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AA210A

Fundamentals of Compressible Flow

Chapter 1 - Introduction to fluid flow

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1.2 Conservation of mass

$$\left\{\begin{array}{c}\text{Rate of mass}\\ \text{accumulation}\\ \text{inside the control}\\ \text{volume}\end{array}\right\} = \left\{\begin{array}{c}\text{Rate of mass}\\ \text{flow}\\ \text{into the control}\\ \text{volume}\end{array}\right\} - \left\{\begin{array}{c}\text{Rate of mass}\\ \text{flow}\\ \text{out of the control}\\ \text{volume}\end{array}\right\}$$

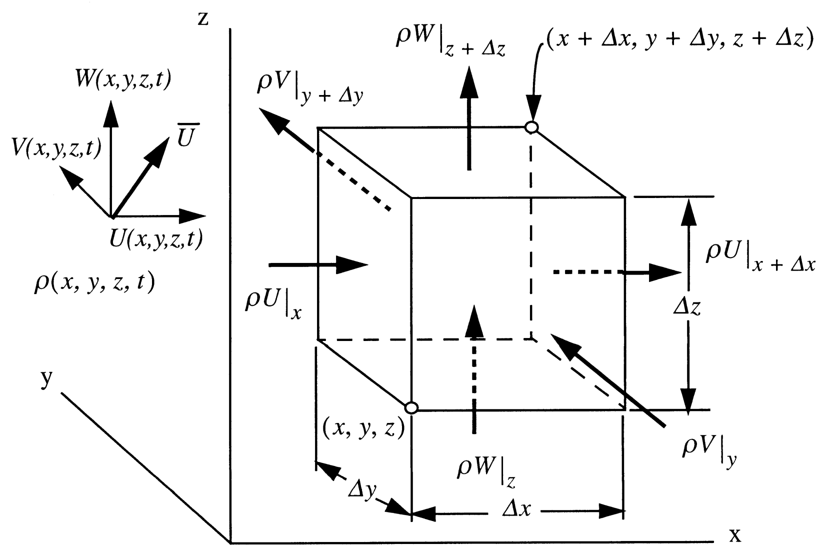

Figure 1.1 Fixed control volume in a moving fluid.

Mass flux in the x-direction

$$[\rho U] = \frac{M\left(\frac{L}{T}\right)}{L^3} = \frac{M}{L^2 T}$$

Momentum per unit volume

Mass per unit area per second

$$\Delta x \Delta y \Delta z\left(\frac{\partial \rho}{\partial t}\right) + \Delta y \Delta z(\rho U|_{x+\Delta x} - \rho U|_x) +$$

$$\Delta x \Delta z(\rho V|_{y+\Delta y} - \rho V|_y) + \Delta x \Delta y(\rho W|_{z+\Delta z} - \rho W|_z) = 0$$

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Divide through by the volume of the control volume.

$$\frac{\partial \rho}{\partial t} + \frac{\rho U|_{x+\Delta x} - \rho U|_x}{\Delta x} + \frac{\rho V|_{y+\Delta y} - \rho V|_y}{\Delta y} + \frac{\rho W|_{z+\Delta z} - \rho W|_z}{\Delta z} = 0$$

Let $(\Delta x \to 0, \Delta y \to 0, \Delta z \to 0)$. In this limit (1.4) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} + \frac{\partial \rho W}{\partial z} = 0$$

1.2.1 Conservation of mass - Incompressible flow

If the density is constant the continuity equation reduces to

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0.$$

Note that this equation applies to both steady and unsteady incompressible flow

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1.2.2 Index notation and the Einstein convention

Make the following replacements

$$(x, y, z) \rightarrow (x_1, x_2, x_3)$$

$$(U, V, W) \rightarrow (U_1, U_2, U_3)$$

Using index notation the continuity equation is

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial(\rho U_i)}{\partial x_i} = 0$$

Einstein recognized that such sums from vector calculus always involve a repeated index. For convenience he dropped the summation symbol.

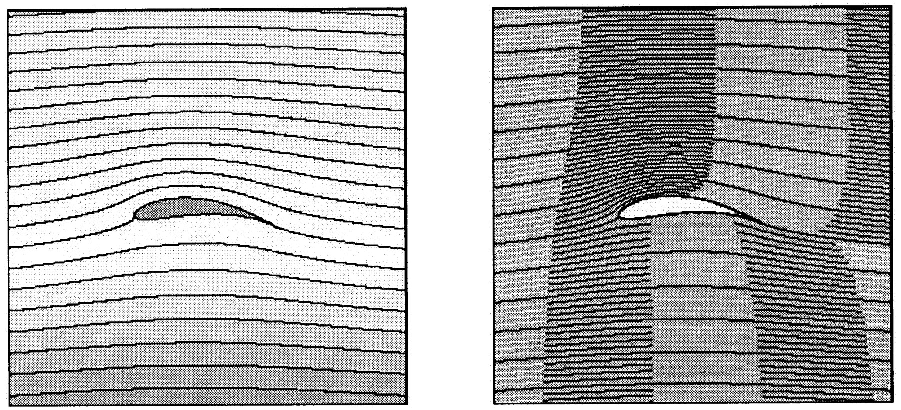
$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U_i)}{\partial x_i} = 0}$$

Coordinate independent form

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \bar{U}) = 0 \qquad \nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

1.3 Particle paths, streamlines and streaklines in 2-D steady flow

The figure below shows the streamlines over a 2-D airfoil.



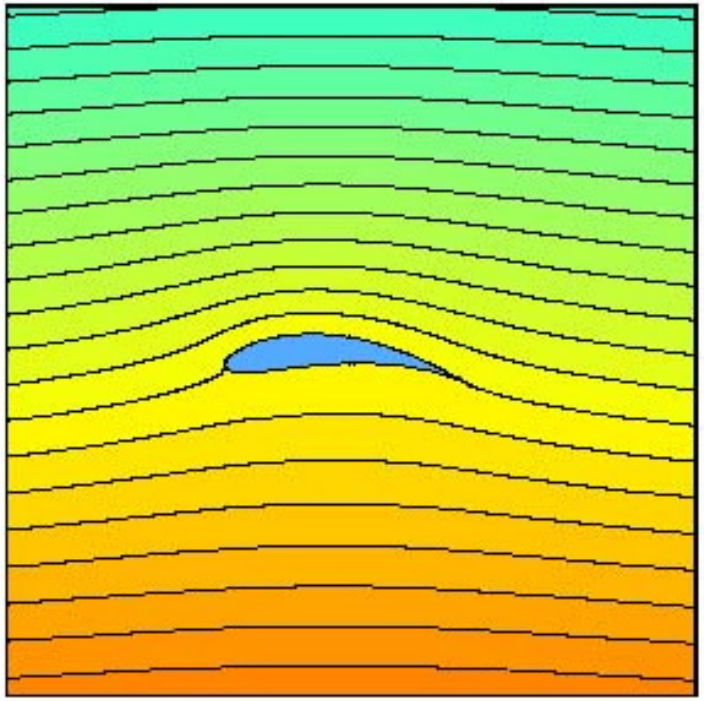
(a) (b)

Figure 1.2 Flow over a 2-D lifting wing; (a) streamlines, (b) streaklines.

