2}d\eta\right\}^2\right]$$

4. The integration of the second equation, in closed form, again succeeds by the same methods as in III (5). For the part of the impressed force quadratic in $e^{-\eta^2}$ a formula with indeterminate coefficients is particularly advisable.

$$(a + b\eta^2 + c\eta^4)e^{-2\eta^2}$$

$$+ (d\eta + e\eta^3 + f\eta^5)e^{-\eta^2}\int_\infty^\eta e^{-\eta^2}d\eta$$

$$+ (g + h\eta^2 + i\eta^4 + k\eta^6)\left\{\int_\infty^\eta e^{-\eta^2}d\eta\right\}^2$$

This formula fails when the impressed force contains terms which exceed $\eta^6 e^{-2\eta^2}$, $\eta^5 e^{-\eta^2}\int_\infty^\eta e^{-\eta^2}d\eta$, $\eta^4\left\{\int_\infty^\eta e^{-\eta^2}d\eta\right\}^2$ (compare III (5)). The coefficients are determined from linear equations. The other portions of $\frac{\partial\zeta_3}{\partial\eta}$ are easier to compute; ζ_3 and $\frac{\partial^2\zeta_3}{\partial\eta^2}$ follow by integration and differentiation. So, when the integration constants are correctly computed, the final result is

$$\frac{\partial^2\zeta_3}{\partial\eta^2} = -\frac{4}{3\sqrt{\pi}}e^{-\eta^2} - \frac{32}{15\pi}\int_\infty^\eta e^{-\eta^2}d\eta$$

$$+ \frac{2}{3\pi}\Big[(-\eta + 2\eta^3)e^{-2\eta^2}$$

$$+ (1 + 2\eta^2 + 8\eta^4)e^{-\eta^2}\int_\infty^\eta e^{-\eta^2}d\eta$$

$$+ (6\eta + 8\eta^3 + 8\eta^5)\left\{\int_\infty^\eta e^{-\eta^2}d\eta\right\}^2\Big]$$

$$+ \frac{1}{5}\left(\frac{5}{6\sqrt{\pi}} - \frac{16}{45\sqrt{\pi^3}}\right)\Big[(16 + 36\eta^2 + 8\eta^4)e^{-\eta^2}$$

$$+ (60\eta + 80\eta^3 + 16\eta^5)\int_\infty^\eta e^{-\eta^2}d\eta\Big]$$

$$\frac{\partial\zeta_3}{\partial\eta} = -\frac{4}{3\sqrt{\pi}}\int_\infty^\eta e^{-\eta^2}d\eta$$

$$-\frac{16}{15\pi}\left[e^{-\eta^2} + 2\eta\int_\infty^\eta e^{-\eta^2}d\eta\right]$$

$$+\frac{1}{9\pi}\left[(8 + \eta^2 + 2\eta^4)e^{-2\eta^2}\right.$$

$$+ (24\eta + 8\eta^3 + 8\eta^5)e^{-\eta^2}\int_\infty^\eta e^{-\eta^2}d\eta$$

$$\left.+ (-9 + 18\eta^2 + 12\eta^4 + 8\eta^6)\left\{\int_\infty^\eta e^{-\eta^2}d\eta\right\}^2\right]$$

$$+\frac{1}{15}\left(\frac{5}{6\sqrt{\pi}} - \frac{16}{45\sqrt{\pi^3}}\right)\left[(33\eta + 28\eta^3 + 4\eta^5)e^{-\eta^2}\right.$$

$$\left.+ (15 + 90\eta^2 + 60\eta^4 + 8\eta^6)\int_\infty^\eta e^{-\eta^2}d\eta\right]$$

hence, by integration

$$\zeta_3 = -\frac{2}{3\sqrt{\pi}}\left[e^{-\eta^2} + 2\eta\int_\infty^\eta e^{-\eta^2}d\eta\right]$$

$$-\frac{8}{15\pi}\left[\eta e^{-\eta^2} + (1 + 2\eta^2)\int_\infty^\eta e^{-\eta^2}d\eta\right]$$

$$+\frac{1}{315\pi}\left[(49\eta + 11\eta^3 + 10\eta^5)e^{-2\eta^2} + 768\int_\infty^\eta e^{-2\eta^2}d\eta\right.$$

