STANFORD
AERONAUTICS & ASTRONAUTICS

AA210A

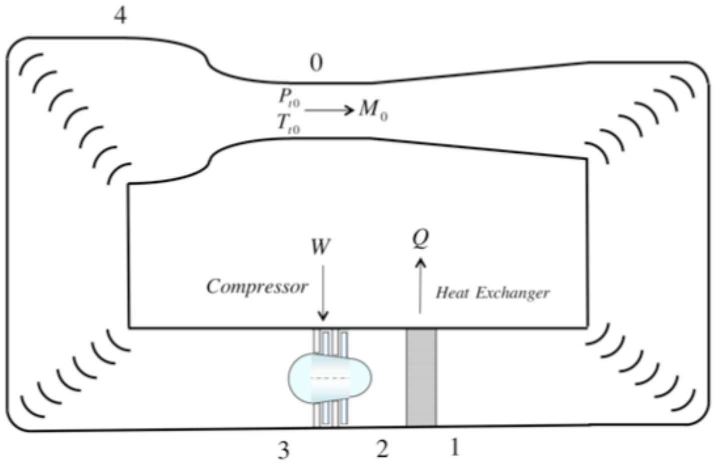
Fundamentals of Compressible Flow

Chapter 7 - Entropy generation and transport

STANFORD
AERONAUTICS &
ASTRONAUTICS

Problem 1 (from a recent midterm) – The figure below shows flow in a wind tunnel circuit driven by a compressor. Low speed air from the settling chamber is accelerated from left to right through a contraction into the tunnel test section. The air enters the test section at station 0 with stagnation temperature T_{t0}, stagnation pressure P_{t0}, and Mach number M_0. From the test section, the air passes into a diffuser designed to decelerate the flow back to low speed. Heat is removed from the flow by a heat exchanger between stations 1 and 2. Work is done on the flow by the compressor between stations 2 and 3. The stagnation temperature ratio across the compressor is $T_{t3}/T_{t2} = 1.15$.

i) Assume the heat removed from the flow by the heat exchanger causes no change in stagnation pressure between stations 1 and 2.
ii) Assume the entire wind tunnel is adiabatic except for the heat exchanger.
iii) Assume the work done on the flow by the compressor causes no change in the entropy of the flow between stations 2 and 3.



1) Determine P_{t3}/P_{t2}.

2) Determine the entropy change across the heat exchanger $(s_2 - s_1)/C_p$.

3) Due to viscous friction, the entropy change from station 3 to station 4 is measured to be one half of the entropy difference between station 1 and station 2. Determine P_{t3}/P_{t4} and P_{t4}/P_{t1}.

4) Wall pressure sensors are used to determine the static pressure ratio across the contraction, $P_4/P_0 = 1.2$. Estimate the test section Mach number.

5)Why is a heat exchanger needed?

Problem (from a recent midterm) – The figure below shows the path of a fluid element from the free stream, over an adiabatic wing, and into the turbulent wake in a compressible viscous flow of air. The path from 0 to the nose of the wing 1 is adiabatic and isentropic. The path from 1 to 2 lies very close to the surface of the wing and the pressure at station 2 is approximately equal to the free stream static pressure. Station 3 is very far downstream of the wing trailing edge. The free-stream pressure at station 0 is 10^5 N/m^2 the free stream temperature is 285K and the flow velocity is 250 m/sec. State any assumptions you need to make to solve the problem.

1) Determine the temperature and pressure of the fluid element at stations 1, 2 and 3.
2) Determine the entropy change of the fluid element between stations 1 and 2.
3) Determine the entropy change of the fluid element between stations 2 and 3.

7.1 Convective form of Gibbs' equation

Gibbs equation following a fluid particle

$$T\frac{Ds}{Dt} = \frac{De}{Dt} + P\frac{Dv}{Dt}.$$

Energy conservation equation

$$\frac{\partial \rho(e+k)}{\partial t} + \nabla \bullet \left(\rho\bar{U}\left(e + \frac{P}{\rho} + k\right) - \bar{\bar{\tau}} \bullet \bar{U} + \bar{Q}\right) - \rho\bar{G} \bullet \bar{U} = 0$$

7.2 The kinetic energy equation

Take the dot product of the velocity vector and the momentum equation.