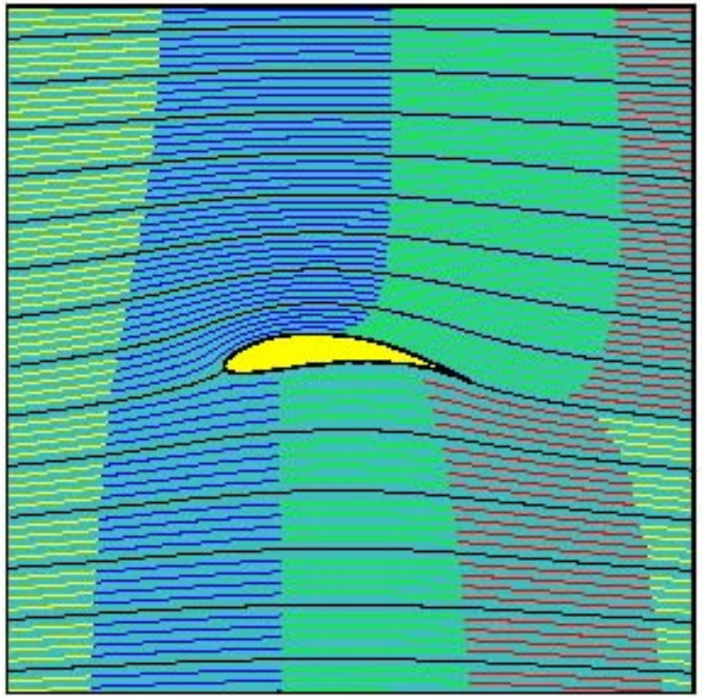
The flow is irrotational and incompressible

$$\nabla \times \bar{U} = 0 \qquad \nabla \bullet \bar{U} = 0.$$

Streamlines



Streaklines



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A vector field that satisfies $\nabla \times \overline{U} = 0$ can always be represented as the gradient of a scalar potential

$$\overline{U} = \nabla \Phi .$$

or

$$(U, V) = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}\right)$$

If the scalar potential is substituted into the continuity equation the result is Laplaces equation.

$$\nabla \bullet \nabla \Phi = \nabla^2 \Phi = 0 .$$

A weakly compressible example - flow over a wing flap.

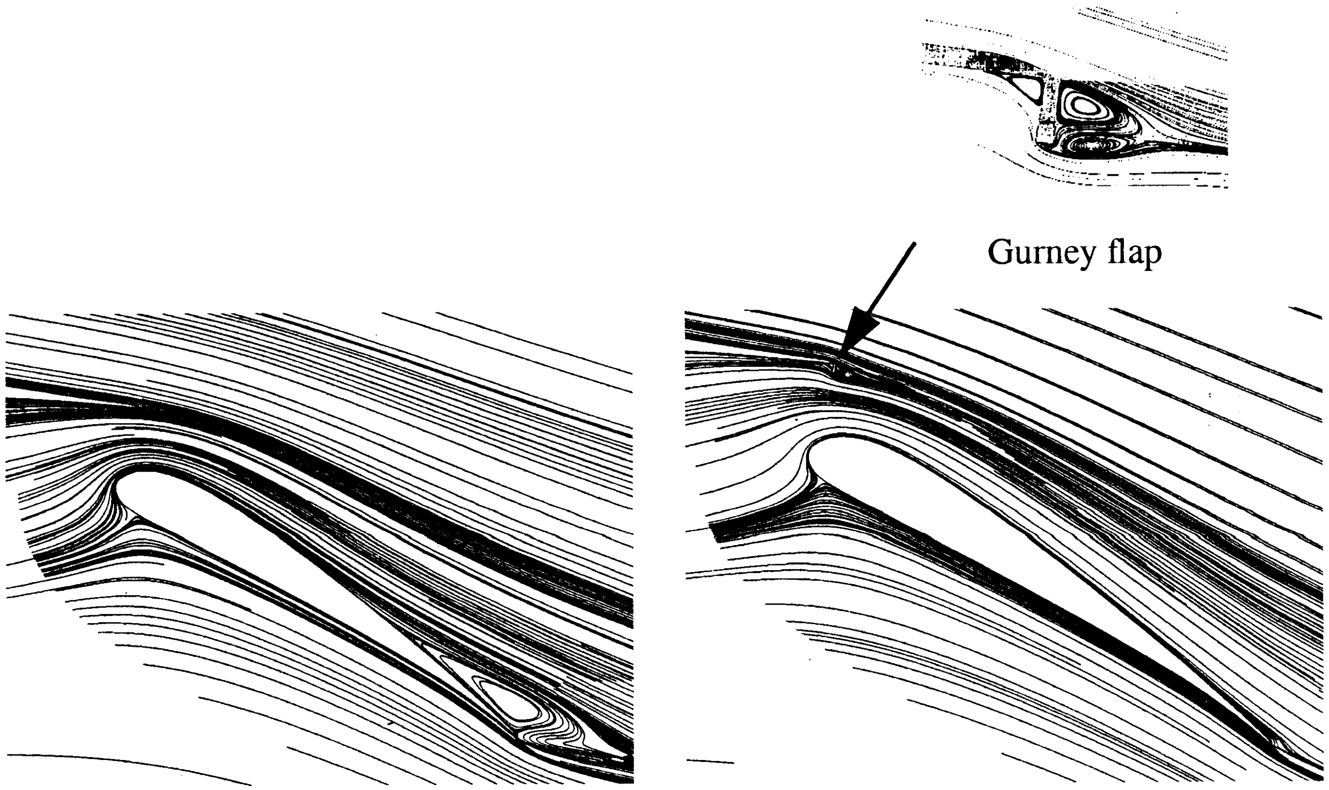
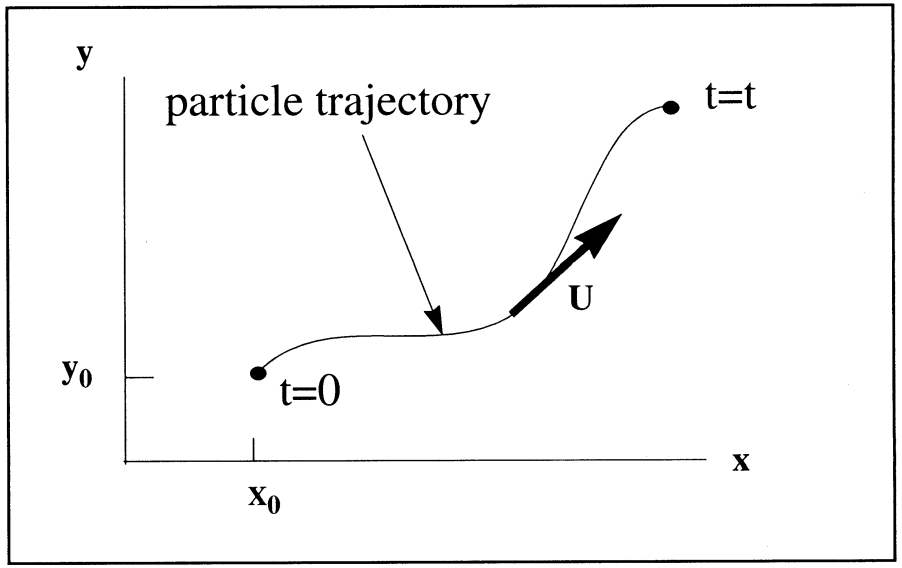


Figure 1.3 Computed streamlines over a wing flap.

Particle paths

The figure below shows the trajectory in space of a fluid element moving under the action of a two-dimensional steady velocity field



The equations that determine the trajectory are:

$$\left.\begin{aligned}\frac{dx(t)}{dt} &= U(x(t), y(t)) \\ \frac{dy(t)}{dt} &= V(x(t), y(t))\end{aligned}\right\}.$$

Formally, these equations are solved by integrating the velocity field in time.

$$\left.\begin{aligned} x(t) &= x_0 + \int_0^t U(x(t), y(t))dt \\ y(t) &= y_0 + \int_0^t V(x(t), y(t))dt \end{aligned}\right\}.$$

Along a particle path

$$x = F(x_0, y_0, t)\ ; \qquad y = G(x_0, y_0, t).$$

Eliminate time between the functions F and G to produce a family of lines. These are the streamlines observed in the figures shown earlier.

$$\psi = \Psi(x, y).$$

The value of a particular streamline is determined by

the initial conditions.

$$\psi_0 = \Psi(x_0, y_0).$$

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This situation is depicted schematically below.

