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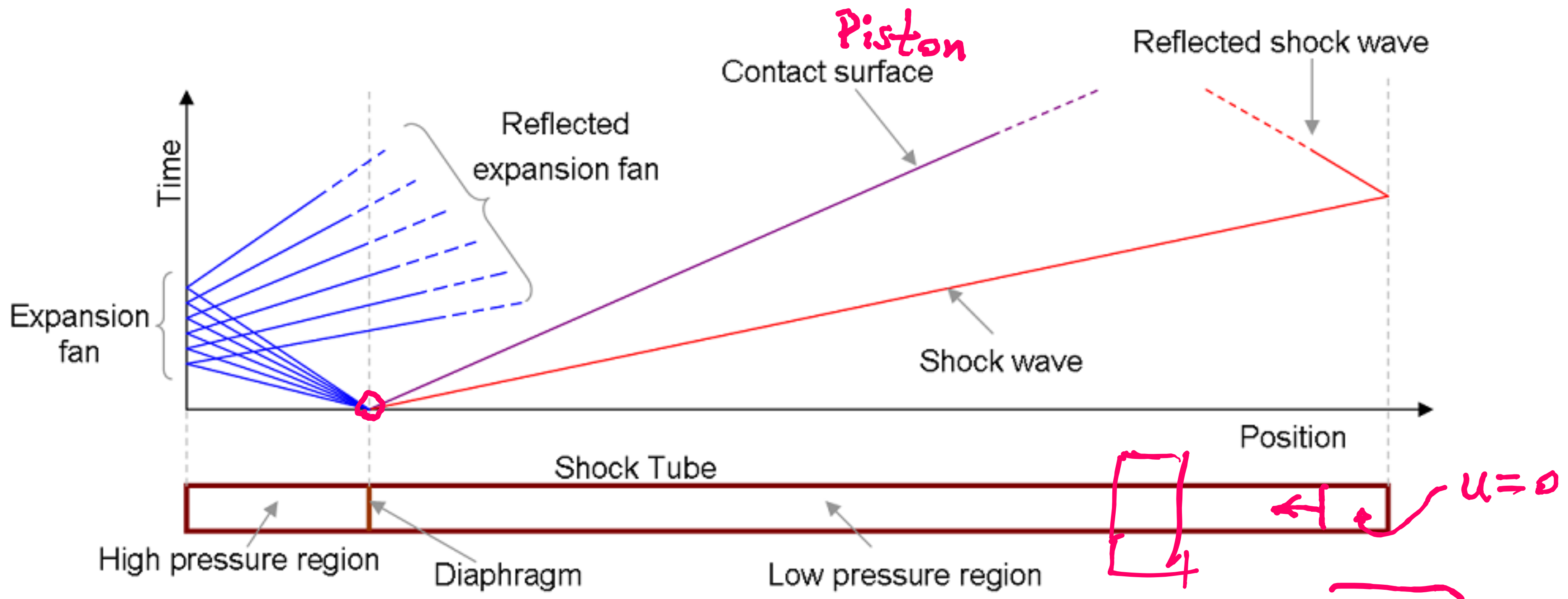
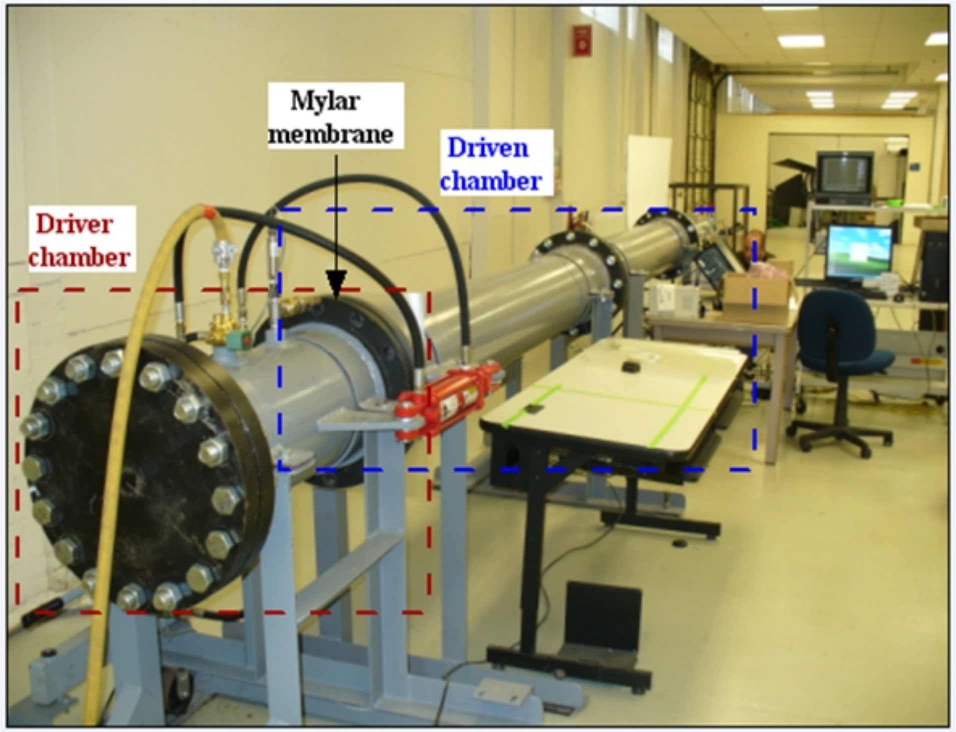
AA210A

Fundamentals of Compressible Flow

chapter 13 - Unsteady Waves in Compressible Flow

The Shock Tube - Wave Diagram




Mylar
membrane
Driven
chamber
Driver
chamber

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13.1 Equations for irrotational, homentropic, unsteady flow

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}\left(\rho U_k\right) = 0$$

$$\rho \frac{\partial U_i}{\partial t} + \rho \frac{\partial}{\partial x_i}\left(\frac{U_k U_k}{2}\right) + \frac{\partial P}{\partial x_i} = 0$$

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

13.2 The acoustic equations

Consider the case where fluctuations in flow variables are very small compared to the mean.

$$\rho = \rho_0 + \rho' \; ; \quad \rho'/\rho_0 \ll 1$$

$$P = P_0 + P' \; ; \quad P'/P_0 \ll 1$$

There is no mean velocity and the velocity disturbance is small. Linearize the equations

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial U_i}{\partial x_i} = 0$$

$$\rho_0 \frac{\partial U_i}{\partial t} + \frac{\partial P}{\partial x_i} = 0$$

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Define the condensation

ρ'

$$r = \frac{\rho - \rho_0}{\rho_0}.$$

The equations of motion in terms of the condensation are.

$$\frac{\partial r}{\partial t} + \frac{\partial U_i}{\partial x_i} = 0$$

$$\rho_0 \frac{\partial U_i}{\partial t} + \frac{\partial P}{\partial x_i} = 0 \quad .$$

$$\frac{P}{P_0} = (1 + r)^{\gamma} \cong 1 + \gamma r$$

Replace the pressure with the density.

$$\frac{\partial r}{\partial t} + \frac{\partial U_i}{\partial x_i} = 0$$

$$\frac{\partial U_i}{\partial t} + a_0^2 \frac{\partial r}{\partial x_i} = 0$$

where

$$a_0^2 = \gamma \frac{P_0}{\rho_0}.$$

Differentiate both equations.

$$\frac{\partial^2 r}{\partial t^2} + \frac{\partial^2 U_i}{\partial x_i \partial t} = 0$$

$$\frac{\partial^2 U_i}{\partial x_i \partial t} + a_0^2 \frac{\partial^2 r}{\partial x_i^2} = 0$$

Subtract one from the other

$$\frac{\partial^2 r}{\partial t^2} - a_0^2 \frac{\partial^2 r}{\partial x_i^2} = 0.$$

The condensation satisfies the linear wave equation.

Consider the velocity disturbance.

$$\frac{\partial^2 r}{\partial t \partial x_j} + \frac{\partial^2 U_i}{\partial x_i \partial x_j} = 0$$

$$\frac{\partial^2 U_i}{\partial t^2} + a_0^2 \frac{\partial^2 r}{\partial x_j \partial_t} = 0$$

Again, subtract one from the other

$$\frac{\partial^2 U_i}{\partial t^2} - a_0^2 \frac{\partial^2 U_i}{\partial x_i \partial x_j} = 0.$$

Use the identity $\nabla \times (\nabla \times \bar{U}) = \nabla(\nabla \bullet \bar{U}) - \nabla^2 \bar{U}$.

$$\frac{\partial^2 U_i}{\partial t^2} - a_0^2 \frac{\partial^2 U_i}{\partial x_j \partial x_j} = 0$$

Similarly, all other variables of the flow satisfy the linear wave equation.

$$\frac{\partial^2 P}{\partial t^2} - a_0^2 \frac{\partial^2 P}{\partial x_i^2} = 0$$

$$\frac{\partial^2 T}{\partial t^2} - a_0^2 \frac{\partial^2 T}{\partial x_i^2} = 0.$$

13.3 Propagation of acoustic waves in one space dimension

In one dimension the acoustic equations are

$$\frac{\partial r}{\partial t} + \frac{\partial U}{\partial x} = 0$$

$$\frac{\partial U}{\partial t} + a_0^2 \frac{\partial r}{\partial x} = 0$$

The wave equation

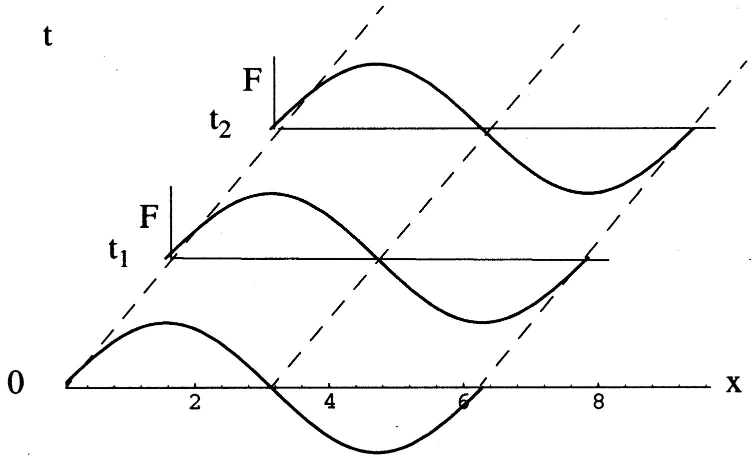
$$\frac{\partial^2 r}{\partial t^2} - a_0^2 \frac{\partial^2 r}{\partial x^2} = 0,$$

Can be solved in the general form

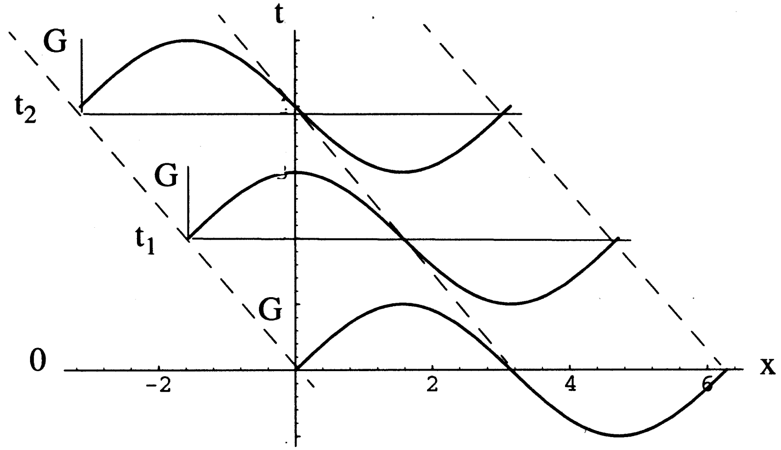
$$r(x, t) = F(x - a_0 t) + G(x + a_0 t)$$

where F and G are arbitrary functions

Right propagating F waves



Left propagating G waves



The pressure disturbance generated by the wave is

$$\frac{P - P_0}{P_0} = \gamma r = \gamma \frac{\rho - \rho_0}{\rho_0}$$

In differential form

$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho}.$$

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The particle velocity induced by an acoustic disturbance can also be written in a very general form

$$U = f(x - a_0 t) + g(x + a_0 t)$$

Substitute the expressions for the condensation and the velocity into the acoustic equations.