$$+ (-537 + 198\eta^2 + 64\eta^4 + 40\eta^6)e^{-\eta^2} \int_\infty^\eta e^{-\eta^2} d\eta$$

$$\left. + (-315\eta + 210\eta^3 + 84\eta^5 + 40\eta^7)\left\{ \int_\infty^\eta e^{-\eta^2} d\eta \right\}^2 \right]$$

$$+ \frac{1}{105}\left(\frac{5}{6\sqrt{\pi}} - \frac{16}{45\sqrt{\pi^3}}\right)\left[(24 + 87\eta^2 + 40\eta^4 + 4\eta^6)e^{-\eta^2}\right.$$

$$\left. + (105\eta + 210\eta^3 + 84\eta^5 + 8\eta^7) \int_\infty^\eta e^{-\eta^2} d\eta \right]$$

$$+ \left(\frac{128}{1575\sqrt{\pi^3}} + \frac{128}{105\sqrt{2\pi}} - \frac{9}{14\sqrt{\pi}}\right)$$

These three functions, plotted in figure 9, rigorously satisfy the differential equations and the boundary conditions for the coefficient χ.

5. The condition for the zone of separation has the form

$$0 = \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{1}{2}\sqrt{\frac{t}{k}}\, w \left[\frac{\partial^2 \zeta_1}{\partial \eta^2} + t^2 \frac{\partial w}{\partial x} \frac{\partial^2 \zeta_3}{\partial \eta^2}\right]_{\eta=0}$$

whence, by the foregoing formulas

$$\frac{\partial^2 \zeta_1}{\partial \eta^2} = \frac{4}{\sqrt{\pi}} \qquad \frac{\partial^2 \zeta_3}{\partial \eta^2} = \frac{31}{15\sqrt{\pi}} - \frac{256}{225\sqrt{\pi^3}}$$

The equation for the separation time $[t]$ reads

$$1 + \left(\frac{31}{60} - \frac{64}{225\pi}\right)[t]^2 \frac{\partial w}{\partial x} = 0$$

The next term in the separation equation $\frac{\partial u}{\partial y} = 0$ would read: $\frac{1}{2\sqrt{\kappa t}} t^5 \frac{\partial^2 \chi_5}{\partial \eta^2}$, and in order to be able to allow for it, too, the coefficient in this separation equation, rather than the total variation of χ_5, is computed.

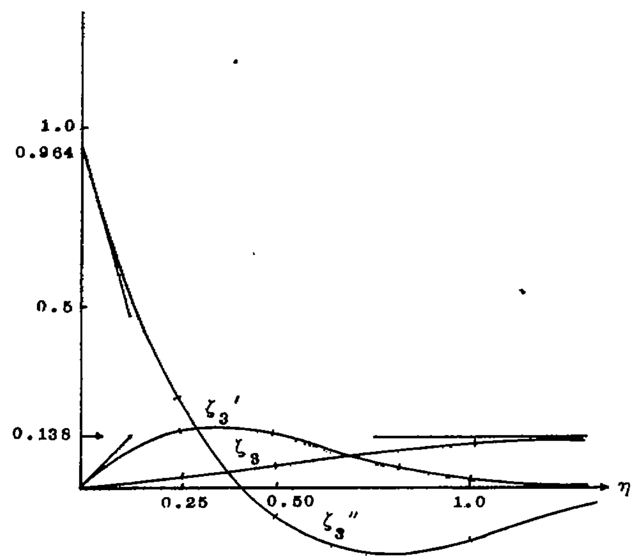


Figure 9

The development term χ_5 satisfied the equation

$$\frac{\partial^3 \chi_5}{\partial \eta^3} + 2\eta \frac{\partial^2 \chi_5}{\partial \eta^2} - 20 \frac{\partial \chi_5}{\partial \eta} = 4\left[\frac{\partial \chi_3}{\partial \eta}\frac{\partial^2 \chi_1}{\partial x \partial \eta} - \frac{\partial \chi_1}{\partial x}\frac{\partial^2 \chi_3}{\partial \eta^2} + \frac{\partial \chi_1}{\partial \eta}\frac{\partial^2 \chi_3}{\partial x \partial \eta} - \frac{\partial \chi_3}{\partial x}\frac{\partial^2 \chi_1}{\partial \eta^2}\right]$$

