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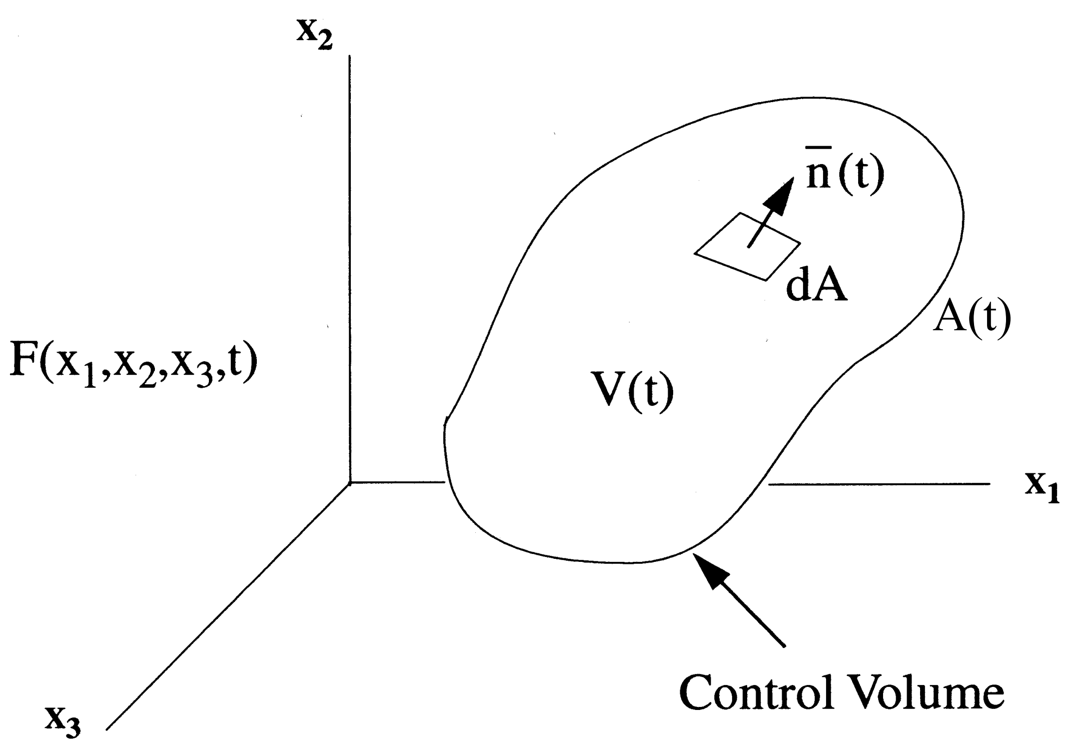
AA210A

Fundamentals of Compressible Flow

Chapter 3 - Control volumes, vector calculus

3.1 Control volume definition



The control volume is a closed, simply connected region in space.

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3.2 Vector calculus

Gradient operator

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x_i}$$

Gradient of a scalar

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \frac{\partial F}{\partial x_i}$$

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Gradient of a vector.

$$\nabla\bar{F} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}, & \frac{\partial F_1}{\partial x_2}, & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1}, & \frac{\partial F_2}{\partial x_2}, & \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_3}{\partial x_1}, & \frac{\partial F_3}{\partial x_2}, & \frac{\partial F_3}{\partial x_3} \end{pmatrix} = \frac{\partial F_i}{\partial x_j}$$

Divergence of a vector.

$$\nabla \bullet \bar{F} = trace(\nabla\bar{F}) = \delta_{ij}\frac{\partial F_i}{\partial x_j}$$

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} = \frac{\partial F_i}{\partial x_i}$$

The Kronecker unit tensor.

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The dot product of a vector and a tensor.

$$\bar{F} \bullet \nabla \bar{F} = F_j \frac{\partial F_i}{\partial x_j} = \begin{pmatrix} F_1 \frac{\partial F_1}{\partial x_1} + F_2 \frac{\partial F_1}{\partial x_2} + F_3 \frac{\partial F_1}{\partial x_3} \\ F_1 \frac{\partial F_2}{\partial x_1} + F_2 \frac{\partial F_2}{\partial x_2} + F_3 \frac{\partial F_2}{\partial x_3} \\ F_1 \frac{\partial F_3}{\partial x_1} + F_2 \frac{\partial F_3}{\partial x_2} + F_3 \frac{\partial F_3}{\partial x_3} \end{pmatrix}$$

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Curl of a vector.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix} =$$

$$\left(\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \right), \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \right), \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) \right)$$

Curl in index notation

$$(\nabla \times \bar{F})_i = \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j}$$

The alternating unit tensor (Levi-Civita tensor)