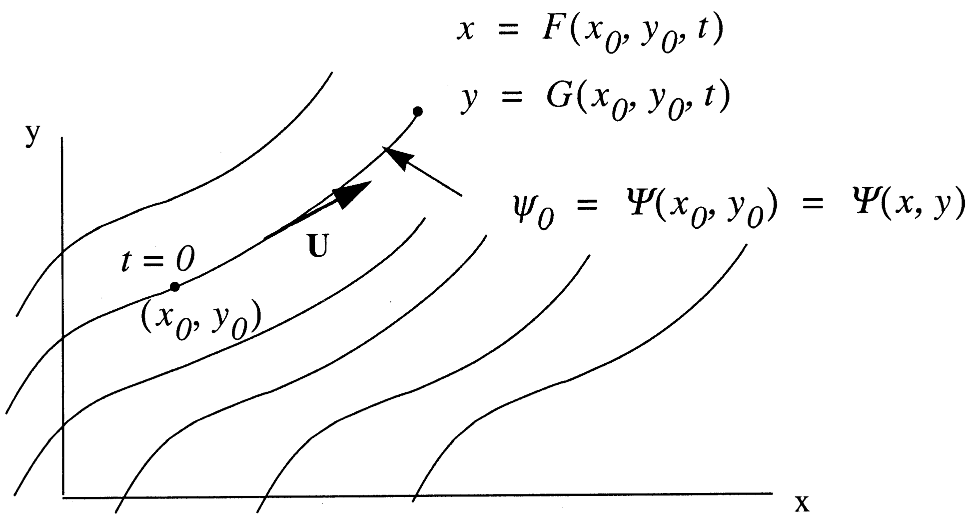

Figure 1.5 Streamlines in steady flow. The value of a particular streamline is determined by the coordinates of a point on the streamline. This can be regarded as the initial position of a fluid particle that traces out the streamline.

The streamfunction can also be determined by solving the first-order ODE generated by eliminating dt from the particle path equations.

$$\frac{dy}{dx} = \frac{V(x, y)}{U(x, y)}.$$

The total differential of the streamfunction is

$$d\psi = \frac{\partial \Psi}{\partial x}dx + \frac{\partial \Psi}{\partial y}dy.$$

Replace the differentials dx and dy.

$$d\psi = \left(U(x,y)\frac{\partial \Psi}{\partial x} + V(x,y)\frac{\partial \Psi}{\partial y} \right) dt .$$

The stream function, can be determined as the solution of a linear, first order PDE.

$$\boldsymbol{U} \bullet \nabla \Psi = U(x,y)\frac{\partial \Psi}{\partial x} + V(x,y)\frac{\partial \Psi}{\partial y} = 0 .$$

This equation is the mathematical expression of the statement that streamlines are parallel to the velocity vector field.

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The first-order ODE governing the stream function can be written as

$$-V(x, y)dx + U(x, y)dy = 0.$$

1.3.1 The integrating factor

On a streamline

$$\frac{\partial \Psi}{\partial x}dx + \frac{\partial \Psi}{\partial y}dy = 0.$$

What is the relationship between these two equations ?

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To be a perfect differential the functions U and V have to satisfy the integrability condition

$$-\frac{\partial V}{\partial y} = \frac{\partial U}{\partial x}.$$

For general functions U and V this condition is not satisfied. The equation $-V(x, y)dx + U(x, y)dy = 0$. must be multiplied by an integrating factor in order to convert it to a perfect differential.

It was shown by the German mathematician Johann Pfaff in the early 1800's that an integrating factor M(x,y) always exists.

$$d\psi = -M(x, y)V(x, y)dx + M(x, y)U(x, y)dy$$

and the partial derivatives are

$$\left.\begin{aligned} \frac{\partial \Psi}{\partial x} &= -M(x, y)V(x, y) \\ \frac{\partial \Psi}{\partial y} &= M(x, y)U(x, y) \end{aligned}\right\}$$

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1.3.2 Incompressible flow in 2 dimensions

The flow of an incompressible fluid in 2-D is constrained by the continuity equation

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

This is exactly the integrability condition . Continuity is satisfied identically by the introduction of the stream function,

$$U = \frac{\partial \Psi}{\partial y} \quad ; \qquad V = -\frac{\partial \Psi}{\partial x}$$

In this case -Vdx+Udy is guaranteed to be a perfect differential and one can write.

$$d\psi = -V dx + U dy .$$

1.3.3 Incompressible, irrotational flow in 2 dimensions

$$\frac{\partial \Psi}{\partial y} = \frac{\partial \Phi}{\partial x}$$

$$-\frac{\partial \Psi}{\partial x} = \frac{\partial \Phi}{\partial y}$$

The Cauchy-Reimann conditions

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1.3.4 Compressible flow in 2 dimensions

The continuity equation for the steady flow of a compressible fluid in two dimensions is

$$\frac{\partial}{\partial x}(\rho U) + \frac{\partial}{\partial y}(\rho V) = 0$$

In this case the required integrating factor is the density and we can write.

$$d\psi = -\rho V dx + \rho U dy$$

The stream function in a compressible flow is proportional to the mass flux and the convergence and divergence of lines in the flow over the flap shown earlier is a reflection of variations of mass flux over different parts of the flow field.

1.4 Particle paths in three dimensions

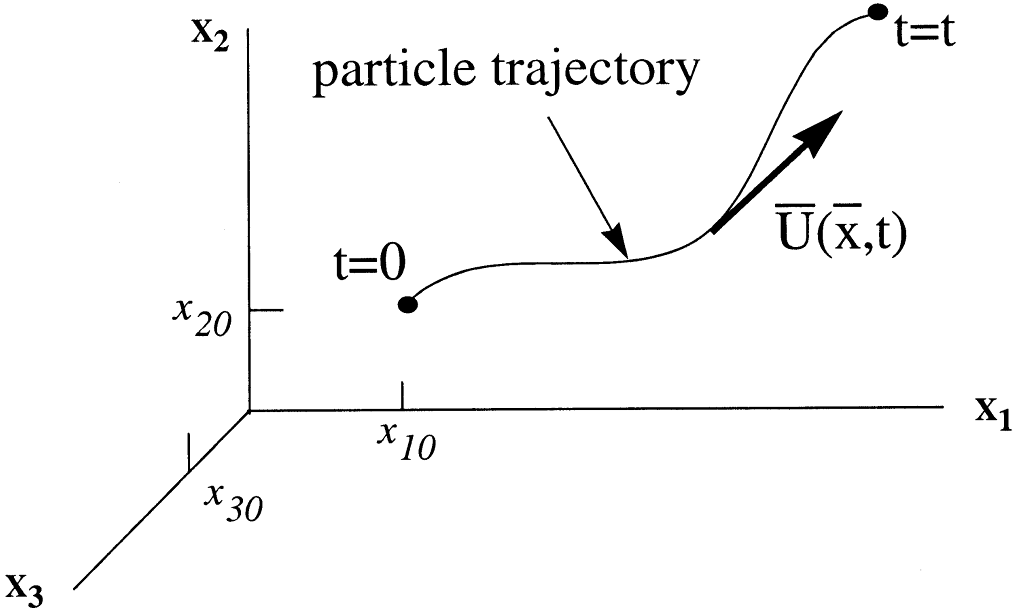

Figure 1.6 Particle trajectory in three dimensions

The figure above shows the trajectory in space traced out by a particle under the action of a general three dimensional unsteady flow, The equations governing the motion of the particle are:

$$\frac{dx_i(t)}{dt} = U_i(x_1(t), x_2(t), x_3(t), t) \quad ; \quad i = 1, 2, 3$$

Formally, these equations are solved by integrating the velocity field.

$$x_i(t) = x_{i0} + \int_0^t U_i(x_1(t), x_2(t), x_3(t), t)dt \quad ; \quad i = 1, 2, 3$$

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1.5 The substantial derivative

The acceleration of a particle is

$$\frac{d^2 x_i(t)}{dt^2} = \frac{d}{dt}U_i(x_1(t), x_2(t), x_3(t), t) = \frac{\partial U_i}{\partial t} + \frac{\partial U_i}{\partial x_k}\frac{dx_k}{dt}$$

Insert the velocities. The result is called the substantial or material derivative and is usually denoted by

$$\frac{DU_i}{Dt} = \frac{\partial U_i}{\partial t} + U_k \frac{\partial U_i}{\partial x_k} = \frac{\partial \bar{U}}{\partial t} + \bar{U} \bullet \nabla \bar{U}$$