The entry of x in χ_1 and χ_3 is known, and calculation of the right-hand side confirms that χ_5 assumes the form

$$X_5 = w\left(\frac{\partial w}{\partial x}\right)^2\zeta_5 + w^2\frac{\partial^2 w}{\partial x^2}\zeta_{5a}$$

Since tw cancels out, the condition of separation reads

$$\left[\frac{\partial^2\zeta_1}{\partial\eta^2}\right]_{\eta=0} + t^2\frac{\partial w}{\partial x}\left[\frac{\partial^2\zeta_3}{\partial\eta^2}\right]_{\eta=0} + t^4\left(\frac{\partial w}{\partial x}\right)^2\left[\frac{\partial^2\zeta_5}{\partial\eta^2}\right]_{\eta=0} + t^4 w\frac{\partial^2 w}{\partial x^2}\left[\frac{\partial^2\zeta_{5a}}{\partial\eta^2}\right]_{\eta=0} = 0$$

6. This leaves the calculation of the coefficients $\left[\frac{\partial^2\zeta_5}{\partial\eta^2}\right]_{\eta=0}$ and $\left[\frac{\partial^2\zeta_{5a}}{\partial\eta^2}\right]_{\eta=0}$

For ζ_5, the differential equation reads

$$\frac{\partial^3\zeta_5}{\partial\eta^3} + 2\eta\frac{\partial^2\zeta_5}{\partial\eta^2} - 20\frac{\partial\zeta_5}{\partial\eta} = 8\frac{\partial\zeta_1}{\partial\eta}\frac{\partial\zeta_3}{\partial\eta} - 4\zeta_1\frac{\partial^2\zeta_3}{\partial\eta^2} - 4\zeta_3\frac{\partial^2\zeta_1}{\partial\eta^2} = f(\eta)$$

and the boundary conditions

$$\zeta_5 = 0, \quad \frac{\partial\zeta_5}{\partial\eta} = 0 \quad \text{for} \quad \eta = 0; \quad \frac{\partial\zeta_5}{\partial\eta} \quad \text{for} \quad \eta = \infty$$

The impressed force $f(\eta)$ is given by the previously written functions. The desired coefficients $\left[\frac{\partial^2\zeta_5}{\partial\eta^2}\right]_{\eta=0}$ are computed by Green's method as follows:

$$\int_0^\infty \vartheta \left(\frac{\partial^3 \zeta_5}{\partial \eta^3} + 2\eta \frac{\partial^2 \zeta_5}{\partial \eta^2} - 20 \frac{\partial \zeta_5}{\partial \eta} \right) d\eta = \left[\vartheta \frac{\partial^2 \zeta_5}{\partial \eta^2} - \frac{\partial \vartheta}{\partial \eta} \frac{\partial \zeta_5}{\partial \eta} + 2\eta \vartheta \frac{\partial \zeta_5}{\partial \eta} \right]_0^\infty$$

$$+ \int_0^\infty \frac{\partial \zeta_5}{\partial \eta} \left(\frac{\partial^2 \vartheta}{\partial \eta^2} - 2 \frac{\partial \eta \vartheta}{\partial \eta} - 20 \vartheta \right) d\eta$$

Then, if ϑ is made to satisfy the adjunct differential equation

$$\frac{\partial^2 \vartheta}{\partial \eta^2} - 2\eta \frac{\partial \vartheta}{\partial \eta} - 22\vartheta = 0$$

and the boundary conditions

$$\vartheta(0) = -1 \qquad \vartheta(\infty) = 0$$

the result is

$$\left[\frac{\partial^2 \zeta_5}{\partial \eta^2} \right]_{\eta=0} = \int_0^\infty \vartheta \cdot f \cdot d\eta$$

$f(\eta)$ is given previously; the influence coefficient ϑ (Green's function) is obtained by integration of his differential equation

$$\vartheta(\eta) = \frac{2}{945\sqrt{\pi}} \left[(2895\eta + 5280\eta^3 + 2352\eta^5 + 352\eta^7 + 16\eta^9) \right.$$

$$\left. + (945 + 9450\eta^2 + 12600\eta^4 + 5040\eta^6 + 720\eta^8 + 32\eta^{10}) e^{\eta^2} \int_\infty^\eta e^{-\eta^2} d\eta \right]$$

The curve of ϑ is shown in figure 10, along with the product $\vartheta \cdot f$. The area of this last curve gives the desired coefficient.