$$U_i \frac{\partial \rho U_i}{\partial t} + U_i \frac{\partial}{\partial x_j}\left(\rho U_i U_j + P\delta_{ij} - \tau_{ij}\right) - \rho U_i G_i = 0.$$

$$U_i \frac{\partial(\rho U_i)}{\partial t} + U_i \frac{\partial(\rho U_i U_j)}{\partial x_j} =$$

$$\underline{U_i U_i \frac{\partial \rho}{\partial t}} + U_i \rho \frac{\partial U_i}{\partial t} + \underline{U_i U_i U_j \frac{\partial \rho}{\partial x_j}} + \underline{\rho U_i U_i \frac{\partial U_j}{\partial x_j}} + \rho U_i U_j \frac{\partial U_i}{\partial x_j}$$

$$U_i \frac{\partial(\rho U_i)}{\partial t} + U_i \frac{\partial(\rho U_i U_j)}{\partial x_j} = \rho \frac{\partial k}{\partial t} + \rho U_j \frac{\partial k}{\partial x_j}$$

STANFORD
AERONAUTICS & ASTRONAUTICS

Use continuity again

$$k\frac{\partial\rho}{\partial t} + kU_j\frac{\partial\rho}{\partial x_j} + k\rho\frac{\partial U_j}{\partial x_j} = 0.$$

$$U_i\frac{\partial(\rho U_i)}{\partial t} + U_i\frac{\partial(\rho U_i U_j)}{\partial x_j} = \frac{\partial(\rho k)}{\partial t} + \frac{\partial}{\partial x_j}(\rho k U_j)$$

Rearrange the pressure term

$$U_i\frac{\partial(P\delta_{ij})}{\partial x_j} = \frac{\partial(PU_j)}{\partial x_j} - P\frac{\partial U_j}{\partial x_j}$$

and the viscous term.

$$U_i\frac{\partial\tau_{ij}}{\partial x_j} = \frac{\partial(U_i\tau_{ij})}{\partial x_j} - \tau_{ij}\frac{\partial U_i}{\partial x_j}.$$

The kinetic energy equation

$$\frac{\partial(\rho k)}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j k + PU_j - U_i\tau_{ij}) = \rho U_j G_j + P\frac{\partial U_j}{\partial x_j} - \tau_{ij}\frac{\partial U_i}{\partial x_j}.$$

STANFORD AERONAUTICS & ASTRONAUTICS

Various terms in the kinetic energy equation

kinetic energy transport by pressure forces.

Kinetic energy transport by viscous forces.

Kinetic energy generation by body forces.

$$\frac{\partial(\rho k)}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j k + P U_j - U_i \tau_{ij}) = \rho U_j G_j + P\frac{\partial U_j}{\partial x_j} - \tau_{ij}\frac{\partial U_i}{\partial x_j}.$$

kinetic energy transport by convection.

Reversible change between kinetic and internal energy due to work of compression or expansion

irreversible loss of kinetic energy due to friction

7.3 Internal energy

Subtract the kinetic energy equation from the energy equation.

$$\frac{\partial \rho e}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j e + Q_j) = -P\frac{\partial U_j}{\partial x_j} + \tau_{ij}\frac{\partial U_i}{\partial x_j}.$$

General conservation form

$$\frac{\partial(\ )}{\partial t} + \nabla \bullet (\quad) = sources.$$

Consider the source term

$$P\frac{\partial U_j}{\partial x_j} = P\nabla \bullet \bar{U}$$

Recall the continuity equation.

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \bullet \bar{U}.$$

Thermodynamic work

$$P\nabla \bullet \bar{U} = -\frac{P}{\rho}\frac{D\rho}{Dt} = \rho\left(P\frac{Dv}{Dt}\right).$$

7.4 Viscous dissipation of kinetic energy

Look at the dissipation term

$$\Phi = \tau_{ij}\frac{\partial U_i}{\partial x_j}.$$

Expand the velocity gradient tensor into symmetric and anti-symmetric parts

$$\Phi = \left(2\mu S_{ij} - \left(\frac{2}{3}\mu - \mu_v\right)\delta_{ij}S_{kk}\right)(S_{ij} + W_{ij})$$

Carry out the sums.

$$\Phi = 2\mu(S_{ij}S_{ij}) - \frac{2}{3}\mu(S_{ii}S_{kk}) + \mu_v(S_{ii}S_{kk}).$$

The dissipation can be written as a sum of squares.