$$\frac{\partial(F(x - a_0 t) + G(x + a_0 t))}{\partial t} + \frac{\partial(f(x - a_0 t) + g(x + a_0 t))}{\partial x} = 0$$

$$\frac{\partial(f(x - a_0 t) + g(x + a_0 t))}{\partial t} + a_0^2 \frac{\partial(F(x - a_0 t) + G(x + a_0 t))}{\partial x} = 0$$

Let

$$\xi = x - a_0 t \qquad \eta = x + a_0 t$$

The equations for r and U become

$$-a_0 F_\xi + a_0 G_\eta + f_\xi + g_\eta = 0$$
$$-f_\xi + g_\eta + a_0 F_\xi + a_0 G_\eta = 0$$

Add and subtract these relations. The result is

$$a_0 G_\eta + g_\eta = 0$$
$$a_0 F_\xi - f_\xi = 0$$

From which we can conclude

$$g = -a_0 G \; ; \; f = a_0 F$$

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This result gives us the relationship between density and particle velocity in left and right running waves.

$$U = a_0 F(x - a_0 t) - a_0 G(x + a_0 t)$$

Now

$$\frac{U}{a_0 r} = \frac{F - G}{F + G}.$$

In a right running wave

$$\frac{U}{a_0 r} = 1$$

In a left running wave

$$\frac{U}{a_0 r} = -1.$$

In differential form

$$dU = \pm a_0 \frac{d\rho}{\rho}.$$

Recall

$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho}$$

13.4 Isentropic, finite amplitude waves

In a general 1-D isentropic flow

$$a = a_0\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}}.$$

Locally, the velocity disturbance can be assumed to be

$$\boxed{dU = \pm a\frac{d\rho}{\rho}.}$$

Integrate this result from an initial to a final state beginning at U_1=0

$$U = \pm\int_{\rho_1}^{\rho} a\frac{d\rho}{\rho} = \pm a_1\int_{\rho_1}^{\rho}\left(\frac{\rho}{\rho_1}\right)^{\frac{\gamma-1}{2}}\frac{d\rho}{\rho}$$

Thus

$$U = \pm\frac{2a_1}{\gamma-1}\left(\left(\frac{\rho}{\rho_1}\right)^{\frac{\gamma-1}{2}} - 1\right) = \pm\frac{2}{\gamma-1}(a-a_1)$$

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The local acoustic speed is

$$a = a_1 \pm \frac{(\gamma - 1)}{2} U$$

The wave speed at any point is

$$c = a \pm U = a_1 \pm \frac{(\gamma - 1)}{2} U \pm U = a_1 \pm \frac{(\gamma + 1)}{2} U$$

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We can put this result on a somewhat more rigorous foundation for a finite amplitude wave as follows.

The equations for 1-D isentropic unsteady flow are:

$$\frac{\partial\rho}{\partial t} + \rho\frac{\partial U}{\partial x} + U\frac{\partial\rho}{\partial x} = 0$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{\gamma P_0}{\rho_0^2}\left(\frac{\rho}{\rho_0}\right)^{\gamma-2}\frac{\partial\rho}{\partial x} = 0$$

Assume

$$\rho = \rho(U)$$

The derivatives of the density are

$$\frac{\partial\rho}{\partial t} = \frac{d\rho}{dU}\frac{\partial U}{\partial t} \qquad \frac{\partial\rho}{\partial x} = \frac{d\rho}{dU}\frac{\partial U}{\partial x}$$

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Substitute the derivatives of the density into the 1-D equations.

$$\frac{d\rho}{dU}\frac{\partial U}{\partial t} + \rho\frac{\partial U}{\partial x} + U\frac{d\rho}{dU}\frac{\partial U}{\partial x} = 0$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{\gamma P_0}{\rho_0^2}\left(\frac{\rho}{\rho_0}\right)^{\gamma-2}\frac{d\rho}{dU}\frac{\partial U}{\partial x} = 0$$

Rearrange

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{\rho}{d\rho/dU}\frac{\partial U}{\partial x} = 0 \quad \text{— continuity}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{\gamma P_0}{\rho_0^2}\left(\frac{\rho}{\rho_0}\right)^{\gamma-2}\frac{d\rho}{dU}\frac{\partial U}{\partial x} = 0 \quad \text{— mom}$$

The continuity and momentum equations become identical except for the coefficient of the last term. To have a solution the coefficients must be equal.

$$\boxed{\frac{\rho}{d\rho/dU} = \frac{\gamma P_0}{\rho_0^2}\left(\frac{\rho}{\rho_0}\right)^{\gamma-2}\frac{d\rho}{dU}}$$

Rearrange

$$\left(\frac{d\rho}{dU}\right)^2 = \frac{\rho_0^2}{a_0^2}\left(\frac{\rho}{\rho_0}\right)^{3-\gamma}$$

Take the square root

$$\frac{d\rho}{dU} = \pm\frac{\rho_0}{a_0}\left(\frac{\rho}{\rho_0}\right)^{\frac{3-\gamma}{2}}$$

Substitute

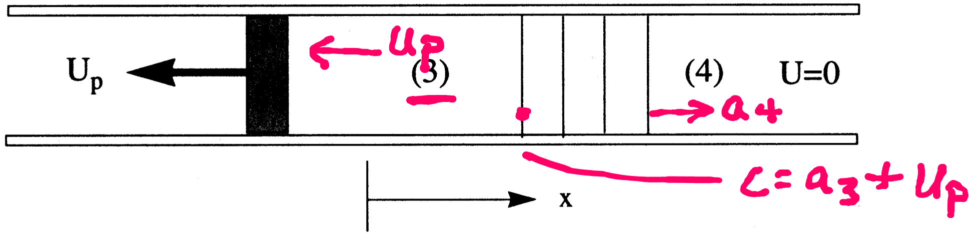
$$\frac{a}{a_0} = \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}}$$

Result

$$dU = \pm a\frac{d\rho}{\rho}$$

13.5 Centered expansion wave

Consider a piston withdrawn from a compressible fluid at rest.



The speed of the fluid in region three is equal to the piston speed.

$$U_p = \frac{2}{\gamma - 1}(a_3 - a_4).$$

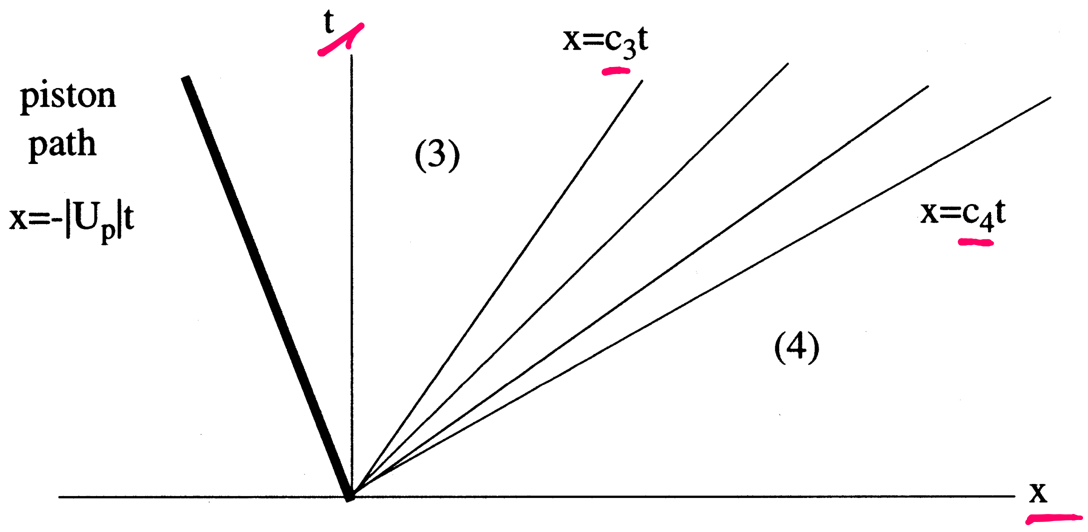
$a_3 < a_4$

Note that the piston speed is negative. In effect, this gives us the sound speed in region 3.

$$a_3 = a_4 + \left(\frac{\gamma - 1}{2}\right)U_p.$$

The leading characteristic propagates to the right with the speed of sound in region 4, the undisturbed gas. The tail of the disturbance moves to the right with wave speed

$$c_3 = a_3 + U_p = a_4 + \left(\frac{\gamma + 1}{2}\right) U_p$$



The density ratio across the wave is given by the isentropic relation.

$$\frac{\rho_3}{\rho_4} = \left(\frac{a_3}{a_4}\right)^{\frac{2}{\gamma-1}}$$

or

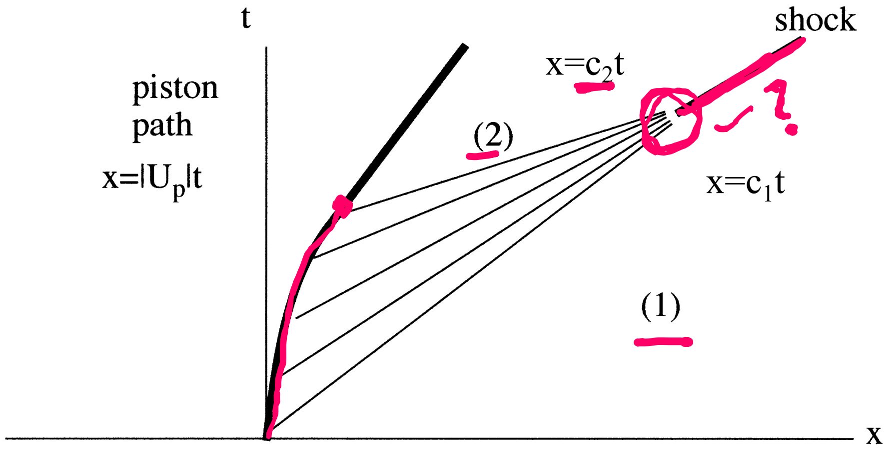
$$\frac{\rho_3}{\rho_4} = \left(1 + \left(\frac{\gamma-1}{2}\right)\frac{U_p}{a_4}\right)^{\frac{2}{\gamma-1}}$$

and the pressure ratio is

$$\frac{P_3}{P_4} = \left(1 + \left(\frac{\gamma-1}{2}\right)\frac{U_p}{a_4}\right)^{\frac{2\gamma}{\gamma-1}}.$$

13.6 Compression wave

Suppose the piston motion is into the gas.



The wave speed at the surface of the piston is

$$c_2 = a_2 + U_p = a_1 + \left(\frac{\gamma + 1}{2}\right) U_p .$$

Once the compression waves catch up the isentropic assumption is no longer valid and there is the formation of a shock wave.