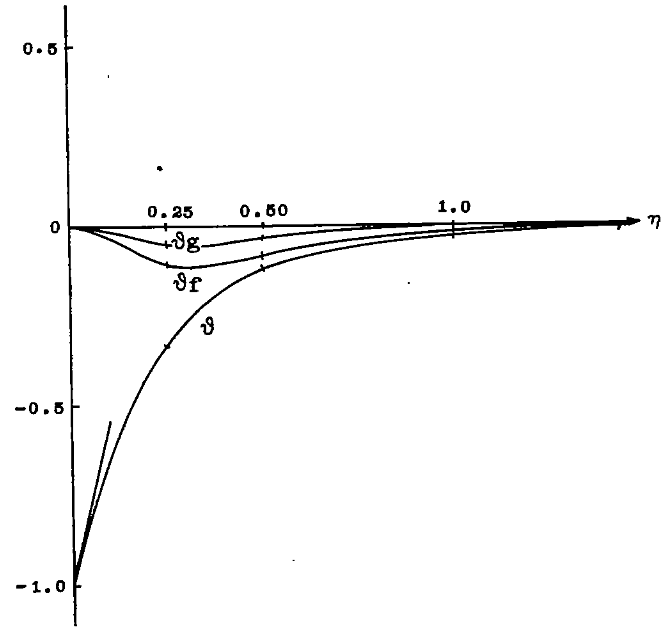


Figure 10

For computing $\left[\frac{\partial^2\zeta_{5\alpha}}{\partial\eta^2}\right]_{\eta=0}$ the equations

$$\frac{\partial^3\zeta_{5\alpha}}{\partial\eta^3} + 2\eta\frac{\partial^2\zeta_{5\alpha}}{\partial\eta^2} - 20\frac{\partial\zeta_{5\alpha}}{\partial\eta} = 4\left[\frac{\partial\zeta_1}{\partial\eta}\frac{\partial\zeta_3}{\partial\eta} - \zeta_3\frac{\partial^2\zeta_1}{\partial\eta^2}\right] = g(\eta)$$

$$\left[\frac{\partial^2\zeta_{5\alpha}}{\partial\eta^2}\right]_{\eta=0} = \int_0^\infty \vartheta \cdot g \cdot d\eta$$

are available; $\vartheta \cdot g$ is plotted in figure 10 according to the values indicated below.

The computed values are the following:

η	0	0.25	0.50	1.00	1.50	∞
ζ_1	0	.061	.211	.638	—	$\eta - 0.376$
$\frac{\partial \zeta_1}{\partial \eta}$	0	.450	.720	.943	—	1
$\frac{\partial^2 \zeta_1}{\partial \eta^2}$	2.26	1.396	.799	.201	0.035	0
ζ_3	0	.022	.060	.115	—	.138
$\frac{\partial \zeta_3}{\partial \eta}$	0	.137	.150	.020	—	0
$\frac{\partial^2 \zeta_3}{\partial \eta^2}$	.964	.231	−.092	−.156	−.05	0
$f(\eta)$	0	.315	.750	.457	—	0
$\vartheta(\eta)$	−1	−.327	−.112	−.018	—	0
$\vartheta \cdot f$	0	−.103	−.084	−.008	—	0
$g(\eta)$	0	.124	.240	−.016	—	0
$\vartheta \cdot g$	0	−.041	−.027	.0003	—	0

The area of the two curves is approximately

$$\left[\frac{\partial^2 \zeta_5}{\partial \eta^2}\right]_{\eta=0} = -0.058 \qquad \left[\frac{\partial^2 \zeta_{5\alpha}}{\partial \eta^2}\right]_{\eta=0} = -0.023$$

7. The equation of separation therefore reads

$$\frac{4}{\sqrt{\pi}} + [t]^2 \frac{\partial w}{\partial x}\left(\frac{31}{15\sqrt{\pi}} - \frac{256}{225\sqrt{\pi^3}}\right) - [t]^4\left(\frac{\partial w}{\partial x}\right)^2 \cdot 0.058 - [t]^4 w \frac{\partial^2 w}{\partial x^2} \cdot 0.023 = 0$$

or

$$1 + 0.427 \cdot [t]^2\frac{\partial w}{\partial x} - 0.026 \cdot [t]^4\left(\frac{\partial w}{\partial x}\right)^2 - 0.01 \cdot [t]^4 w\frac{\partial^2 w}{\partial x^2} = 0$$

Since the newly added correction term is even negative, the existence of the zero position appears to be certain.