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{if any two indices are the same} \\ 1, & ijk \text{ an even permutation of } 1,2,3 \\ -1, & ijk \text{ an odd permutation of } 1,2,3 \end{cases}$$

Useful identities involving the Kronecker and alternating tensors

$$\begin{aligned}
\delta_{ij}\varepsilon_{ijk} &= 0 \\
\varepsilon_{ipq}\varepsilon_{jpq} &= 2\delta_{ij} \\
\varepsilon_{ijk}\varepsilon_{ijk} &= 6 \\
\varepsilon_{ijk}\varepsilon_{pqk} &= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}
\end{aligned}$$

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Some useful vector identities.

$$\nabla(\psi\phi) = \psi\nabla\phi + \phi\nabla\psi$$

$$\nabla\cdot\left(\phi\bar{U}\right) = \phi\nabla\cdot\bar{U} + \bar{U}\cdot\nabla\phi$$

$$\nabla\times\left(\phi\bar{U}\right) = \phi\nabla\times\bar{U} + \nabla\phi\times\bar{U}$$

$$\nabla\left(\bar{U}\cdot\bar{V}\right) = \left(\bar{U}\cdot\nabla\right)\bar{V} + \left(\bar{V}\cdot\nabla\right)\bar{U} + \bar{U}\times\left(\nabla\times\bar{V}\right) + \bar{V}\times\left(\nabla\times\bar{U}\right)$$

$$\nabla\cdot\left(\bar{U}\times\bar{V}\right) = \bar{V}\cdot\left(\nabla\times\bar{U}\right) - \bar{U}\cdot\left(\nabla\times\bar{V}\right)$$

$$\nabla\times\left(\bar{U}\times\bar{V}\right) = \bar{U}\left(\nabla\cdot V\right) + \left(\bar{V}\cdot\nabla\right)\bar{U} - \bar{V}\left(\nabla\cdot\bar{U}\right) - \left(\bar{U}\cdot\nabla\right)\bar{V}$$

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Some more vector identities - this time involving second derivatives.

$$\nabla \bullet (\nabla F) = \nabla^2 F$$

$$\nabla \bullet (\nabla \bar{F}) = \nabla^2 \bar{F}$$

$$\nabla \bullet (\nabla \times \bar{F}) = 0$$

$$\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \bullet \bar{F}) - \nabla^2 \bar{F}$$

Show that $\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \bullet \bar{F}) - \nabla^2 \bar{F}$

Let $A_k = \varepsilon_{kpq} \frac{\partial F_q}{\partial x_p}$ $\quad B_i = \varepsilon_{ijk} \frac{\partial A_k}{\partial x_j}$

$$\nabla \times (\nabla \times F)\big|_i = B_i = \varepsilon_{ijk} \frac{\partial A_k}{\partial x_j} = \varepsilon_{ijk} \varepsilon_{kpq} \frac{\partial^2 F_q}{\partial x_j \partial x_p}$$

Use the identity $\varepsilon_{ijk} \varepsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}$

Which is the same as $\varepsilon_{ijk} \varepsilon_{kpq} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}$

$$\nabla \times (\nabla \times F)\big|_i = \delta_{ip} \delta_{jq} \frac{\partial^2 F_q}{\partial x_j \partial x_p} - \delta_{iq} \delta_{jp} \frac{\partial^2 F_q}{\partial x_j \partial x_p}$$

$$\nabla \times (\nabla \times F)\big|_i = \frac{\partial^2 F_j}{\partial x_j \partial x_i} - \frac{\partial^2 F_i}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial F_j}{\partial x_j}\right) - \frac{\partial^2 F_i}{\partial x_j \partial x_j} = \left(\nabla\left(\nabla \bullet \bar{F}\right) - \nabla^2 \bar{F}\right)\Big|_i$$

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The Laplacian.

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \frac{\partial^2}{\partial x_i \partial x_i}.$$

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3.3 Gauss' theorem

This famous theorem in vector calculus can be used to convert a volume integral involving the gradient to a surface integral.

$$\int_V \nabla F \, dV = \int_A F \bar{n} \, dA$$

$$\int_V (\nabla \bullet \bar{F}) dV = \int_A \bar{F} \bullet \bar{n} \, dA$$

$$\int_V \frac{\partial F_{ij}}{\partial x_j} dV = \int_A F_{ij} n_j \, dA$$

The variable *F* can be a scalar, vector or tensor.