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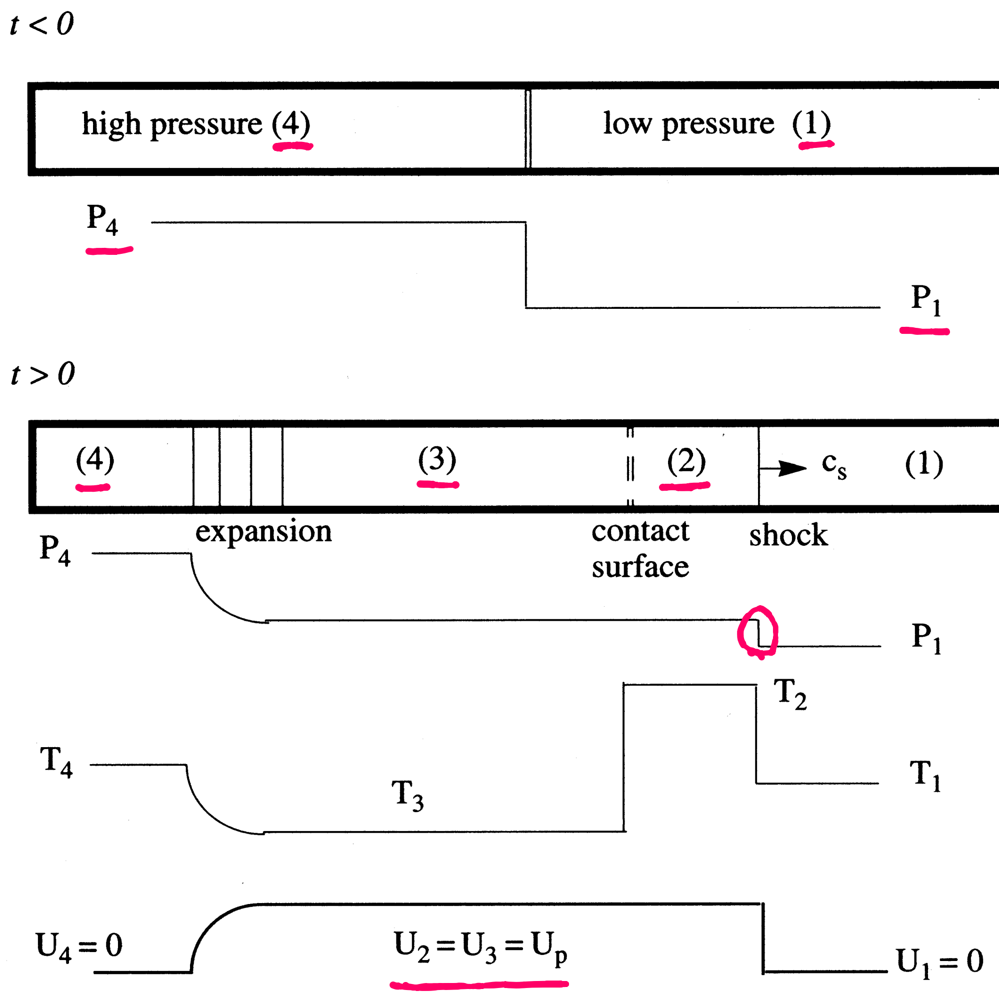
13.7 The shock tube



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The conditions at the contact surface are

$$P_2 = P_3$$

$$U_2 = U_3 = U_p$$

In a frame of reference moving with the shock

$$U'_1 = -c_s$$

$$U'_2 = -c_s + U_p$$

The shock jump conditions give

$$\frac{U'_2}{U'_1} = \frac{1 + \frac{\gamma_1 - 1}{2} {M_1}^2}{\frac{\gamma_1 + 1}{2} {M_1}^2}$$

$$\frac{P_2}{P_1} = \frac{\gamma_1 {M_1}^2 - \frac{\gamma_1 - 1}{2}}{\frac{\gamma_1 + 1}{2}}$$

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The shock Mach number can be written in terms of the piston speed.

$$\frac{U'_2 - U'_1}{U'_1} = \frac{1 + \frac{\gamma_1 - 1}{2} {M_1}^2}{\frac{\gamma_1 + 1}{2} {M_1}^2} - 1 = \frac{1 - {M_1}^2}{\frac{\gamma_1 + 1}{2} {M_1}^2}$$

$$U_p = U'_2 - U'_1 = U'_1 \left(\frac{1 - {M_1}^2}{\frac{\gamma_1 + 1}{2} M_1 \left(-\frac{U'_1}{a_1} \right)} \right) = a_1 \left(\frac{{M_1}^2 - 1}{\frac{\gamma_1 + 1}{2} M_1} \right)$$

where U_p is positive.

This can be written in terms of the shock pressure ratio.

$$U_p = a_1 \left(\frac{P_2}{P_1} - 1 \right) \left(\frac{2}{\gamma_1 (\gamma_1 + 1) \left(\frac{P_2}{P_1} \right) + \gamma_1 (\gamma_1 - 1)} \right)^{1/2}$$

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The velocity behind the expansion is

$$U_3 = U_p = \frac{2a_4}{\gamma_4 - 1}\left(1 - \left(\frac{P_3}{P_4}\right)^{\frac{\gamma_4-1}{2\gamma_4}}\right)$$

Note that

$$\frac{P_3}{P_4} = \frac{P_3}{P_2}\frac{P_2}{P_1}\frac{P_1}{P_4}$$

$$a_1\left(\frac{P_2}{P_1} - 1\right)\left(\frac{2}{\gamma_1(\gamma_1+1)\left(\frac{P_2}{P_1}\right) + \gamma_1(\gamma_1-1)}\right)^{1/2} = \frac{2a_4}{\gamma_4 - 1}\left(1 - \left(\frac{P_3}{P_2}\frac{P_2}{P_1}\frac{P_1}{P_4}\right)^{\frac{\gamma_4-1}{2\gamma_4}}\right)$$

Use the pressure condition at the contact surface.

$$P_3/P_2 = 1$$

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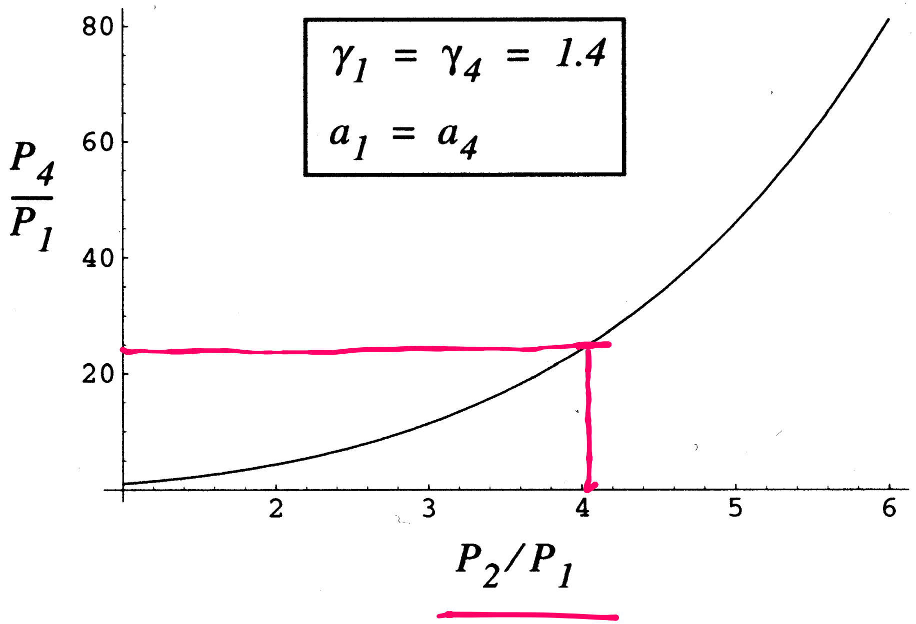
The result is the basic shock tube equation

$$\frac{P_4}{P_1} = \frac{P_2}{P_1}\left(1 - \frac{(\gamma_4 - 1)\left(\frac{a_1}{a_4}\right)\left(\frac{P_2}{P_1} - 1\right)}{\left(4\gamma_1{}^2 + 2\gamma_1(\gamma_1+1)\left(\frac{P_2}{P_1} - 1\right)\right)^{1/2}}\right)^{-\left(\frac{2\gamma_4}{\gamma_4 - 1}\right)}$$



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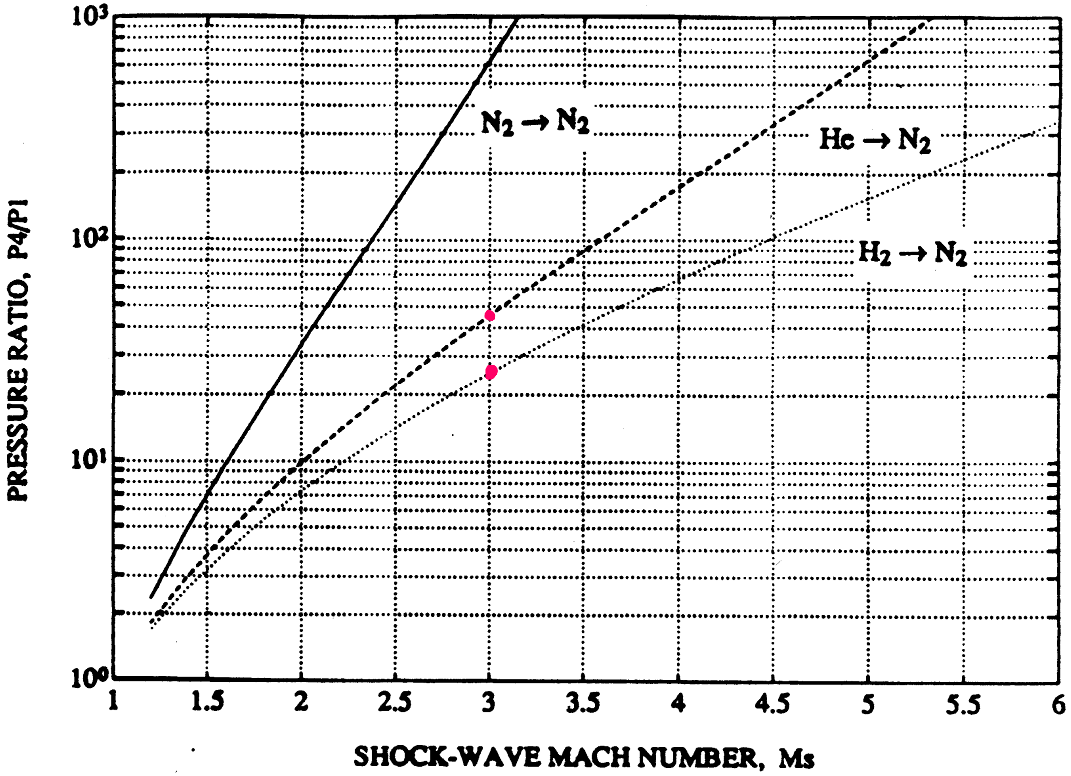
The shock Mach number is determined from

$$\frac{P_2}{P_1} = \frac{2\gamma_1}{\gamma_1 + 1} {M_s}^2 - \left(\frac{\gamma_1 - 1}{\gamma_1 + 1}\right)$$

or

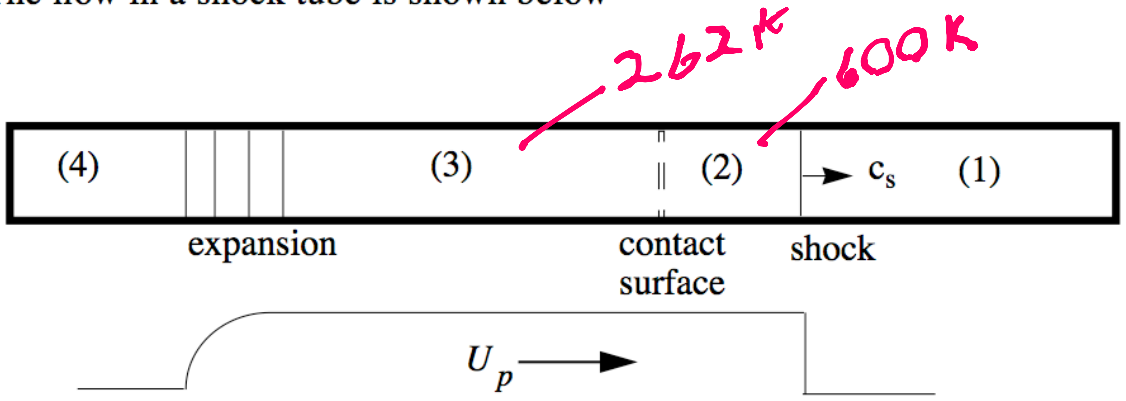
$${M_s}^2 = \frac{\gamma_1 + 1}{2\gamma_1}\left(\frac{P_2}{P_1} + \frac{\gamma_1 - 1}{\gamma_1 + 1}\right)$$



PRESSURE RATIO, P4/P1
10^3
10^2
10^1
10^0
$N_2 \rightarrow N_2$
$He \rightarrow N_2$
$H_2 \rightarrow N_2$
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
SHOCK-WAVE MACH NUMBER, Ms

13.7.1 EXAMPLE - FLOW INDUCED BY THE SHOCK IN A SHOCK TUBE

The flow in a shock tube is shown below



The shock wave induces a gas velocity in the laboratory frame, $U_p = U_2 = U_3$ where

$$U_2 = a_1((P_2/P_1) - 1)\left(\frac{2/\gamma_1}{(\gamma_1 + 1)(P_2/P_1) + (\gamma_1 - 1)}\right)^{1/2} \qquad (13.67)$$

Suppose the gas in the driver section and test section is Air initially at $300\,°K$. A shock wave with a Mach number of 2 is produced in the tube.