The position and time of separation is according to the earlier approximation (without the term computed last)

$$[t]^2\frac{\partial w}{\partial x} = -2.34$$

For the case of a cylinder symmetrical to the direction flow, $w\frac{\partial^2 w}{\partial x^2} = 0$ at the rear point where the separation starts, the newly computed correction gives

$$[t]^2\frac{\partial w}{\partial x} = -2.08$$

equivalent to an error of about 10 percent. From this the quality of the approximation made in the other problems, where only the first powers were taken, can probably be also appraised.

V. APPLICATION OF THE RESULTS OF THE SEPARATION PROBLEM TO THE CIRCULAR CYLINDER

1. On the circular cylinder

$$\overline{u} = 2V \sin \frac{X}{R}$$

$\frac{X}{R}$ is called the reduced coordinate X; V is the velocity at which the parallel flow flows toward the right, and the cylinder moves toward the left, respectively. In the steady case, the separation starts according to Part II, Section 7 at $x_{sep.} = 0.65\sqrt{\frac{q_o}{q_1}}$; the maximum velocity lies at

$$x_{max} = 0.577\sqrt{\frac{q_o}{q_1}}$$

where

$$\bar{u} = q_o x - q_1 x^3$$

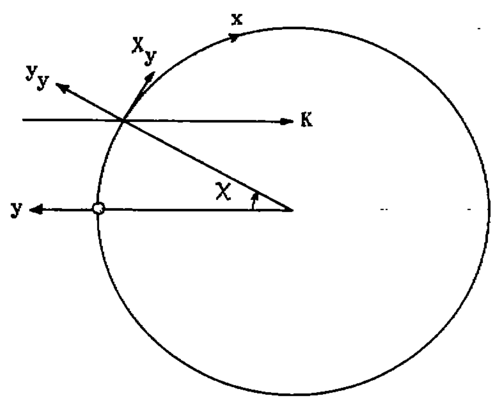


Figure 11

Taking the ordinary development in powers of sine

$$q_o = \frac{2V}{R}, \qquad q_1 = \frac{2V}{6R^3}$$

$$x_{sep.} = 1.59 \cdot R,\ X_{sep.} = 91\tfrac{1}{4}^\circ; \qquad x_{max} = 1.41 \cdot R;\ X_{max} = 81^\circ$$

But approximating the sine in the interval $0 - \pi$ by the method of least squares, gives

$$q_o = \frac{2V}{R} \cdot 0.856, \qquad q_1 = \frac{2V}{R^3} \cdot 0.093$$

$$x_{sep.} = 1.97 \cdot R, \qquad X_{sep.} = 113^\circ$$

$$x_{max} = 1.75 \cdot R, \qquad X_{max} = 101^\circ$$

In any case, the point of separation lies, by the present calculation, at from 11 percent to 12 percent of the total boundary-layer length behind the maximum of the velocity. This statement makes, of course, no claim to accuracy, since only the first two powers of x are taken into consideration. Besides, test records of the pressure difference indicate that the state near the separation is difficult to attain by development from starting point of the boundary layer, because it is too strongly affected by the pressure distribution of the turbulent bodies behind the cylinder. The sole purpose of the present calculations is to indicate that separation is actually obtained by the hydrodynamic equations. Further development of the calculating methods, especially for the more important problems of solids of revolution, promises, therefore, success.

2. If the cylinder with constant velocity is suddenly set in motion

$$\overline{u} = 2V \sin X \qquad \frac{\partial \overline{u}}{\partial x} = \frac{2V}{R} \cos X$$

The time of separation $[t]$ is, according to III (6), given by

$$\left(1 + \frac{4}{3\pi}\right) [t] \frac{\partial u}{\partial x} = -1$$

$$[t] = -0.35 \frac{R}{V \cos X}$$

The separation starts for $X = \pi$, $\cos X = -1$ at time

$$t_o = 0.35\frac{R}{V}$$

Up to that, the cylinder has travelled a distance

$$S = Vt_o = 0.35 \cdot R$$

All this is independent of velocity, density, and friction coefficient (little friction assumed).