$$\Phi = 2\mu\left(S_{ij} - \frac{1}{3}\delta_{ij}S_{kk}\right)\left(S_{ij} - \frac{1}{3}\delta_{ij}S_{kk}\right) + \mu_v(S_{ii}S_{kk})$$

Stokes' hypothesis

Mean normal stress

$$\sigma_{mean} = (1/3)\sigma_{ii} = -P + \mu_v S_{kk}.$$

Note also

$$trace(S_{ij} - (1/3)\delta_{ij} S_{kk}) = 0.$$

STANFORD AERONAUTICS & ASTRONAUTICS

7.5 Entropy

Recall the Gibbs equation

$$\rho\frac{De}{Dt}-\left(\frac{P}{\rho}\right)\frac{D\rho}{Dt} = \rho T\frac{Ds}{Dt}$$

Internal energy

$$\frac{\partial\rho e}{\partial t}+\frac{\partial}{\partial x_j}(\rho U_j e+Q_j) = -P\frac{\partial U_j}{\partial x_j}+\Phi$$

$$\rho\frac{De}{Dt}+P\frac{\partial U_j}{\partial x_j} = -\frac{\partial Q_j}{\partial x_j}+\Phi.$$

$$\left(\frac{P}{\rho}\right)\frac{D\rho}{Dt} = -P\frac{\partial U_j}{\partial x_j}$$

$$\underline{\rho\frac{De}{Dt}-\left(\frac{P}{\rho}\right)\frac{D\rho}{Dt}} = -\frac{\partial Q_j}{\partial x_j}+\Phi.$$

Use the Gibbs equation to replace the left-hand-side.

$$\underline{\rho T \frac{Ds}{Dt}} = -\frac{\partial Q_j}{\partial x_j} + \Phi.$$

Put this result into conservation equation form. Use continuity again.

$$\rho \frac{\partial s}{\partial t} + \rho U_j \frac{\partial s}{\partial x_j} = -\frac{1}{T}\frac{\partial Q_j}{\partial x_j} + \frac{\Phi}{T}$$

$$s\frac{\partial \rho}{\partial t} + sU_j \frac{\partial \rho}{\partial x_j} + s\rho \frac{\partial U_j}{\partial x_j} = 0$$

Add these two equations

$$\frac{\partial \rho s}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j s) = -\frac{1}{T}\frac{\partial Q_j}{\partial x_j} + \frac{\Phi}{T}.$$

Heat flux term

$$\frac{\partial}{\partial x_j}\left(\frac{Q_j}{T}\right) = \frac{1}{T}\frac{\partial Q_j}{\partial x_j} - \frac{Q_j}{T^2}\frac{\partial T}{\partial x_j}$$

Linear heat conducting material

$$Q_j = -k\frac{\partial T}{\partial x_j}.$$

Let

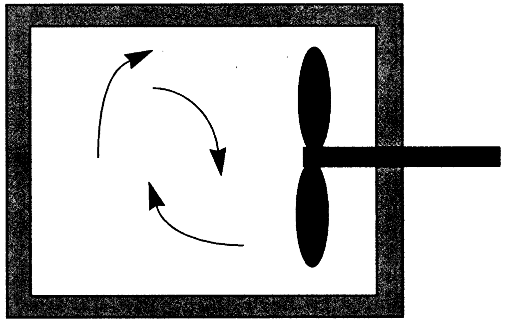
$$\Upsilon = \frac{k}{T}\left(\frac{\partial T}{\partial x_j}\frac{\partial T}{\partial x_j}\right).$$

Conservation equation for entropy

The source terms are always positive.

$$\boxed{\frac{\partial \rho s}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho U_j s - \frac{k}{T}\frac{\partial T}{\partial x_j}\right) = \frac{\Upsilon + \Phi}{T}}.$$

Entropy rise in an adiabatic box stirred by a fan.



$$\frac{d}{dt}\int_V \rho s\, dV + \int_A \left(\rho U_j s - \frac{k}{T}\frac{\partial T}{\partial x_j}\right) n_j\, dA = \int_V \left(\frac{\Upsilon + \Phi}{T}\right) dV .$$