1) Determine the stagnation temperature of the gas in region (2) in the laboratory frame.

2) Determine the stagnation temperature of the gas in region (3) in the laboratory frame.

Answer

The pressure ratio across a Mach number 2 shock in Air is,

$$P_2/P_1 = 4.5 \tag{13.68}$$

and the temperature ratio is,

$$T_2/T_1 = 1.687. \tag{13.69}$$

The speed of sound in Air is $a_1 = \sqrt{\gamma RT} = 347\ M/sec$ and the heat capacity at constant pressure is, $C_p = 1005\ M^2/Sec^2 - K$. The piston velocity is,

$$U_2 = a_1((P_2/P_1) - 1)\left(\frac{2/\gamma_1}{(\gamma_1 + 1)(P_2/P_1) + (\gamma_1 - 1)}\right)^{1/2} = \\ 347(3.5)\left(\frac{2}{15.68}\right)^{1/2} = 433.75\ M/sec \tag{13.70}$$

The temperature in region (3) is obtained from (13.53).

$$\frac{T_3}{T_4} = \left(1 + \left(\frac{\gamma - 1}{2}\right)\frac{U_p}{a_4}\right)^2 = \left(1 - 0.2\left(\frac{433.75}{347}\right)\right)^2 = 0.5625 \tag{13.71}$$

Now, in region (2)

$$T_{t2} = T_2 + \frac{1}{2}\frac{U_p^2}{C_p} = 1.687 \times 300 + \frac{433.75^2}{2 \times 1005} = 506.1 + 93.6 = 599.7\,°K \tag{13.72}$$

and in region (3)

$$T_{t3} = T_3 + \frac{1}{2}\frac{U_p^2}{C_p} = 0.5625 \times 300 + \frac{433.75^2}{2 \times 1005} = 168.75 + 93.6 = 262.35\,°K \tag{13.73}$$

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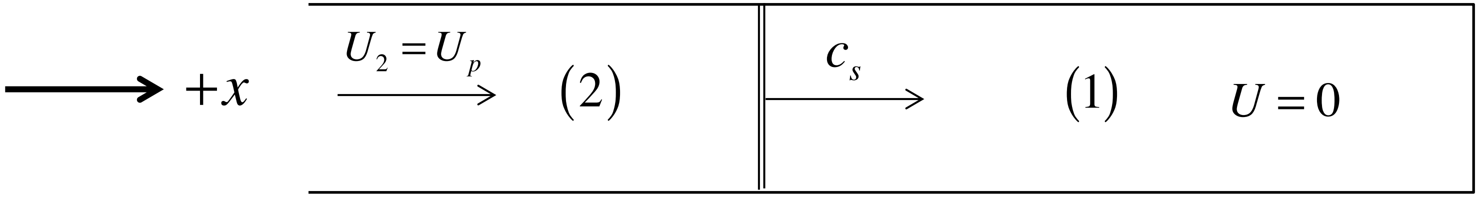
Properties of a reflected shock

$$\frac{P_4}{P_1} = f\left(\gamma_1, \gamma_4, \frac{a_1}{a_4}, \frac{P_2}{P_1}\right)$$

Incident shock $c_s > 0$



$$M_s = \frac{c_s}{a_1} = M_1$$

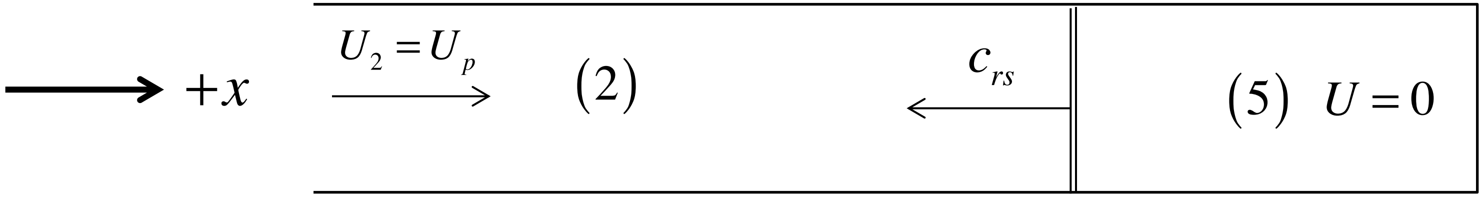
$$M_1 = M_s = \left(\frac{\gamma_1+1}{2\gamma_1}\right)^{1/2}\left(\frac{P_2}{P_1} + \frac{\gamma_1-1}{\gamma_1+1}\right)^{1/2}$$

$$U_p = a_1\left(\frac{M_1^{\,2}-1}{\left(\frac{\gamma_1+1}{2}\right)M_1}\right)$$

$$\frac{a_2}{a_1} = \frac{\left(\gamma_1 M_1^{\,2} - \frac{\gamma_1-1}{2}\right)^{1/2}\left(1+\frac{\gamma_1-1}{2}M_1^{\,2}\right)^{1/2}}{\left(\frac{\gamma_1+1}{2}\right)M_1}$$

$$\frac{P_4}{P_1} \rightarrow \frac{P_2}{P_1} \rightarrow M_1 \rightarrow U_p, \frac{a_2}{a_1}$$

$c_{rs} < 0$ Reflected shock



$$M_{rs} = \frac{-c_{rs} + U_p}{a_2} = M_2$$

$$U_p = a_2\left(\frac{M_2^{\,2}-1}{\left(\frac{\gamma_1+1}{2}\right)M_2}\right) \rightarrow M_2^{\,2} - \frac{U_p}{a_2}\left(\frac{\gamma_1+1}{2}\right)M_2 - 1 = 0$$

Take the positive root

$$\frac{P_5}{P_2} = \frac{2\gamma_1}{\gamma_1+1}M_2^{\,2} - \frac{\gamma_1-1}{\gamma_1+1}$$

$$\frac{a_5}{a_2} = \frac{\left(\gamma_1 M_2^{\,2} - \frac{\gamma_1-1}{2}\right)^{1/2}\left(1+\frac{\gamma_1-1}{2}M_2^{\,2}\right)^{1/2}}{\left(\frac{\gamma_1+1}{2}\right)M_2}$$

$$U_p, \frac{a_2}{a_1} \rightarrow M_2 \rightarrow \frac{a_5}{a_2}, \frac{P_5}{P_2}$$

Reflected shock with a high Mach number incident shock

$$M_2^{\,2} - \frac{U_p}{a_2}\left(\frac{\gamma_1+1}{2}\right)M_2 - 1 = 0$$

$$M_2^{\,2} - \frac{\left(\dfrac{\left(\dfrac{M_1^{\,2}-1}{\left(\dfrac{\gamma_1+1}{2}\right)M_1}\right)\left(\dfrac{\gamma_1+1}{2}\right)}{\dfrac{\left(\gamma_1 M_1^{\,2}-\dfrac{\gamma_1-1}{2}\right)^{1/2}\left(1+\dfrac{\gamma_1-1}{2}M_1^{\,2}\right)^{1/2}}{\left(\dfrac{\gamma_1+1}{2}\right)M_1}}\right)}{} M_2 - 1 = 0$$

Take the limit $\lim_{M_1\to\infty}$

$$M_2^{\,2} - \frac{\left(\dfrac{\gamma_1+1}{2}\right)}{(\gamma_1)^{1/2}\left(\dfrac{\gamma_1-1}{2}\right)^{1/2}}M_2 - 1 = 0$$

Take the positive root

$$M_2 = \frac{\left(\dfrac{\gamma_1+1}{2}\right)}{2(\gamma_1)^{1/2}\left(\dfrac{\gamma_1-1}{2}\right)^{1/2}} + \frac{1}{2}\left(\frac{(\gamma_1+1)^2+8(\gamma_1)(\gamma_1-1)}{2(\gamma_1)(\gamma_1-1)}\right)^{1/2}$$

$$M_2 = \left(\frac{2\gamma_1}{\gamma_1-1}\right)^{1/2}$$

$$\frac{a_5}{a_2} = \frac{\left(\gamma_1 M_2^{\,2}-\dfrac{\gamma_1-1}{2}\right)^{1/2}\left(1+\dfrac{\gamma_1-1}{2}M_2^{\,2}\right)^{1/2}}{\left(\dfrac{\gamma_1+1}{2}\right)M_2}$$

$$\frac{P_5}{P_2} = \frac{\dfrac{2\gamma_1}{\gamma_1-1}M_2^{\,2}-1}{\dfrac{\gamma_1+1}{\gamma_1-1}}$$

$$U_p, \frac{a_2}{a_1} \to M_2 \to \frac{a_5}{a_2}, \frac{P_5}{P_2}$$

The free piston shock tube
A scheme for increasing the pressure and temperature of the driver gas

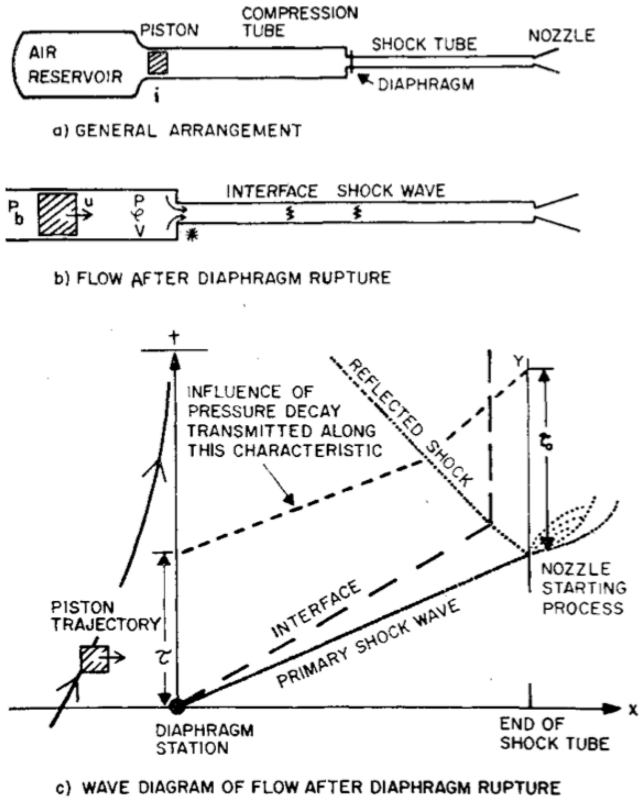
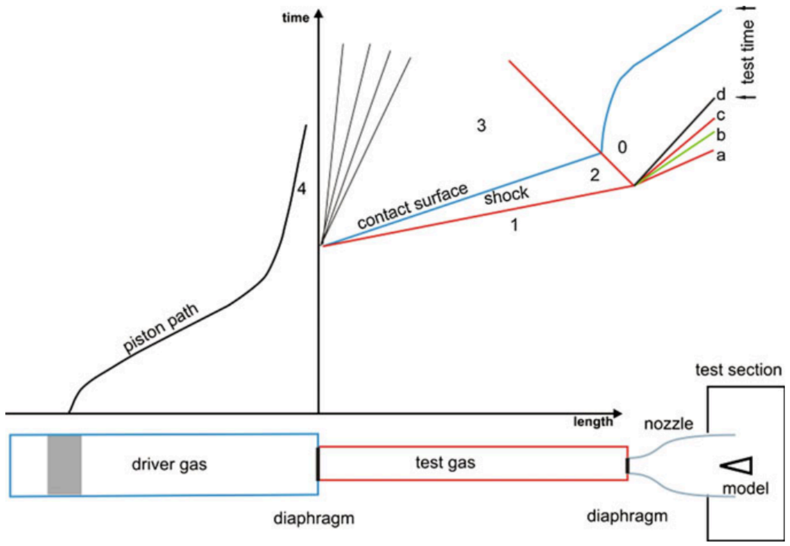
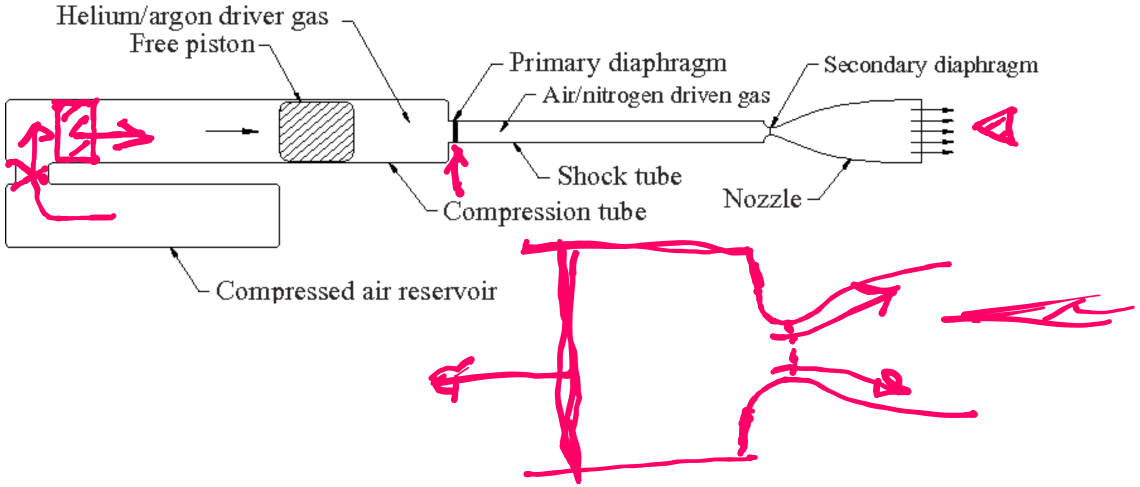