3. At constant acceleration

$$\overline{u} = tw(x) = 2Vt \sin \frac{x}{R}$$

where V is then the acceleration of the cylinder in the flow. The separation time is (IV,7) for the start of separation

$$[t]^2\frac{\partial w}{\partial x} = -2.34 \quad \text{or} \quad = -2.08$$

respectively, or

$$[t]^2 = -1.17\frac{R}{V \cos X} \quad \text{or} \quad = -1.04\frac{R}{V \cos X}$$

respectively. The distance covered by the cylinder is

$$S = \frac{1}{2}Vt^2$$

at start of separation $(X = \pi)$

$$S = 0.59 \cdot R \quad \text{or} \quad = 0.52 \cdot R$$

respectively.

4. The resistance which the cylinder experiences at constant acceleration is computed next. The stress components are

$$Y_y = -p + 2k\frac{\partial v}{\partial y}$$

$$X_y = k\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Owing to the smallness of the friction, $\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}$ cancel with respect to $\frac{\partial u}{\partial y}$, leaving as force in direction of the outside flow

$$K = 2 \cdot B \int_0^{\pi}\left(p \cos X + k\frac{\partial u}{\partial y} \sin X\right) \cdot RdX$$

B is the width of the layer (height of immersed part of cylinder).

The pressure portion is computed as follows;

$$K_{pressure} = 2BR \int_0^{\pi} p \cos XdX = -2BR^2 \int_0^{\pi} \frac{\partial p}{\partial x} \sin XdX$$

Then

$$-\frac{\partial p}{\partial x} = \rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x}\right) \qquad \bar{u} = tw$$

$$= \rho\left(w + t^2 w\frac{\partial w}{\partial x}\right) \qquad w = 2V \sin X$$

The second term cancels out in the integration; the first gives

$$K_{pressure} = 2\pi\rho BR^2 V$$

hence, an increase in inertia by twice the amount of displaced fluid. The friction portion is

$$K_{friction} = \frac{2kBR}{2\sqrt{\kappa t}} \int_0^{\pi} \left(tw \frac{\partial^2 \zeta_1}{\partial \eta^2} + t^3 w \frac{\partial w}{\partial x} \frac{\partial^2 \zeta_3}{\partial \eta^2} \right) \sin X dX$$

where $\kappa = k/\rho$. Again, the second term disappears because $\frac{\partial^2 \zeta_1}{\partial \eta^2}$ and $\frac{\partial^2 \zeta_3}{\partial \eta^2}$ are merely constants, leaving

$$K_{friction} = 4\sqrt{\pi\rho k t} \cdot BRV$$

5. To give a picture of the flow conditions corresponding to these formulas, the flow curves for a specific state of motion of the uniformly accelerated cylinder are represented in a diagram. The parameters R, V, κ are arbitrary; hence, necessitate the introduction of reduced quantities for x, y, t, ψ, and u, so that R, V, κ disappear. It is accomplished by

$$x = RX, \quad t = \sqrt{\frac{R}{V}} T, \quad y = \sqrt[4]{\frac{R\kappa^2}{V}} Y$$

$$\psi = \sqrt[4]{R^3 \kappa^2 V}\, \Psi, \qquad u = \sqrt{RV} U$$

by which the formulas (compare IV (2) and V(3))

$$\psi = 2\sqrt{\kappa}\,t^{3/2}w\left(\zeta_1 + t^2\frac{\partial w}{\partial x}\zeta_3\right)$$

$$u = tw\left(\frac{\partial \zeta_1}{\partial \eta} + t^2\frac{\partial w}{\partial x}\,\frac{\partial \zeta_3}{\partial \eta}\right)$$

$$\eta = \frac{y}{2\sqrt{\kappa t}}$$

$$w = 2V \sin\frac{x}{R}$$

$$t^2\frac{\partial w}{\partial x} = 2\frac{Vt^2}{R}\cos\frac{x}{R}$$

become the following reduced equations

$$\Psi = 4T^{3/2}\sin X\cdot\left(\zeta_1 + 2T^2\cos X\cdot\zeta_3\right)$$

$$U = 2T\sin X\cdot\left(\frac{\partial \zeta_1}{\partial \eta} + 2T^2\cos X\cdot\frac{\partial \zeta_3}{\partial \eta}\right)$$