$$S = \int_V \rho s\, dV$$

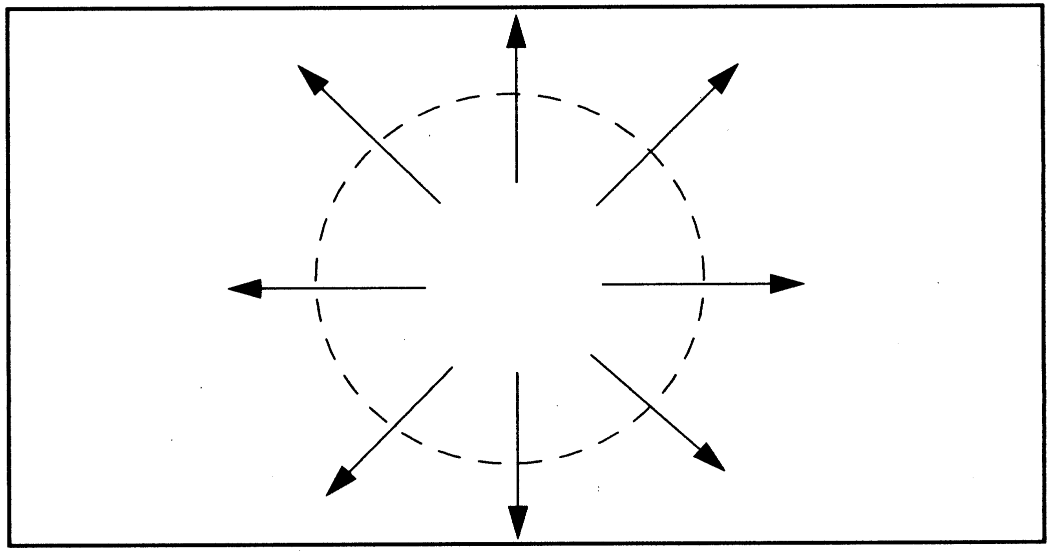
$$\frac{dS}{dt} = \int_V \left(\frac{\Upsilon + \Phi}{T}\right) dV > 0$$

STANFORD
AERONAUTICS &
ASTRONAUTICS

7.6 Problems

Problem 1 - Show that (7.17) can be expressed in the form (7.18).

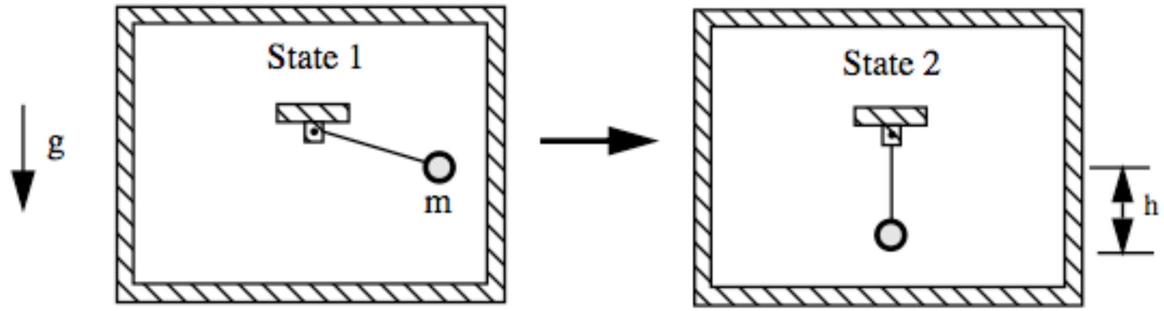
Problem 2 - Imagine a spherically symmetric flow in the absence of shear.



Let the fluid velocity in the radial direction be $f(r)$. Work out the expression for the kinetic energy dissipation in terms of f and the two viscosities μ, μ_v. Suppose the fluid is a monatomic gas for which $\mu_v = 0$. Does there exist a function f for which the dissipation is zero?

Problem 3 - Suppose the fluid in the box shown above is Air initially at 300K and one atmosphere. Let the tip velocity of the propellor be 50 m/sec and the typical boundary layer thickness over the surface of the propellor be 1 millimeter. The surface area of the propellor is $1m^2$ .and the volume of the box is $1m^3$. Roughly estimate the rate at which the temperature of the air in the box increases due to viscous dissipation.

Problem 4 - A simple pendulum of mass $m = 1.0\ kilogram$ is placed inside a rigid adiabatic container filled with $0.1\ kilograms$ of Helium gas initially at rest (state 1). The gas pressure is one atmosphere and the temperature is 300°K. The acceleration due to gravity is $g = 9.8\ meters/sec^2$.



At $t = 0$ the pendulum is released from an initial displacement height $h = 0.1\ meters$ and begins to oscillate and stir the gas. Eventually the pendulum and gas all come to rest (state 2).