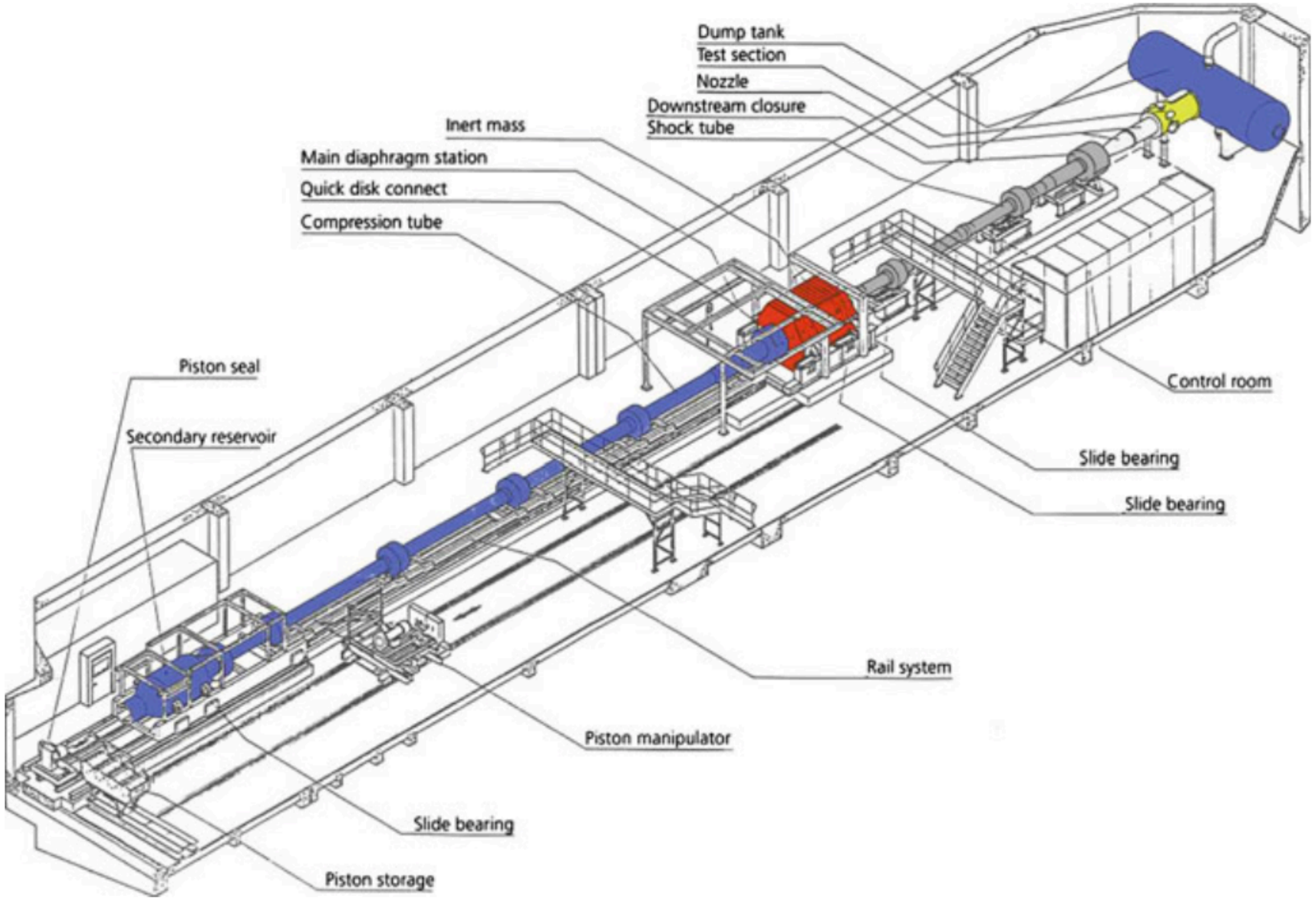
Fig. 1 Operation of free-piston shock tunnel.





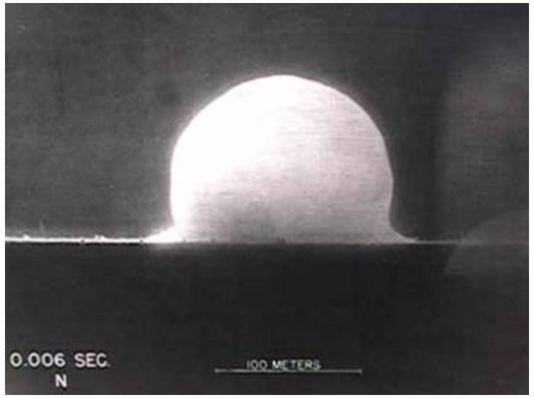
Main valve

A free piston shock tube facility in Goettingen Germany



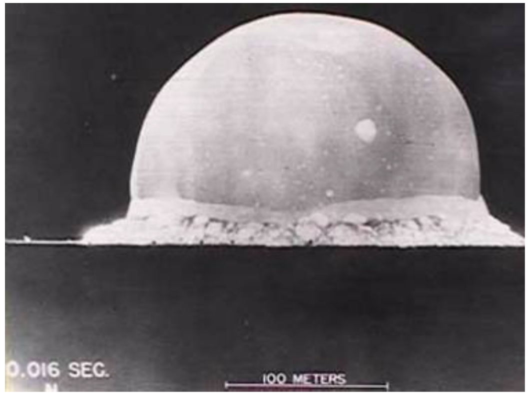
Shock wave produced by a very strong point explosion

Images from the trinity test, Alamogordo N.M., July 16, 1945 made public in 1947



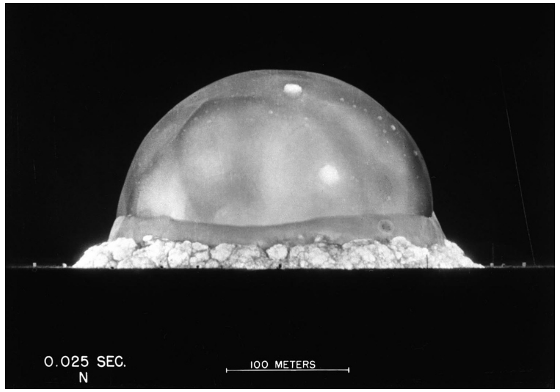
0.006 sec

100 meters



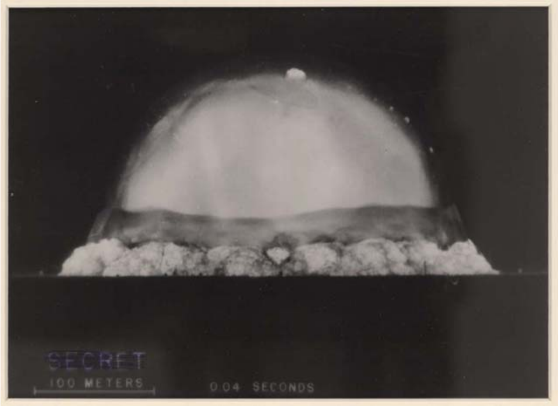
0.016 sec

100 meters



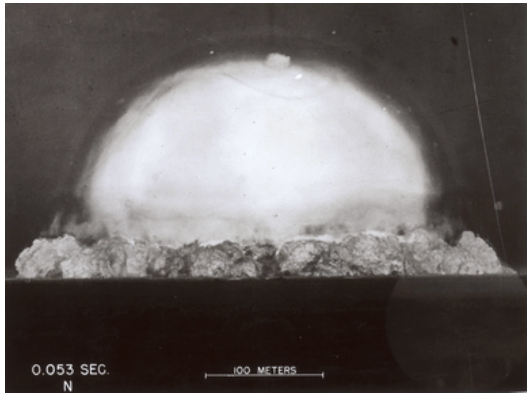
0.025 sec

100 meters



0.040 sec

100 meters



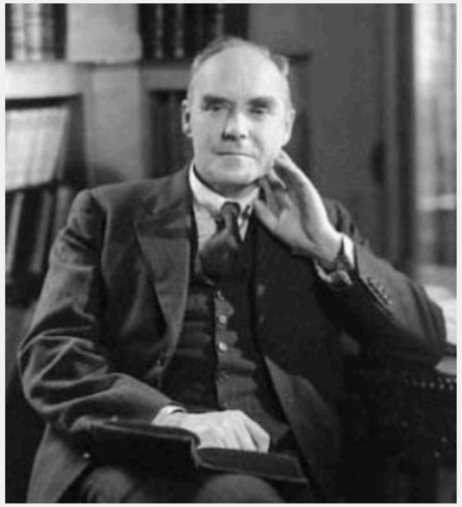
0.053 sec

100 meters

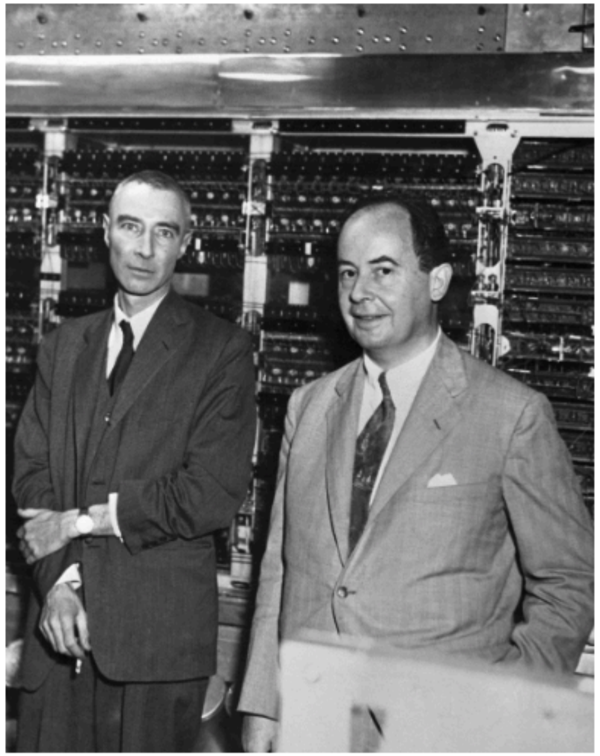
0.062 sec

100 meters

G.I Taylor UK 1941 and 1950, John Von Neumann USA 1941 and 1947 and Leonid Sedov USSR 1946



Geoffrey Ingram Taylor (1886-1975).



Dr. John Von Neumann (right) stands with Dr. J. Robert Oppenheimer in front of an early computer.