$$\eta = \frac{Y}{2\sqrt{T}}$$

The curve Ψ is then plotted against $Y = 2\sqrt{T}\cdot\eta$ for a fixed time T for a number of coordinate values X, and the position of the values Ψ = constant read from these curves. In figure 12, the cylinder is shown from $X = \pi/2$ to $X = \pi$. The separation time is given by $2T^2\cos X = -2.34$, that is, the start of the separation by $T = 1.08$. In figure 12 $2T^2 = 5$, hence $T = 1.58$, was chosen. For this chosen time, the separation point has already progressed up to beyond 60° at the cylinder; nevertheless the boundary layer still is fairly thin, the relative sizes correspond to the values $R = 10$ cm, $\kappa = 0.01 \frac{cm^2}{sec}$ (water), $V = 0.1 \frac{cm}{sec^2}$, that is, to a very small acceleration. Accordingly, $t = 15.8$ sec.

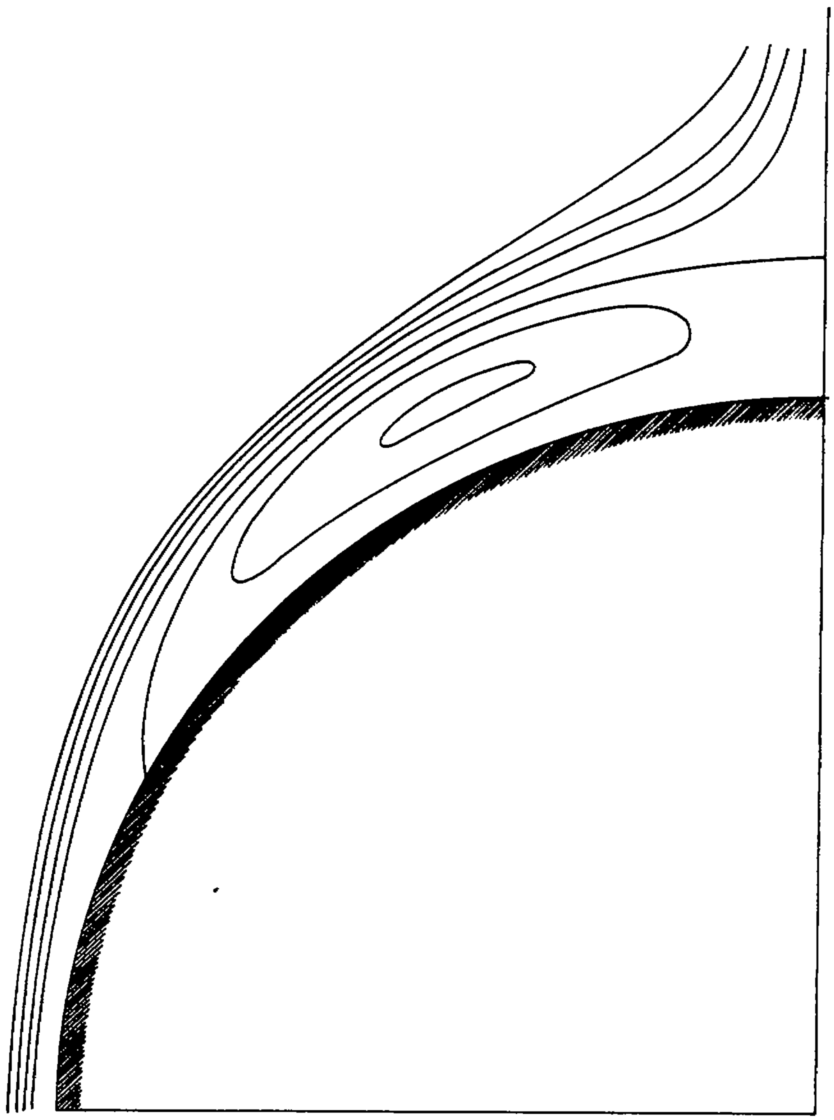


Figure 12

The picture obtained by the previous reduction formulas for $V = 10 \frac{cm}{sec^2}$ after 1.58 sec. is represented in figure 12. The thickening of the boundary layer would be diminished in the ratio of $1 : \sqrt{10}$.

Translated by J. Vanier
National Advisory Committee
for Aeronautics