1) Assume all the potential energy of the pendulum is converted to internal energy of the gas. What is the change in temperature of the Helium in going from state 1 to state 2?

2) What is the change in entropy per unit mass of the Helium in going from state 1 to state 2?

Sample problems from previous midterm exams

Problem 1 (10 points) – Consider two small parcels of air passing over an airfoil. Parcel A follows an adiabatic, isentropic path from A1 to A2. Parcel B follows a path that takes it very close to the surface of the airfoil where it comes nearly to rest.

At state 1 the velocity of both parcels is 100 m/sec, the temperature is 300°K and the pressure is 10^5 Pascals. The speed of parcel A at state 2 is 150 m/sec. The pressure of parcel B at state 2 is 6×10^4 Pascals.

1) Determine the temperature and pressure of parcel A at state 2.

Solution:

$$T_{A2} = T_{A1} + \frac{1}{2C_p}\left(U_{A1}^2 - U_{A2}^2\right) = 300 + \frac{1}{2\times1005}\left(100^2 - 150^2\right) = 293.781$$

$$P_{A2} = P_{A1}\left(\frac{T_{A2}}{T_{A1}}\right)^{\frac{\gamma}{\gamma-1}} = 10^5\left(\frac{293.781}{300}\right)^{3.5} = 9.29306\times10^4$$

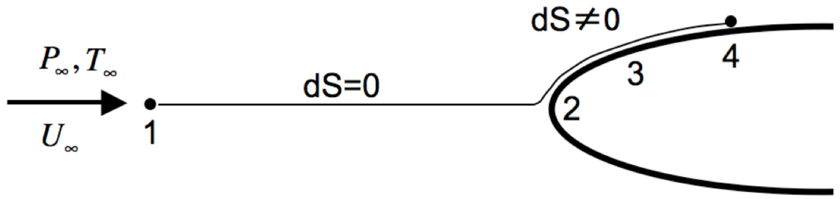
2) Determine the entropy change of parcel B between state 1 and state 2, $\left(s_{B2} - s_{B1}\right)/C_p$.

Solution: Flow stagnation temperature: $T_t = T_{A1} + \frac{U_{A1}}{2C_p}^2 = 300 + \frac{100^2}{2\times1005} = 304.975$

$$\left(\frac{s_{B2} - s_{B1}}{C_p}\right) = Ln\left(\frac{T_{B2}}{T_{B1}}\right) - \left(\frac{\gamma-1}{\gamma}\right)Ln\left(\frac{P_{B2}}{P_{B1}}\right) = Ln\left(\frac{304.975}{300}\right) - \left(\frac{0.4}{1.4}\right)Ln(0.6) = 0.162398$$

STANFORD
AERONAUTICS & ASTRONAUTICS

Problem 1 (15 points) – Consider a parcel of air passing over an adiabatic airfoil. The parcel follows an adiabatic, isentropic path from 1 to 2 where point 2 is very close to the stagnation point at the leading edge of the airfoil. From 2 the parcel slowly follows an adiabatic, non-isentropic path to 3 and 4 that is very close to the surface of the airfoil.



At state 1 the velocity of the parcel is $U_\infty = 100$ m/sec, the temperature is $T_\infty = 300K$ and the pressure is $P_\infty = 10^5$ Pascals. The pressure at 3 is the same as 1 (10^5 Pascals) and the pressure at 4 is half that at 1 (0.5×10^5 Pascals).

1) (**5 points**) Determine the temperature and pressure of the parcel at 2.
Solution
The stagnation temperature and pressure are

$$\frac{T_t}{T_\infty} = 1+\left(\frac{\gamma-1}{2}\right)M^2{}_\infty = 1+0.2\left(\frac{100}{347}\right)^2 = 1+0.2(0.288)^2 = 1.017$$

$$T_2 = 1.017\times300 = 305\,\mathrm{K}$$

$$\frac{P_t}{P_\infty} = \left(1+\left(\frac{\gamma-1}{2}\right)M^2{}_\infty\right)^{\frac{\gamma}{\gamma-1}} = \left(1+0.2(0.288)^2\right)^{3.5} = 1.059$$

$$P_2 = 1.059\times10^5\,\mathrm{Pa}$$

2) (**5 points**) Determine the entropy change of the parcel between state 3 and state 1, $(s_3 - s_1)/C_p$ and between state 4 and state 1 $(s_4 - s_1)/C_p$.