The time derivative of any flow variable evaluated on a fluid element is given by a similar formula. For example the rate of change of density following a fluid particle is

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + U_k \frac{\partial \rho}{\partial x_k} = \frac{\partial \rho}{\partial t} + \bar{U} \bullet \nabla \rho$$

1.5.1 Frames of reference

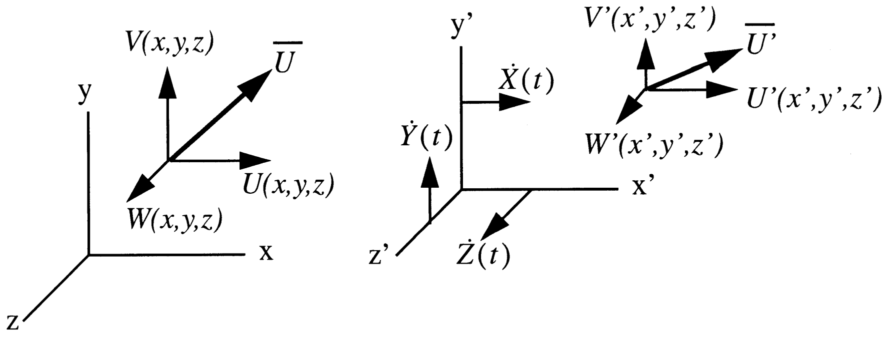


Figure 1.7 Fixed and moving frames of reference

Transformation of position and velocity

$$x' = x - X(t)$$
$$y' = y - Y(t)$$
$$z' = z - Z(t)$$
$$U' = U - \dot{X}(t)$$
$$V' = V - \dot{Y}(t)$$
$$W' = W - \dot{Z}(t)$$

Transformation of momentum

$$m\overline{U}' = m\overline{U} - m d\overline{X}/dt$$

momentum in moving coordinates

momentum in fixed coordinates

Transformation of kinetic energy

$$\text{kinetic energy in moving coordinates} = \frac{1}{2}m(U'^2 + V'^2 + W'^2)$$

$$\text{kinetic energy in fixed coordinates} = \frac{1}{2}m(U^2 + V^2 + W^2)$$

$$\frac{1}{2}m(U'^2 + V'^2 + W'^2) = \frac{1}{2}m((U - \dot{X})^2 + (V - \dot{Y})^2 + (W - \dot{Z})^2).$$

$$\frac{1}{2}m(U'^2 + V'^2 + W'^2) = \frac{1}{2}m(U^2 + V^2 + W^2) +$$

$$\frac{1}{2}m\dot{X}(\dot{X} - 2U) + \frac{1}{2}m\dot{Y}(\dot{Y} - 2V) + \frac{1}{2}m\dot{Z}(\dot{Z} - 2W)$$

$$k' = k + \frac{1}{2}m\dot{X}(\dot{X} - 2U) + \frac{1}{2}m\dot{Y}(\dot{Y} - 2V) + \frac{1}{2}m\dot{Z}(\dot{Z} - 2W).$$

Thermodynamic properties such as density, temperature and pressure do not depend on the frame of reference.

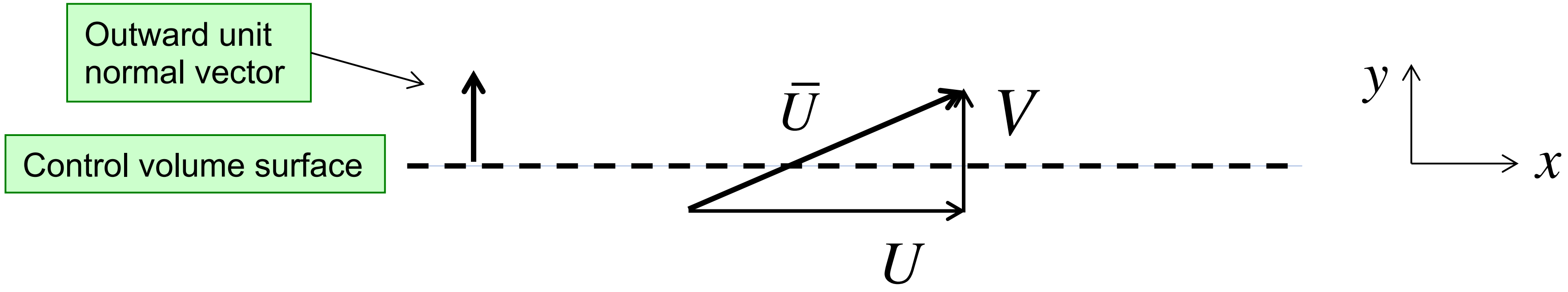
1.6 Momentum transport due to convection

Density

$$[\rho] = \frac{M}{L^3}$$

Volume flux in the y direction

$$[V] = \frac{L}{T} = \frac{L^3}{L^2 T} = \frac{Volume}{Area \cdot Sec}$$



Momentum flux

$$[\rho U V] = \frac{M\left(\frac{L}{T}\right)}{L^3}\left(\frac{L}{T}\right) = \frac{M\left(\frac{L}{T}\right)}{L^2 T}$$

- $\frac{M\left(\frac{L}{T}\right)}{L^3}$: x-momentum per unit volume
- $\left(\frac{L}{T}\right)$: Volume per unit area per second
- $\frac{M\left(\frac{L}{T}\right)}{L^2 T}$: x-momentum convected in the y-direction per unit area per second

The conservation equation for momentum

$$\left\{\begin{array}{c}\textit{Rate of}\\ \textit{momentum}\\ \textit{accumulation}\\ \textit{inside the}\\ \textit{control volume}\end{array}\right\} = \left\{\begin{array}{c}\textit{Rate of}\\ \textit{momentum flow}\\ \textit{into the}\\ \textit{control volume}\end{array}\right\} - \left\{\begin{array}{c}\textit{Rate of}\\ \textit{momentum flow}\\ \textit{out of the}\\ \textit{control volume}\end{array}\right\} + \left\{\begin{array}{c}\textit{Sum of}\\ \textit{forces acting}\\ \textit{on the}\\ \textit{control volume}\end{array}\right\}$$

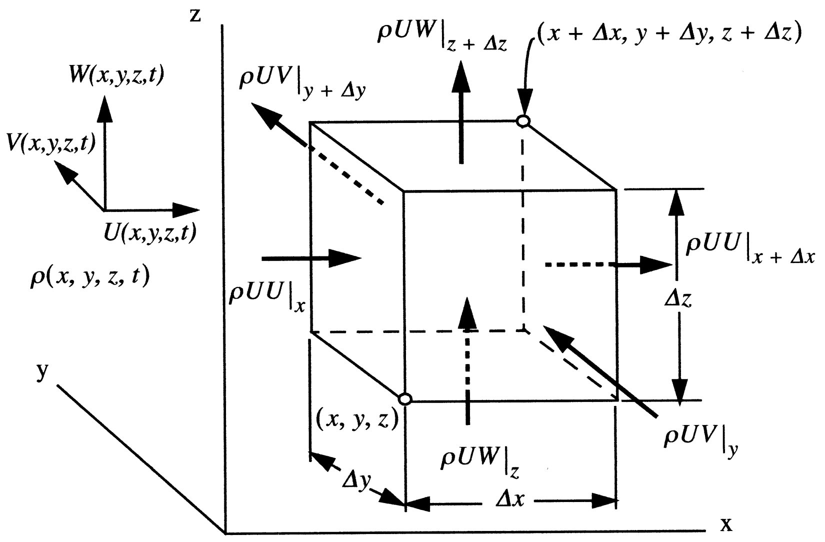


Figure 1.8 Fluxes of x-momentum through a fixed control volume. Arrows denote the velocity component carrying momentum into or out of the control volume.