Sedov

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Radius of a sphere of ambient gas with the same total internal energy as the explosion energy

$$E = \tfrac{4}{3}\pi R_0^3 \rho_\infty C_v T_\infty$$

$$R_0 = \left(\frac{3}{4}\frac{\gamma - 1}{\pi}\frac{E}{p_\infty}\right)^{1/3}$$

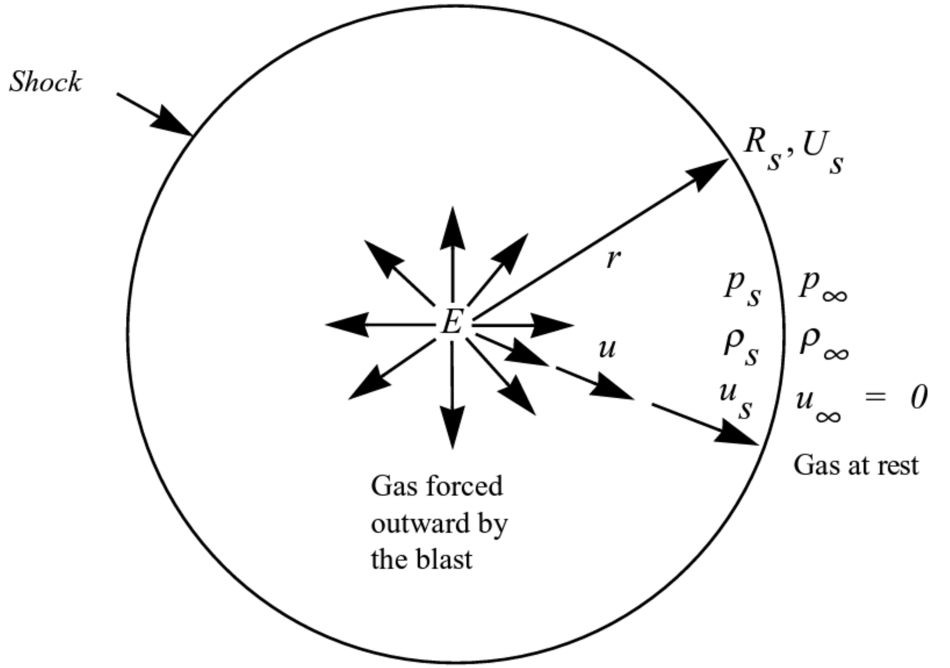

Fig. 12.5. Spherical point explosion.

Gas properties behind the shock come from normal shock theory with M>>1.

$$\rho_s = \frac{\gamma + 1}{\gamma - 1}\rho_\infty,$$

$$p_s = \frac{2}{\gamma + 1}\rho_\infty U_s^2,$$

$$u_s = \frac{2}{\gamma + 1}U_s,$$

As long as the radius of propagation of the shock satisfies $R_s \ll R_0$, the following assumptions hold:

(i) The thermal energy per unit volume of the ambient gas can be neglected compared to the energy per unit volume of the gas within the wave.
(ii) The pressure ratio across the shock is large: $p_S/p_\infty \gg 1$.

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Notice that the limiting density ratio across the shock is finite. Therefore, in contrast to the ambient pressure, the effect of the ambient density on the shock speed cannot be ignored. Thus the flow pattern generated by the blast wave is completely determined by only two physical parameters, the energy E and the ambient density ρ_∞. Dimensional analysis applied to these parameters, including the time and the shock radius, with dimensions

$$\hat{E} = ML^2/T^2, \qquad \hat{\rho} = M/L^3, \qquad \hat{R}_s = L, \qquad \hat{t} = T, \quad (12.48)$$

leads to

$$\left(\frac{\rho_\infty}{E}\right)^{1/5} \frac{R_s}{t^{2/5}} = \text{constant} = \alpha_s. \quad (12.49)$$

The shock speed is

$$U_s = \frac{dR_s}{dt} = \alpha_s \frac{2}{5}\left(\frac{E}{\rho_\infty}\right)^{1/5} t^{-3/5}. \quad (12.50)$$

α_s needs to be determined from theory

Governing equations

inviscid, isentropic flow behind the shock

$$\rho_t + u\rho_r + \rho\left(u_r + \frac{2u}{r}\right) = 0,$$
$$u_t + uu_r + \frac{p_r}{\rho} = 0, \qquad (12.52)$$
$$p_t + up_r + \gamma p\left(u_r + \frac{2u}{r}\right) = 0.$$

The last equation in (12.52) is derived from the equation for conservation of entropy. When combined with the continuity equation it can be written as

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\ln[p/\rho^{\gamma}] = 0. \qquad (12.53)$$

Noting the formula (12.29) for the entropy of an ideal gas, we can write (12.53) as

$$\frac{DS}{Dt} = 0, \qquad (12.54)$$

If we neglect the internal energy of the ambient fluid being continuously enclosed by the outwardly moving shock, then the total energy of the gas between the origin and the shock front is approximately constant,

$$E = \int_0^{R_s} \left(\rho C_v T + \frac{1}{2}\rho u^2 \right) 4\pi r^2 \, dr = \int^{R_s} \left(\frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \right) 4\pi r^2 \, dr. \tag{12.55}$$

Similarity Solution

The problem is invariant under a one-parameter dilation group.

$$\tilde{r} = e^{a} r, \qquad \tilde{t} = e^{(5/2)a} t, \qquad \tilde{u} = e^{-(3/2)a} u, \qquad \tilde{p} = e^{-3a} p, \qquad \tilde{\rho} = \rho$$

Similarity variables

$$r = \left(\frac{E}{\rho_\infty}\right)^{1/5} t^{2/5} \ \alpha$$

$$u = \frac{2}{5}\left(\frac{2}{\gamma+1}\right)\left(\frac{E}{\rho_\infty}\right)^{1/5} t^{-3/5} \ \alpha U(\alpha)$$

$$p = \frac{4}{25}\left(\frac{2}{\gamma+1}\right)\left(\rho_\infty{}^{3} E^{2}\right)^{1/5} t^{-6/5} \ \alpha^{2} P(\alpha)$$

$$\rho = \rho_\infty\left(\frac{\gamma+1}{\gamma-1}\right) G(\alpha)$$

Solution

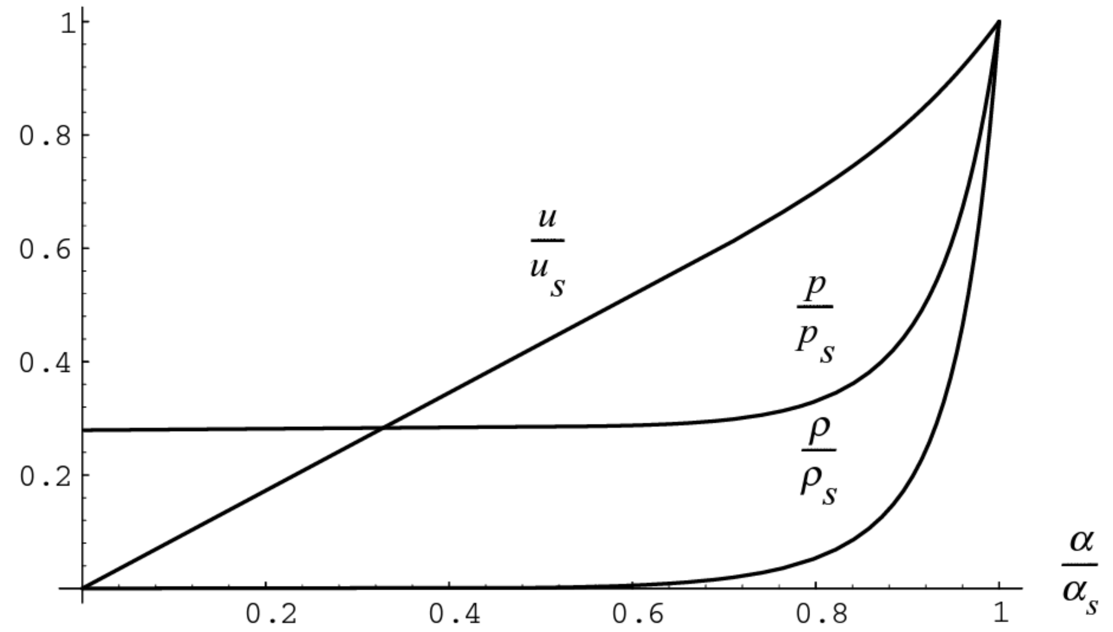


Fig. 12.6. Velocity, density, and pressure for $\gamma = 1.4$.

$$\rho_s = \frac{\gamma + 1}{\gamma - 1}\rho_\infty,$$

$$p_s = \frac{2}{\gamma + 1}\rho_\infty U_s^2 = \alpha_s^2\left(\frac{4}{25}\right)\left(\frac{2}{\gamma + 1}\right)(\rho_\infty^{3/5}E^{2/5})t^{-6/5}$$

$$u_s = \frac{2}{\gamma + 1}U_s = \alpha_s\left(\frac{2}{5}\right)\left(\frac{2}{\gamma + 1}\right)\left(\frac{E}{\rho_\infty}\right)^{1/5}t^{-3/5}.$$

Solution for the unknown constant as a function of gamma

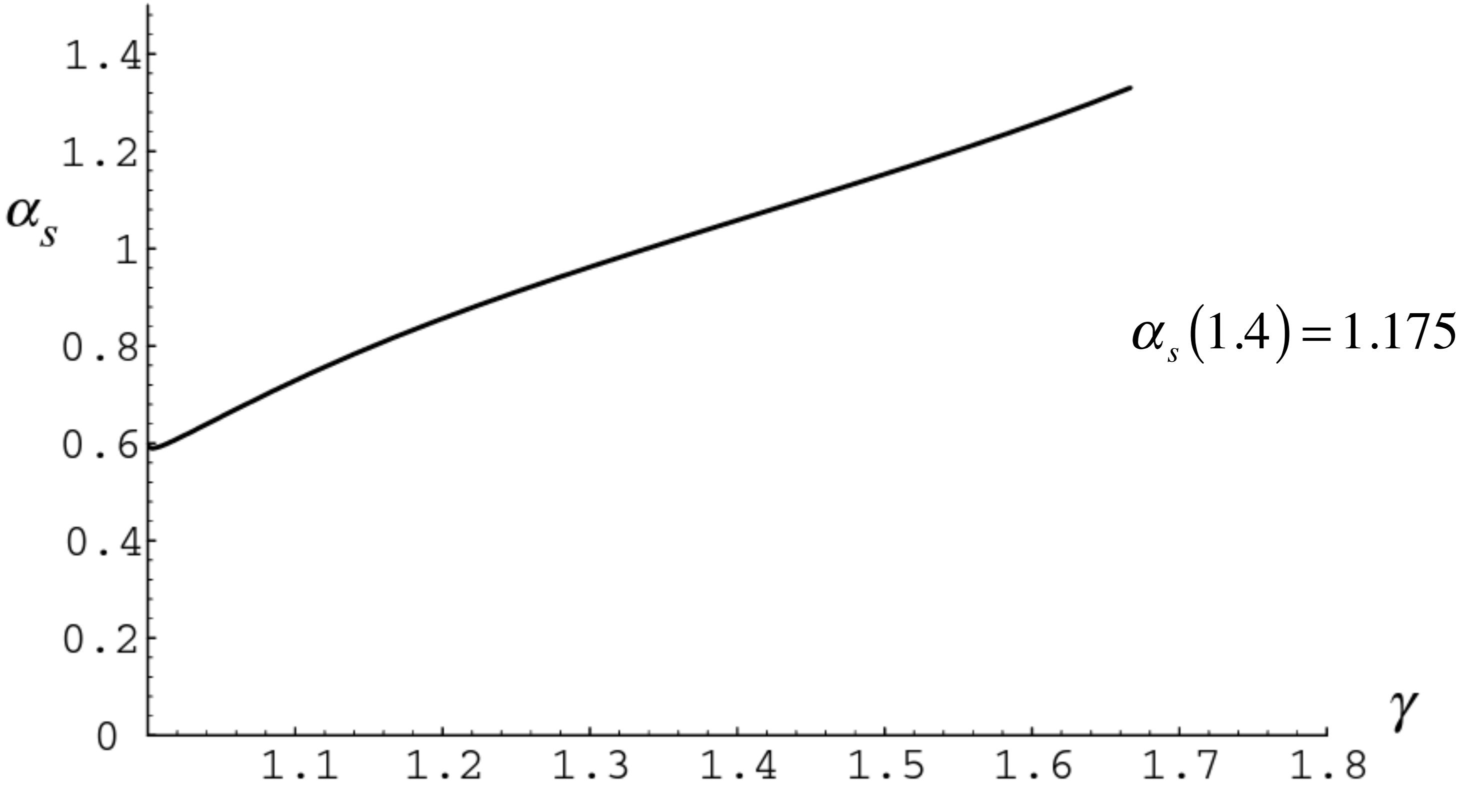


Fig. 12.8. The shock speed parameter as a function of γ.