A volume integral involving the curl can be converted to a surface integral.

$$\int_V (\nabla \times \bar{F})\, dV = \int_A \bar{n} \times \bar{F}\, dA$$

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Recall the development of the continuity equation

$$\Delta x \Delta y \Delta z\left(\frac{\partial \rho}{\partial t}\right) + \Delta y \Delta z(\rho U|_{x+\Delta x} - \rho U|_x) +$$
$$\Delta x \Delta z(\rho V|_{y+\Delta y} - \rho V|_y) + \Delta x \Delta y(\rho W|_{z+\Delta z} - \rho W|_z) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\rho U|_{x+\Delta x} - \rho U|_x}{\Delta x} + \frac{\rho V|_{y+\Delta y} - \rho V|_y}{\Delta y} + \frac{\rho W|_{z+\Delta z} - \rho W|_z}{\Delta z} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} + \frac{\partial \rho W}{\partial z} = 0$$

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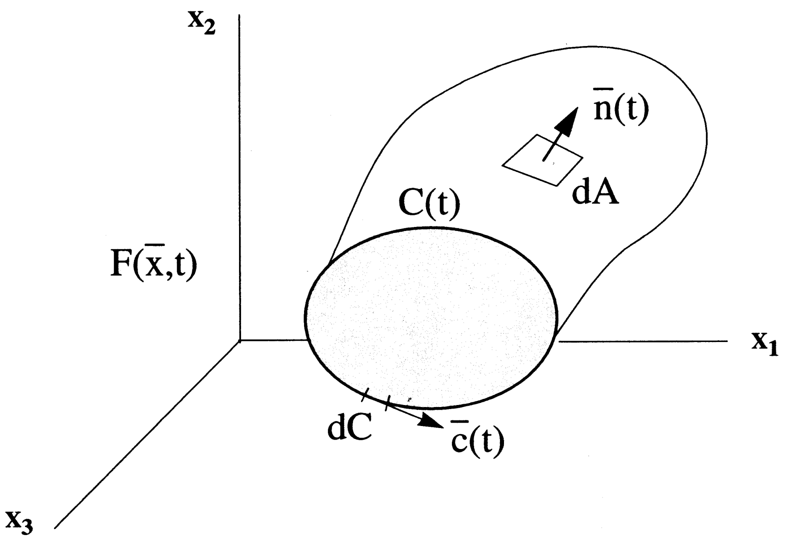
3.4 Stokes' theorem



$$\int_A (\nabla \times \bar{F}) \bullet \bar{n} \, dA = \oint_C \bar{F} \bullet \bar{c} \, dC$$

$$\int_A \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j} n_i \, dA = \oint_C F_k c_k \, dC$$

$$\int_A (\bar{n} \times \nabla F) \, dA = \oint_C F \bar{c} \, dC .$$

3.5 Problems

Problem 1 - Working in Cartesian coordinates and using index notation, prove each of the following the vector identities

$$\nabla \bullet (\rho \bar{F}) = \bar{F} \bullet \nabla\rho + \rho\nabla \bullet \bar{F} \tag{3.26}$$

$$\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \bullet \bar{F}) - \nabla^2 \bar{F} \tag{3.27}$$

$$\bar{F} \bullet \nabla\bar{F} = (\nabla \times \bar{F}) \times \bar{F} + \nabla\left(\frac{\bar{F} \bullet \bar{F}}{2}\right) \tag{3.28}$$

Problem 2 - Let $\bar{e}_i$, $\bar{e}_j$ and $\bar{e}_k$ be the unit vectors in a right hand orthogonal coordinate system. Show that

$$\varepsilon_{ijk} = \bar{e}_i \bullet (\bar{e}_j \times \bar{e}_k) \tag{3.29}$$

Problem 3 - Demonstrate Stokes' theorem by integration of the curl of some smooth vector field variable over a square boundary.

Problem 4 - Find a unit vector normal to each of the following surfaces.

i) $x + y + z = 2$

ii) $ax^2 + by^2 + cz^2 = 1$

iii) $xyz = 1$

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Problem 5 - Show that the unit vector normal to the plane

$$ax + by + cz = d \tag{3.30}$$

has the components

$$\bar{n} = \left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right) \tag{3.31}$$

Why doesn't $\bar{n}$ depend on d?

Problem 6 - Verify Gauss's theorem

$$\int_V (\nabla \cdot \bar{F}) dV = \int_A \bar{F} \cdot \bar{n} dA \tag{3.33}$$

in each of the following cases,

i) $\bar{F} = (x, y, z)$ and V is a cube of side b aligned with the x, y, z axes,

ii) $\bar{F} = \bar{n}_r r^2$ where n_r is a unit vector in the radial direction, V is a sphere of radius b surrounding the origin and $r^2 = x^2 + y^2 + z^2$.

Problem 7 - Verify Stokes' theorem

$$\int_A (\nabla \times \bar{F}) \cdot \bar{n} dV = \oint_C \bar{F} \cdot \bar{c} dC \tag{3.34}$$

where $\bar{F} = (x, y, z)$ and A is the surface of a cube of side b aligned with the x, y, z axes. The open face of the cube has an outward normal aligned with the positive x-axis.