$$\Delta x\Delta y\Delta z\left(\frac{\partial \rho U}{\partial t}\right) + \Delta y\Delta z(\rho UU|_{x+\Delta x} - \rho UU|_{x}) +$$

$$\Delta x\Delta z(\rho UV|_{y+\Delta y} - \rho UV|_{y}) + \Delta x\Delta y(\rho UW|_{z+\Delta z} - \rho UW|_{z}) =$$

$$\{\textit{the sum of x-component forces acting on the system}\}$$

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Divide through by the volume

$$\frac{\partial \rho U}{\partial t} + \frac{\rho UU|_{x+\Delta x} - \rho UU|_x}{\Delta x} + \frac{\rho UV|_{y+\Delta y} - \rho UV|_y}{\Delta y} + \frac{\rho UW|_{z+\Delta z} - \rho UW|_z}{\Delta z} = \left\{\begin{matrix} \textit{the sum of x-component forces acting on the system} \\ \textit{per unit volume} \end{matrix}\right\}$$

Let $(\Delta x \to 0, \Delta y \to 0, \Delta z \to 0)$. In this limit (1.54) becomes

$$\frac{\partial \rho U}{\partial t} + \frac{\partial \rho UU}{\partial x} + \frac{\partial \rho UV}{\partial y} + \frac{\partial \rho UW}{\partial z} = \left\{\begin{matrix} \textit{The sum of} \\ \textit{x-component forces} \\ \textit{per unit volume acting} \\ \textit{on the control volume} \end{matrix}\right\}$$

x - component

In the y and z directions

$$\frac{\partial \rho V}{\partial t} + \frac{\partial \rho VU}{\partial x} + \frac{\partial \rho VV}{\partial y} + \frac{\partial \rho VW}{\partial z} = \left\{\begin{matrix} \textit{The sum of} \\ \textit{y-component forces} \\ \textit{per unit volume acting} \\ \textit{on the control volume} \end{matrix}\right\}$$

$$\frac{\partial \rho W}{\partial t} + \frac{\partial \rho WU}{\partial x} + \frac{\partial \rho WV}{\partial y} + \frac{\partial \rho WW}{\partial z} = \left\{\begin{matrix} \textit{The sum of} \\ \textit{z-component forces} \\ \textit{per unit volume acting} \\ \textit{on the control volume} \end{matrix}\right\}$$

In index notation the momentum conservation equation is

$$\frac{\partial \rho U_i}{\partial t} + \frac{\partial(\rho U_i U_j)}{\partial x_j} = \left\{ \begin{matrix} \textit{Sum of the} \\ \textit{ith-component forces} \\ \textit{per unit volume acting} \\ \textit{on the control volume} \end{matrix} \right\} \quad ; \quad i = 1,2,3$$

Rearrange

$$\rho \frac{\partial U_i}{\partial t} + \rho U_j \frac{\partial (U_i)}{\partial x_j} + U_i \left(\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U_j)}{\partial x_j} \right) = \left\{ \begin{matrix} \textit{The sum of} \\ \textit{ith-component forces} \\ \textit{per unit volume acting} \\ \textit{on the control volume} \end{matrix} \right\}$$

$$\rho \frac{DU_i}{Dt} = \left\{ \begin{matrix} \textit{The sum of} \\ \textit{ith-component forces} \\ \textit{per unit volume acting} \\ \textit{on the control volume} \end{matrix} \right\}.$$

In words,

$$\left\{ \begin{matrix} \textit{The rate of momentum change} \\ \textit{of a fluid element} \end{matrix} \right\} = \left\{ \begin{matrix} \textit{The vector sum of} \\ \textit{forces acting} \\ \textit{on the fluid element} \end{matrix} \right\}$$

1.7 Momentum transport due to molecular motion

1.7.1 Pressure

1.7.2 Viscous friction - Plane Couette Flow

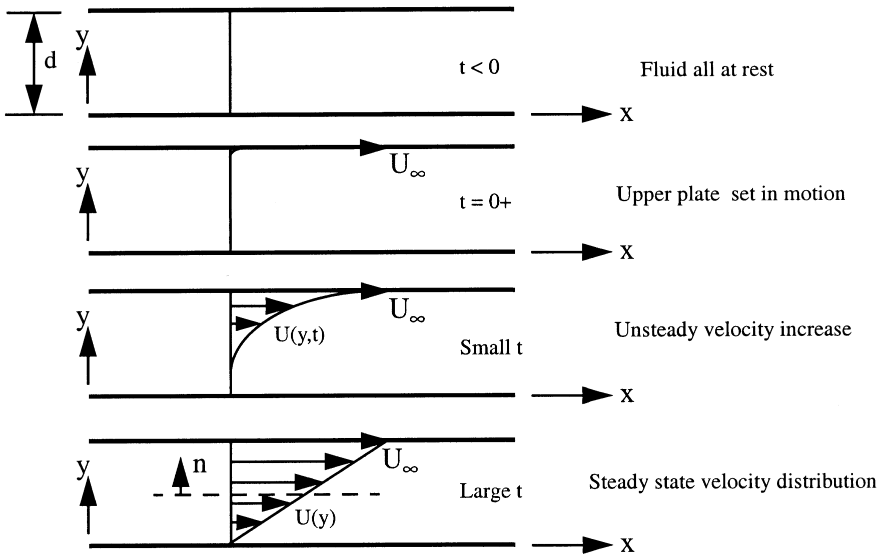

Figure 1.9 Build-up to a steady laminar velocity profile for a viscous fluid contained between two parallel plates. At t=0 the upper plate is set into motion at a constant speed U_∞.

Force/Area needed to maintain the motion of the upper plate

$$\frac{F}{A} = \mu \frac{U_\infty}{d} \qquad \tau_{xy} = \mu \frac{dU}{dy}$$

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1.7.3 A question of signs

1.7.4 Newtonian fluids

1.7.5 Forces acting on a fluid element

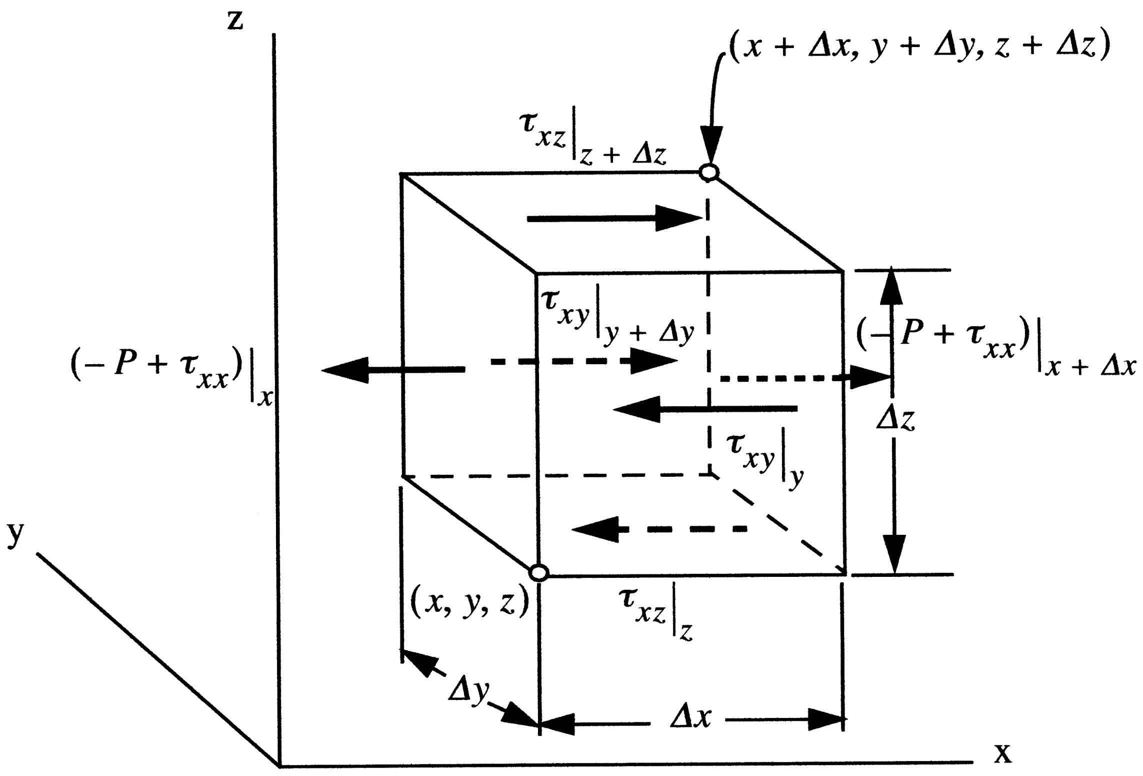

Figure 1.10 Pressure and viscous stresses acting in the x-direction

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Pressure-viscous-stress force components