Knowing α_s determine the energy of the explosion

$$\left(\frac{\rho_\infty}{E}\right)^{1/5}\frac{R_s}{t^{2/5}} = \text{constant} = \alpha_s, \tag{12.80}$$

or

$$R_s = \alpha_s\left(\frac{E}{\rho_\infty}\right)^{1/5} t^{2/5}, \tag{12.81}$$

or

$$\ln R_s = \ln\left[\alpha_s\left(\frac{E}{\rho_\infty}\right)^{1/5}\right] + \frac{2}{5}\ln t. \tag{12.82}$$

When $\ln R_s$ is plotted versus $\ln t$ with α_s estimated from Figure 12.8 the result is a value for E.

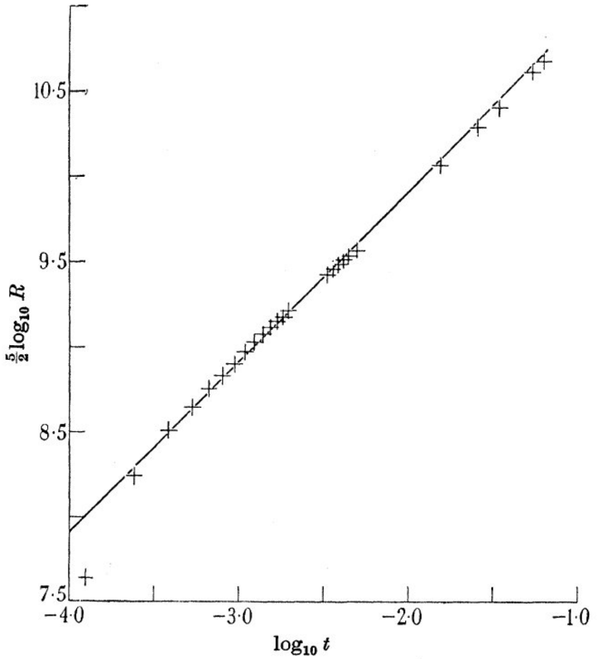


FIGURE 1. Logarithmic plot showing that $R^{\frac{5}{2}}$ is proportional to t.

$$E = 7.19\times 10^{13} J$$

Taylor Wave

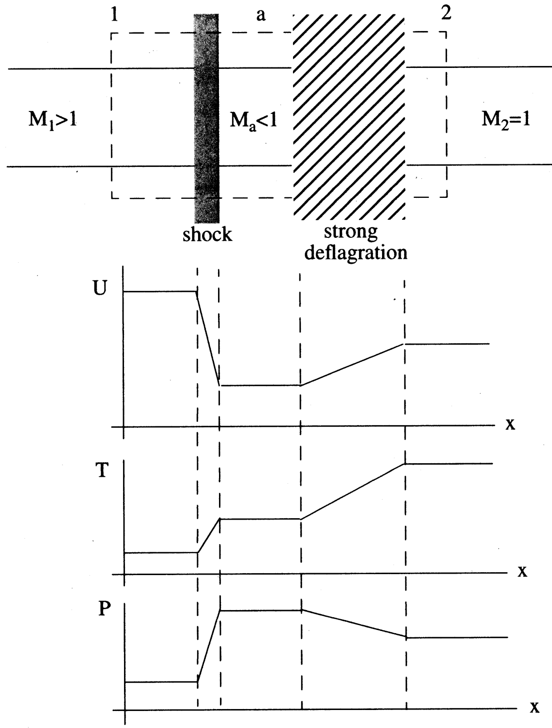
This is an expansion wave that occurs behind a detonation wave. Recall from Chapter 11 the properties of a Chapman-Jouget wave can be determined from the heat addition jump conditions.

$$\rho_1 U_1 = \rho_2 U_2$$

$$P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2$$

$$C_p T_1 + U_1^2/2 + \Delta h_t = C_p T_2 + U_2^2/2$$

$$P = \rho R T$$



Recall the ZND model in a frame of reference moving with the detonation.

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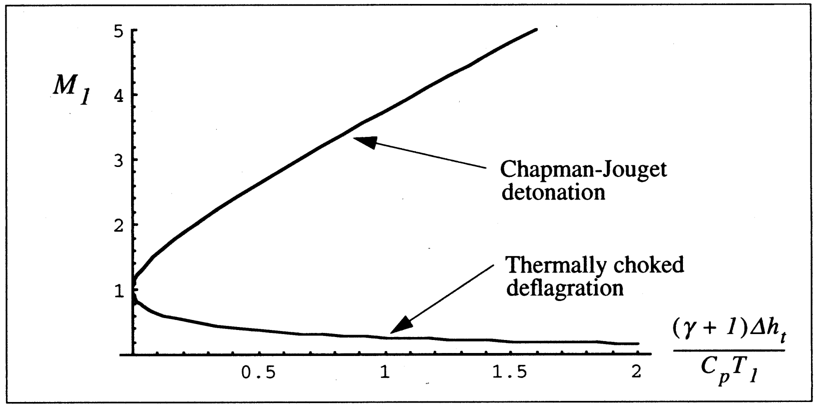
Detonation Wave Mach number

$$M_1^{\,2} = \left(1 + \frac{(\gamma+1)\Delta h_t}{C_p T_1}\right) \pm \left(\left(1 + \frac{(\gamma+1)\Delta h_t}{C_p T_1}\right)^2 - 1\right)^{1/2}.$$

Subsonic and supersonic roots.



$$\frac{U}{U^*} = \frac{(\gamma+1)M^2}{1+\gamma M^2} = \frac{\rho^*}{\rho}$$

$$\frac{T^*}{T} = \frac{(1+\gamma M^2)^2}{(\gamma+1)^2 M^2}$$

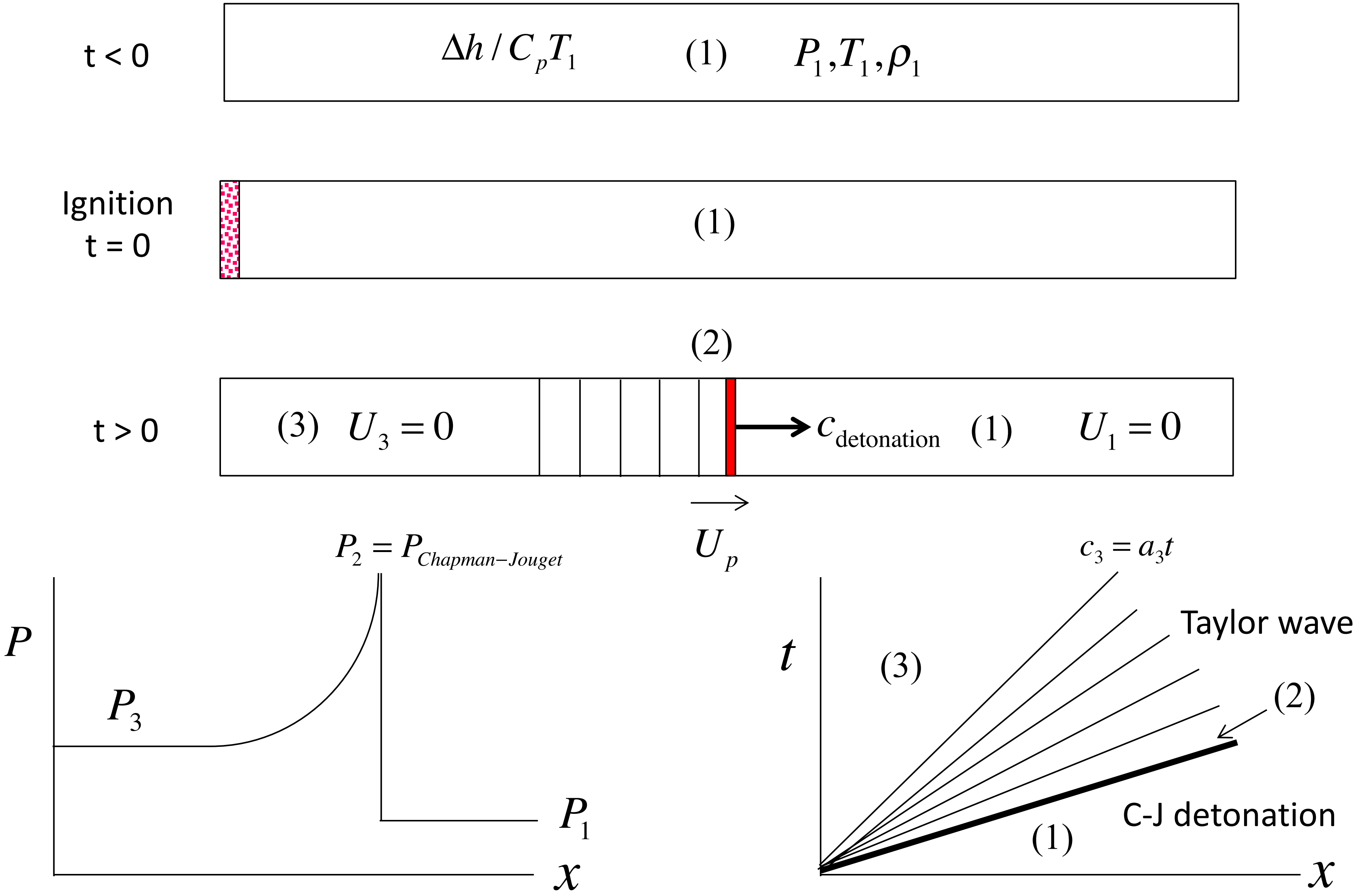
$$\frac{P^*}{P} = \frac{1+\gamma M^2}{1+\gamma}$$

$$\frac{P_t^*}{P_t} = \left(\frac{1+\gamma M^2}{1+\gamma}\right)\left(\frac{\frac{\gamma+1}{2}}{1+\left(\frac{\gamma-1}{2}\right)M^2}\right)^{\frac{\gamma}{(\gamma-1)}}$$

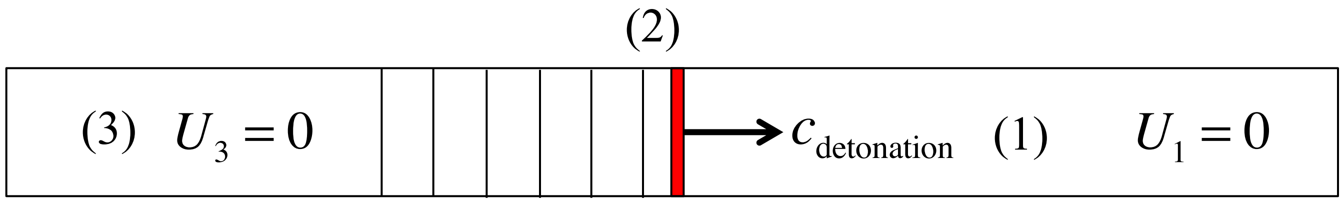
We assume gamma does not change through the wave

Pressure, temperature and density directly behind the wave come from Rayleigh theory for thermally choked heat addition.

Suppose a detonation is initiated in a tube filled with a fuel/oxidizer mixture.