Solution

$$(s_3 - s_1)/C_p = Ln\left(\frac{T_3}{T_1}\right) - \left(\frac{\gamma-1}{\gamma}\right) Ln\left(\frac{P_3}{P_1}\right) = Ln\left(\frac{T_3}{T_1}\right) = Ln(1.017) = 0.017$$

$$(s_4 - s_1)/C_p = Ln\left(\frac{T_4}{T_1}\right) - \left(\frac{\gamma-1}{\gamma}\right) Ln\left(\frac{P_4}{P_1}\right) = Ln(1.017) - \left(\frac{0.4}{1.4}\right) Ln\left(\frac{1}{2}\right) = 0.215$$

3) (**5 points**) Consider an observer on the ground who sees the airfoil moving to the left at 100 m/sec. What would be the stagnation temperature and stagnation pressure of a fluid element at station 2 as measured by that observer?

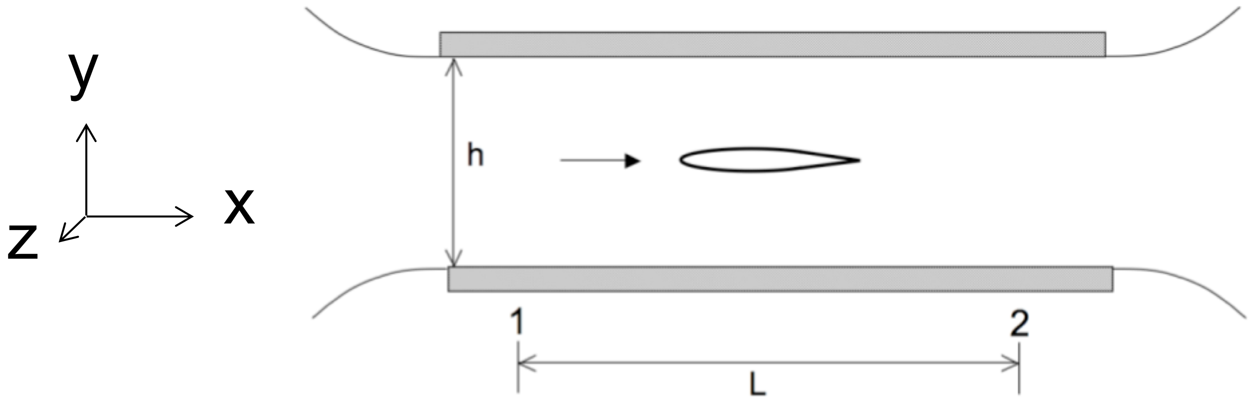
Solution

The temperature and pressure of the fluid element are $T_2 = 1.017 \times 300 = 305$ K and $P_2 = 1.059 \times 10^5$ Pa. In the frame of reference of the observer on the ground the stagnation temperature of the fluid element is

$$T_t = T_2 + \left(\frac{U_2^2}{2C_p}\right) = 305 + \left(\frac{100^2}{2010}\right) = 310 \text{ K}$$

$$P_t = P_2\left(1 + \left(\frac{U_2^2}{2C_p T_2}\right)\right)^{\frac{\gamma}{\gamma-1}} = 1.059\left(\frac{310}{305}\right)^{3.5} \times 10^5 = 1.121 \times 10^5 \text{ Pa}$$

Problem 2 (10 points) – The figure below depicts steady flow past a 3-D airfoil in a wind tunnel test section. Let the width of the tunnel be W.



1) **(5 points)** Use a control volume balance to relate the drag of the airfoil to the integrated flow at stations 1 and 2 as well as the integral of the wall shear stress between 1 and 2.

Solution

$$D = \int_{-W/2}^{W/2}\int_{-h/2}^{h/2}\left(\left(\rho_1 U_1^{\,2} + P_1 - \tau_{xx1}\right) - \left(\rho_2 U_2^{\,2} + P_2 - \tau_{xx2}\right)\right)dydz - \int_{-W/2}^{W/2}\int_0^L \tau_{xy}\left(x,0,z\right)dz - \int_{-W/2}^{W/2}\int_0^L \tau_{xy}\left(x,h,z\right)dz -$$

$$-\int_{-h/2}^{h/2}\int_0^L \tau_{xy}\left(x,y,0\right)dz - \int_{-h/2}^{h/2}\int_0^L \tau_{xy}\left(x,y,W\right)dz$$