$$F_x = \Delta y \Delta z\left((-P+\tau_{xx})\big|_{x+\Delta x} - (-P+\tau_{xx})\big|_x\right) + \Delta x \Delta z\left(\tau_{xy}\big|_{y+\Delta y} - \tau_{xy}\big|_y\right) +$$
$$\Delta x \Delta y\left(\tau_{xz}\big|_{z+\Delta z} - \tau_{xz}\big|_z\right)$$

$$F_y = \Delta y \Delta z\left(\tau_{xy}\big|_{x+\Delta x} - \tau_{xy}\big|_x\right) + \Delta x \Delta z\left((-P+\tau_{yy})\big|_{y+\Delta y} - (-P+\tau_{yy})\big|_y\right) +$$
$$\Delta x \Delta y\left(\tau_{yz}\big|_{z+\Delta z} - \tau_{yz}\big|_z\right)$$

$$F_z = \Delta y \Delta z\left(\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x\right) + \Delta x \Delta z\left(\tau_{yz}\big|_{y+\Delta y} - \tau_{yz}\big|_y\right) +$$
$$\Delta x \Delta y\left((-P+\tau_{zz})\big|_{z+\Delta z} - (-P+\tau_{zz})\big|_z\right)$$

Momentum balance in the x-direction

$$\Delta x \Delta y \Delta z\left(\frac{\partial \rho U}{\partial t}\right) = \Delta y \Delta z(\rho UU|_x - \rho UU|_{x+\Delta x}) +$$
$$\Delta x \Delta z(\rho UV|_y - \rho UV|_{y+\Delta y}) + \Delta x \Delta y(\rho UW|_z - \rho UW|_{z+\Delta z}) +$$
$$\Delta y \Delta z\left((-P+\tau_{xx})\big|_{x+\Delta x} - (-P+\tau_{xx})\big|_x\right) +$$
$$\Delta x \Delta z\left(\tau_{xy}\big|_{y+\Delta y} - \tau_{xy}\big|_y\right) + \Delta x \Delta y\left(\tau_{xz}\big|_{z+\Delta z} - \tau_{xz}\big|_z\right)$$

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Divide by the volume

$$\frac{\partial \rho U}{\partial t} + \frac{\rho UU|_{x+\Delta x} - \rho UU|_x + (P-\tau_{xx})|_{x+\Delta x} - (P-\tau_{xx})|_x}{\Delta x} +$$

$$\frac{\rho UV|_{y+\Delta y} - \rho UV|_y - \left(\tau_{xy}|_{y+\Delta y} - \tau_{xy}|_y\right)}{\Delta y} +$$

$$\frac{\rho UW|_{z+\Delta z} - \rho UW|_z - \left(\tau_{xz}|_{z+\Delta z} - \tau_{xz}|_z\right)}{\Delta z} = 0$$

Let $(\Delta x \to 0, \Delta y \to 0, \Delta z \to 0)$. In this limit (1.65) becomes

$$\frac{\partial \rho U}{\partial t} + \frac{\partial(\rho UU + P - \tau_{xx})}{\partial x} + \frac{\partial(\rho UV - \tau_{xy})}{\partial y} + \frac{\partial(\rho UW - \tau_{xz})}{\partial z} = 0$$

x - component

In the y and z directions

$$\frac{\partial \rho V}{\partial t} + \frac{\partial(\rho VU - \tau_{xy})}{\partial x} + \frac{\partial(\rho VV + P - \tau_{yy})}{\partial y} + \frac{\partial(\rho VW - \tau_{yz})}{\partial z} = 0$$

$$\frac{\partial \rho W}{\partial t} + \frac{\partial(\rho WU - \tau_{xz})}{\partial x} + \frac{\partial(\rho WV - \tau_{yz})}{\partial y} + \frac{\partial(\rho WW + P - \tau_{zz})}{\partial z} = 0$$

In index notation the equation for conservation of momentum is

$$\frac{\partial \rho U_i}{\partial t} + \frac{\partial(\rho U_i U_j)}{\partial x_j} + \frac{\partial P}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} = 0 \quad ; \qquad i = 1,2,3 .$$

Coordinate independent form

$$\frac{\partial \rho \bar{U}}{\partial t} + \nabla \bullet (\rho \bar{U} \bar{U}) + \nabla P - \nabla \bullet \bar{\bar{\tau}} = 0 .$$

1.7 Conservation of energy

$$\left\{\begin{array}{c}\textit{Rate of}\\ \textit{energy}\\ \textit{accumulation}\\ \textit{inside the}\\ \textit{control volume}\end{array}\right\} = \left\{\begin{array}{c}\textit{Rate of}\\ \textit{energy flow}\\ \textit{into the}\\ \textit{control volume}\\ \textit{by convection}\end{array}\right\} - \left\{\begin{array}{c}\textit{Rate of}\\ \textit{energy flow}\\ \textit{out of the}\\ \textit{control volume}\\ \textit{by convection}\end{array}\right\} +$$

$$\left\{\begin{array}{c}\textit{Work done on the}\\ \textit{control volume}\\ \textit{by pressure and}\\ \textit{viscous forces}\end{array}\right\} + \left\{\begin{array}{c}\textit{Rate of energy}\\ \textit{addition due to}\\ \textit{heat conduction}\end{array}\right\} + \left\{\begin{array}{c}\textit{Energy generation}\\ \textit{due to sources}\\ \textit{inside the}\\ \textit{control volume}\end{array}\right\}$$

$$k = (1/2)(U^2 + V^2 + W^2)$$

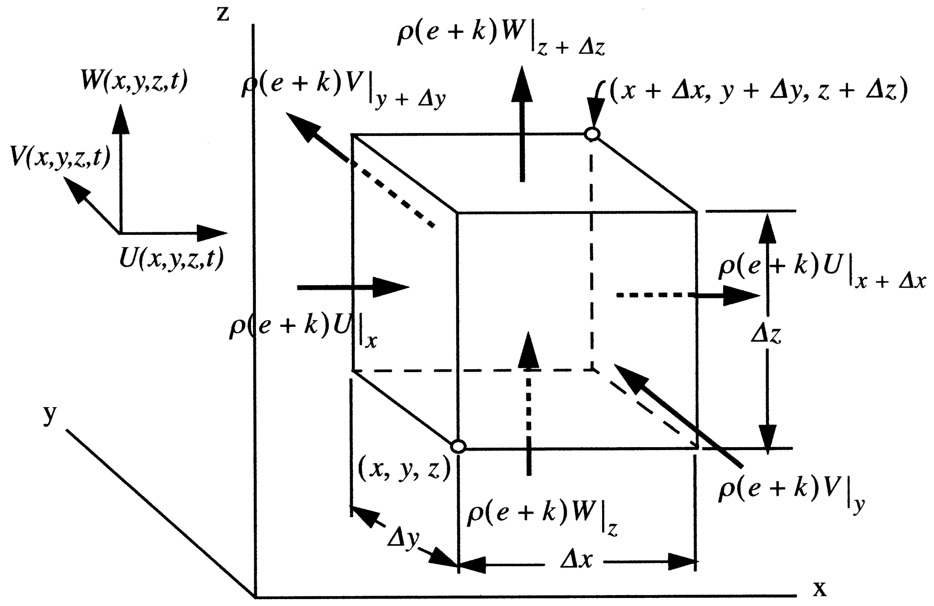

Figure 1.11 Convection of energy into and out of a control volume.

1.8.1 Pressure and viscous work

$$\textit{Power input to the control volume} = \bar{F} \bullet \bar{U}$$

Fully written out this relation is

$$\textit{Power input to the control volume} =$$

$$\Delta y \Delta z \left\{ \left((-P+\tau_{xx})\Big|_{x+\Delta x} - (-P+\tau_{xx})\Big|_{x} \right) U + \left(\tau_{xy}\Big|_{x+\Delta x} - \tau_{xy}\Big|_{x} \right) V + \right.$$

$$\left. \left(\tau_{xz}\Big|_{x+\Delta x} - \tau_{xz}\Big|_{x} \right) W \right\} +$$

$$\Delta x \Delta z \left\{ \left(\tau_{xy}\Big|_{y+\Delta y} - \tau_{xy}\Big|_{y} \right) U + \left((-P+\tau_{yy})\Big|_{y+\Delta y} - (-P+\tau_{yy})\Big|_{y} \right) V + \right.$$

$$\left. \left(\tau_{yz}\Big|_{y+\Delta y} - \tau_{yz}\Big|_{y} \right) W \right\} +$$

$$\Delta x \Delta y \left\{ \left(\tau_{xz}\Big|_{z+\Delta z} - \tau_{xz}\Big|_{z} \right) U + \left(\tau_{yz}\Big|_{z+\Delta z} - \tau_{yz}\Big|_{z} \right) V + \right.$$

$$\left. \left((-P+\tau_{zz})\Big|_{z+\Delta z} - (-P+\tau_{zz})\Big|_{z} \right) W \right\}$$

The previous equation can be rearranged to read in terms of energy fluxes.

Power input to the control volume =

$$\Delta y\Delta z\left\{(-PU+\tau_{xx}U+\tau_{xy}V+\tau_{xz}W)\Big|_{x+\Delta x}-(-PU+\tau_{xx}U+\tau_{xy}V+\tau_{xz}W)\Big|_{x}\right\}+$$

$$\Delta x\Delta z\left\{(-PV+\tau_{xy}U+\tau_{yy}V+\tau_{yz}W)\Big|_{y+\Delta y}-(-PV+\tau_{xy}U+\tau_{yy}V+\tau_{yz}W)\Big|_{y}\right\}+$$

$$\Delta x\Delta y\left\{(-PW+\tau_{zx}U+\tau_{zy}V+\tau_{zz}W)\Big|_{z+\Delta z}-(-PW+\tau_{zx}U+\tau_{zy}V+\tau_{zz}W)\Big|_{z}\right\}$$

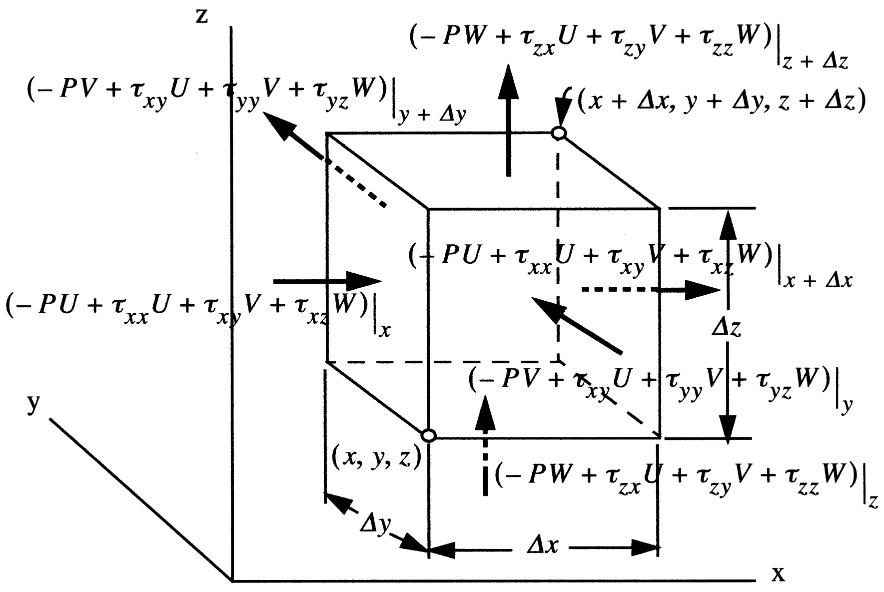

Figure 1.12 Energy fluxes due to the work done on the control volume by pressure and viscous forces.

Energy balance.

$$\Delta x\Delta y\Delta z\left(\frac{\partial\rho(e+k)}{\partial t}\right) = \Delta y\Delta z\left(\rho(e+k)U\big|_{x} - \rho(e+k)U\big|_{x+\Delta x}\right) +$$
$$\Delta x\Delta z\left(\rho(e+k)V\big|_{y} - \rho(e+k)V\big|_{y+\Delta y}\right) +$$
$$\Delta x\Delta y\left(\rho(e+k)W\big|_{z} - \rho(e+k)W\big|_{z+\Delta z}\right) +$$
$$\Delta y\Delta z\left((-PU+\tau_{xx}U+\tau_{xy}V+\tau_{xz}W)\big|_{x+\Delta x} - (-PU+\tau_{xx}U+\tau_{xy}V+\tau_{xz}W)\big|_{x}\right)$$
$$\Delta x\Delta z\left((-PV+\tau_{xy}U+\tau_{yy}V+\tau_{yz}W)\big|_{y+\Delta y} - (-PV+\tau_{xy}U+\tau_{yy}V+\tau_{yz}W)\big|_{y}\right) \qquad (1.76)$$
$$\Delta x\Delta y\left((-PW+\tau_{zx}U+\tau_{zy}V+\tau_{zz}W)\big|_{z+\Delta z} - (-PW+\tau_{zx}U+\tau_{zy}V+\tau_{zz}W)\big|_{z}\right) +$$
$$\Delta y\Delta z\left(Q_x\big|_{x} - Q_x\big|_{x+\Delta x}\right) + \Delta x\Delta z\left(Q_y\big|_{y} - Q_y\big|_{y+\Delta y}\right) + \Delta x\Delta y\left(Q_z\big|_{z} - Q_z\big|_{z+\Delta z}\right) +$$

{*Power generation due to sources inside the control volume*}

Divide (1.76) through by the infinitesimal volume $\Delta x\Delta y\Delta z$ and take the limit $(\Delta x \rightarrow 0, \Delta y \rightarrow 0, \Delta z \rightarrow 0)$. The conservation equation for the energy becomes

$$\frac{\partial\rho(e+k)}{\partial t} + \frac{\partial(\rho(e+k)U)}{\partial x} + \frac{\partial(\rho(e+k)U)}{\partial y} + \frac{\partial(\rho(e+k)U)}{\partial z} +$$
$$\frac{\partial(PU-\tau_{xx}U-\tau_{xy}V-\tau_{xz}W)}{\partial x} + \frac{\partial(PV-\tau_{xy}U-\tau_{yy}V-\tau_{yz}W)}{\partial y} + \qquad (1.77)$$
$$\frac{\partial(PW-\tau_{zx}U-\tau_{zy}V-\tau_{zz}W)}{\partial z} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} =$$

{*Power generation due to sources inside the control volume*}

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In index notation the equation for conservation of energy is

$$\frac{\partial \rho(e+k)}{\partial t} + \frac{\partial(\rho(e+k)U_j)}{\partial x_j} + \frac{\partial P U_j}{\partial x_j} - \frac{\partial(U_i \tau_{ij})}{\partial x_j} + \frac{\partial Q_j}{\partial x_j} = \{\mathit{Power\ sources}\}.$$

Coordinate independent form

$$\frac{\partial \rho(e+k)}{\partial t} + \nabla \bullet (\rho(e+k)\overline{U} + P\overline{U} - \bar{\bar{\tau}} \bullet \overline{U} + \overline{Q}) = \{\mathit{Power\ sources}\}$$

1.9 Summary - the equations of motion

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} + \frac{\partial \rho W}{\partial z} = 0$$

Conservation of momentum

$$\frac{\partial \rho U}{\partial t} + \frac{\partial(\rho UU + P - \tau_{xx})}{\partial x} + \frac{\partial(\rho UV - \tau_{xy})}{\partial y} + \frac{\partial(\rho UW - \tau_{xz})}{\partial z} = 0$$

$$\frac{\partial \rho V}{\partial t} + \frac{\partial(\rho VU - \tau_{xy})}{\partial x} + \frac{\partial(\rho VV + P - \tau_{yy})}{\partial y} + \frac{\partial(\rho VW - \tau_{yz})}{\partial z} = 0$$

$$\frac{\partial \rho W}{\partial t} + \frac{\partial(\rho WU - \tau_{xz})}{\partial x} + \frac{\partial(\rho WV - \tau_{yz})}{\partial y} + \frac{\partial(\rho WW + P - \tau_{zz})}{\partial z} = 0 .$$

Conservation of energy

$$\frac{\partial \rho(e+k)}{\partial t} + \frac{\partial(\rho(e+k)U)}{\partial x} + \frac{\partial(\rho(e+k)U)}{\partial y} + \frac{\partial(\rho(e+k)U)}{\partial z} +$$

$$\frac{\partial(PU - \tau_{xx}U - \tau_{xy}V - \tau_{xz}W)}{\partial x} + \frac{\partial(PV - \tau_{xy}U - \tau_{yy}V - \tau_{yz}W)}{\partial y} +$$

$$\frac{\partial(PW - \tau_{zx}U - \tau_{zy}V - \tau_{zz}W)}{\partial z} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} = \{Power\ sources\}$$

Some remarks on the pressure field

Two dimensional steady, inviscid, incompressible flow

Conservation of mass

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0$$

Conservation of momentum

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-\frac{\partial}{\partial x}\left(\frac{P}{\rho}\right)$$

$$U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}=-\frac{\partial}{\partial y}\left(\frac{P}{\rho}\right)$$

Vorticity

$$\Omega=\frac{\partial V}{\partial x}-\frac{\partial U}{\partial y}$$

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For any steady, inviscid, incompressible, irrotational velocity field the pressure field exists!

$$\Omega = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{P}{\rho}\right) = -\left(U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right) = -U\frac{\partial U}{\partial x} - V\frac{\partial V}{\partial x} = -\frac{1}{2}\frac{\partial}{\partial x}\left(U^2 + V^2\right)$$

$$\frac{\partial}{\partial y}\left(\frac{P}{\rho}\right) = -\left(U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y}\right) = -U\frac{\partial U}{\partial y} - V\frac{\partial V}{\partial y} = -\frac{1}{2}\frac{\partial}{\partial y}\left(U^2 + V^2\right)$$

$$\frac{\partial}{\partial x}\left(\frac{P}{\rho} + \frac{1}{2}\left(U^2 + V^2\right)\right) = 0$$

$$\frac{\partial}{\partial y}\left(\frac{P}{\rho} + \frac{1}{2}\left(U^2 + V^2\right)\right) = 0$$

$$\frac{P}{\rho} + \frac{1}{2}\left(U^2 + V^2\right) = const$$

This is the incompressible Bernoulli pressure

1.10 Problems

Problem 1 - Show that the continuity equation can be expressed as

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{\partial U_j}{\partial x_j} = 0 \tag{1.94}$$

Problem 2 - Use direct measurements from the streamlines in Figure 2.13 to estimate the percent change from the free stream velocity at points A , B , C and D.

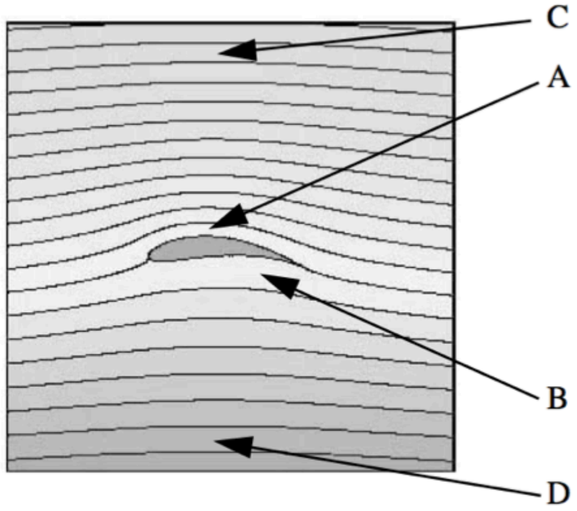

Figure 1.13: *Streamlines about a wing in potential flow.*

Problem 3 - The general, first order, linear ODE

$$\frac{dy}{dx} = -g(x)\,y + f(x) \tag{1.95}$$

can be written as the differential form

$$(g(x)\,y - f(x))\,dx + dy = 0 \tag{1.96}$$

Show that (1.96) can be converted to a perfect differential by multiplying by the integrating factor.

$$M = e^{\int g(x)dx} \tag{1.97}$$

Work out the solution of (1.95) in terms of integrals. What is the solution for the case $g = \sin(x)$, $f = \cos(x)$? Sketch the corresponding streamline pattern.

Problem 4 - Solve

$$y\frac{\partial \Psi}{\partial x} - \frac{x}{3}\frac{\partial \Psi}{\partial y} = 0. \tag{1.98}$$

Sketch the resulting streamline pattern.

Problem 5 - Show that the following expression is a perfect differential.

$$-\sin(x)\sin(y)\,dx + \cos(x)\cos(y)\,dy = 0 \tag{1.99}$$

Integrate (1.99) to determine the stream function and sketch the corresponding flow pattern. Work out the substantial derivatives of the velocity components and sketch the acceleration vector field.

Problem 6 - Determine the acceleration of a particle in the 1-D velocity field

$$\bar{U} = \left(k\frac{x}{t}, 0, 0\right) \tag{1.100}$$

where k is constant.

Problem 7 - In a fixed frame of reference a fluid element has the velocity components

$$(U, V, W) = (100, 60, 175)\ meters/\sec. \tag{1.101}$$

Suppose the same fluid element is observed in a frame of reference moving at

$$\dot{X} = (25, 110, 90)\ meters/\sec \tag{1.102}$$

with respect to the fixed frame. Determine the velocity components measured by the observer in the moving frame. Determine the kinetic energy per unit mass in each frame.

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Problem 8 - The stream function of a steady, 2-D compressible flow in a corner is shown in Figure 2.14.

$$\psi = \frac{xy}{1 + x + y}$$

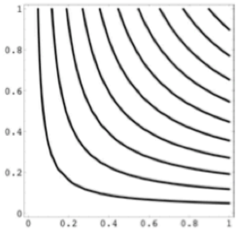

Figure 1.14: *Streamlines for potential flow in a corner.*

Determine plausible expressions for the velocity components and density field. Does a pressure field exist for this flow if it is assumed to be inviscid?

Problem 9 - The expansion into vacuum of a spherical cloud of a monatomic gas such as helium has a well-known exact solution of the equations for compressible isentropic flow. The velocity field is

$$U = \frac{xt}{t_0^2 + t^2} \qquad V = \frac{yt}{t_0^2 + t^2} \qquad W = \frac{zt}{t_0^2 + t^2}. \tag{1.103}$$

The density and pressure are

$$\frac{\rho}{\rho_0} = \frac{t_0^3}{(t_0^2 + t^2)^{3/2}} \left(1 - \frac{t_0^2}{R_{initial}^2} \left(\frac{x^2 + y^2 + z^2}{t_0^2 + t^2}\right)\right)^{3/2}$$
$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{5/3} \tag{1.104}$$

where $R_{initial}$ is the initial radius of the cloud. This problem has served as a model of the expanding gas nebula from an exploding star.

1) Determine the particle paths (x(t), y(t), z(t)).

2) Work out the substantial derivative of the density $D\rho/Dt$.

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Problem 10 - A moving fluid contains a passive non-diffusing scalar contaminant. Smoke in a wind tunnel would be a pretty good example of such a contaminant. Let the concentration of the contaminant be $C(x, y, z, t)$. The units of C are

$$mass\ of\ contaminant/unit\ mass\ of\ fluid. \tag{1.105}$$

Derive a conservation equation for C.

Problem 11 - Include the effects of gravity in the equations of motion (1.93). You can check your answer with the equations derived in Chapter 5.