Properties of the Taylor wave connecting regions 2 and 3



$$U'_2 = -c_{\text{detonation}} + U_p = -a_2 \qquad U'_1 = -c_{\text{detonation}}$$

$$U_p = c_{\text{detonation}} - a_2 = M_1 a_1 - a_2$$

Velocities in a frame of reference moving with the detonation

$$\frac{U'_1}{U'_2} = \frac{(1+\gamma)M_1^2}{(1+\gamma M_1^2)} = \frac{\rho_2}{\rho_1} \qquad \frac{T_2}{T_1} = \frac{(1+\gamma M_1^2)^2}{(1+\gamma)^2 M_1^2}$$

$$\frac{a_2}{a_1} = \frac{(1+\gamma M_1^2)}{(1+\gamma)M_1} \qquad \frac{P_2}{P_1} = \frac{(1+\gamma M^2)}{(1+\gamma)}$$

Pressure, temperature and density in region 2 just behind the detonation wave comes from steady flow Rayleigh line theory.

$$a_3 = a_2 - \left(\frac{\gamma-1}{2}\right) U_p$$

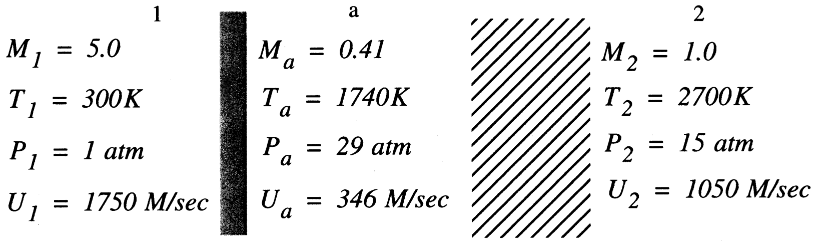
$$\frac{\rho_3}{\rho_2} = \left(\frac{a_3}{a_2}\right)^{\frac{2}{\gamma-1}} = \left(1 - \left(\frac{\gamma-1}{2}\right)\frac{U_p}{a_2}\right)^{\frac{2}{\gamma-1}}$$

$$\frac{P_3}{P_2} = \left(1 - \left(\frac{\gamma-1}{2}\right)\frac{U_p}{a_2}\right)^{\frac{2\gamma}{\gamma-1}}$$

Pressure and temperature in region 3 comes from unsteady expansion wave and isentropic flow theory.

Example - mixture of 5% hydrogen and 5%fluorine in 90% nitrogen.

$$\frac{\Delta h_t}{C_p T_1} = \frac{1.36 \times 10^7}{929 \times 300 \times 10} = 4.88$$



Behind the Taylor wave expansion.

$$U_p = 700\ m/\sec$$

$$a_2 = 1050\ m/\sec$$

$$\gamma = 1.4$$

$$a_3 = a_2 - \left(\frac{\gamma-1}{2}\right)U_p = 1050 - 0.2 \times 700 = 910\ m/\sec$$

$$\frac{\rho_3}{\rho_2} = \left(\frac{a_3}{a_2}\right)^{\frac{2}{\gamma-1}} = \left(1-\left(\frac{\gamma-1}{2}\right)\frac{U_p}{a_2}\right)^{\frac{2}{\gamma-1}} = \left(1-0.2\frac{700}{1050}\right)^5 = 0.489$$

$$\frac{P_3}{P_2} = \left(1-\left(\frac{\gamma-1}{2}\right)\frac{U_p}{a_2}\right)^{\frac{2\gamma}{\gamma-1}} = \left(1-0.2\frac{700}{1050}\right)^7 = 0.367$$

$$\frac{T_3}{T_2} = \frac{P_3}{P_2}\frac{\rho_2}{\rho_1} = 0.751$$

13.8 Problems

Problem 1 - Consider the expression ρU^n; n=1 corresponds to the mass flux, n=2 corresponds to the momentum flux and n=3 corresponds to the energy flux of a compressible gas. Use the acoustic relation,

$$dU = \pm a \frac{d\rho}{\rho} \tag{13.74}$$

to determine the Mach number (as a function of n) at which ρU^n is a maximum in a one-dimensional, unsteady expansion wave. The steady case is considered in one of the problems at the end of Chapter 9.

Problem 2 - In the shock tube example discussed above, determine the stagnation pressure of the gas in regions (2) and (3). Determine the stagnation pressure in both the laboratory frame and in the frame of reference moving with the shock wave.

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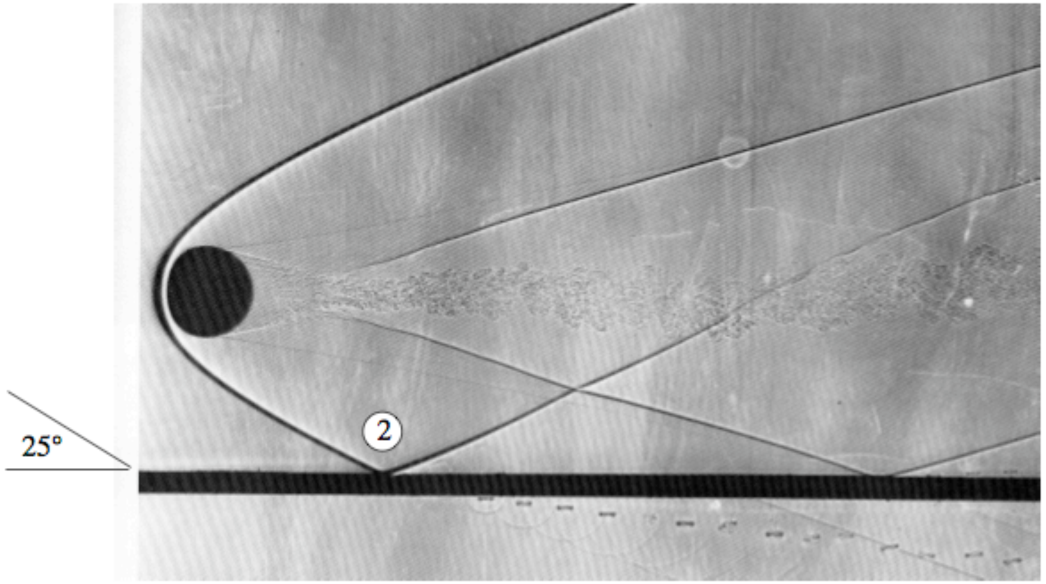
Problem 3 - The figure below shows a sphere moving over a flat plate in a ballistic range (Van Dyke figure 271). The sphere has been fired from a gun and is translating to the left at a Mach number of 3. The static temperature of the air upstream of the sphere is 300°K and the pressure is one atmosphere. On the plane of symmetry of the flow (the plane of the photo) the shock intersects the plate at an angle of 25 degrees as indicated.



i) Determine the flow Mach number, speed and angle behind the shock in region 2 close to the intersection with the plate. Work out your results in a frame of reference moving with the sphere. In this frame the upstream air is moving to the right at 3 times the speed of sound.

ii) What are the streamwise and plate normal velocity components of the flow in region 2 referred to a frame of reference at rest with respect to the upstream gas.

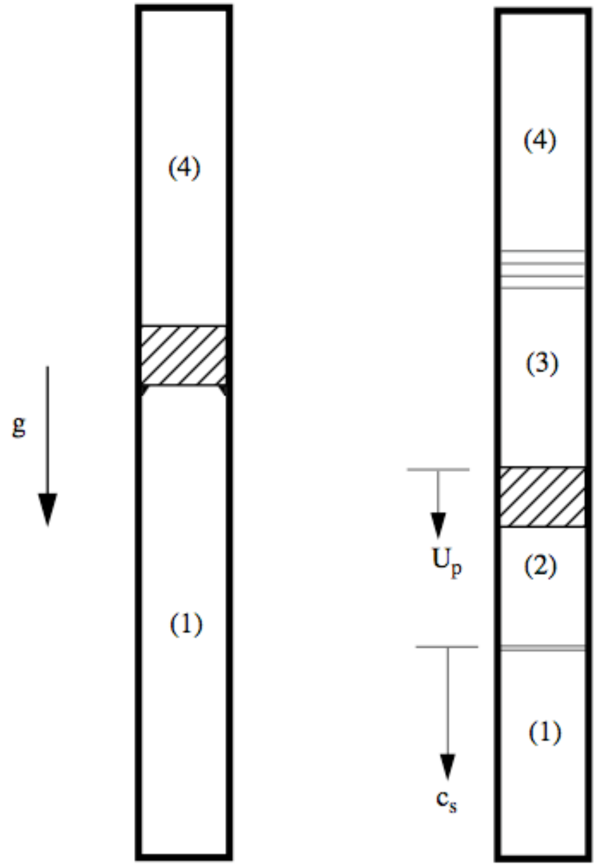
iii) Determine the stagnation temperature and pressure of the flow in region 2 referred to a frame of reference at rest with respect to the upstream gas.

Problem 4 - We normally think of the shock tube as a device that can be used to study relatively strong shock waves. But shock tubes have also been used to study weak shocks relevant to the sonic boom problem. Suppose the shock tube is used to generate weak shock waves with $P_2/P_1 = 1 + \varepsilon$. Show that for small ε the relationship between P_2/P_1 and P_4/P_1 is approximated by

$$P_4/P_1 = 1 + A\varepsilon . \qquad (13.75)$$

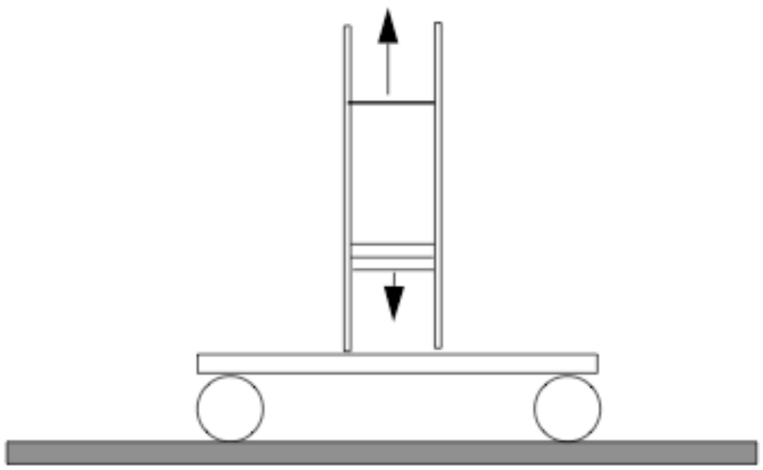
How does A depend on the properties of the gases in regions 1 and 4? Use the exact theory to determine the strength of a shock wave generated in a air-air shock tube operated at $P_4/P_1 = 1.2$. Compare with the approximate result.

Problem 5 - The sketch below shows a one meter diameter tube filled with Air and divided into two volumes by a heavy piston of weight W. The piston is held in place by a mechanical stop and the pressure and temperature are uniform throughout the tube at one atmosphere and 300°K. Body force effects in the gas may be neglected.

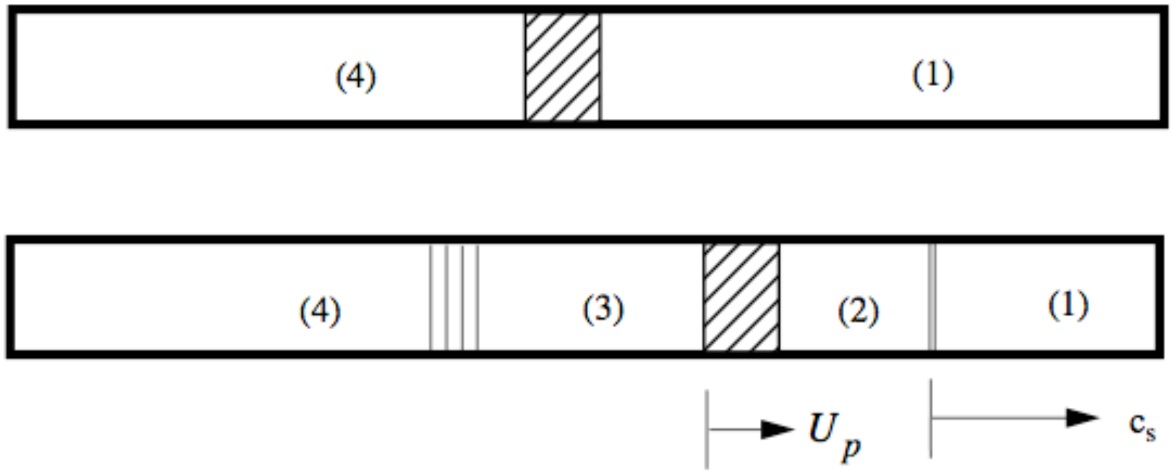


The piston is released and accelerates downward due to gravity. After a short transient the piston reaches a constant velocity U_p. The piston drives a shock wave ahead of it at a wave speed c_s equal to twice the sound speed in region 1. What is the weight of the piston?

Problem 6 - Each time Stanford makes a touchdown an eight inch diameter, open ended shock tube is used to celebrate the score. Suppose the shock wave developed in the tube is required to have a pressure ratio of 2. What pressure is needed in the driver section? Assume the driver gas is Air. What is the shock Mach number? Suppose the shock tube is mounted vertically on a cart as shown in the figure below. Estimate the force that the cart must withstand when the tube fires.



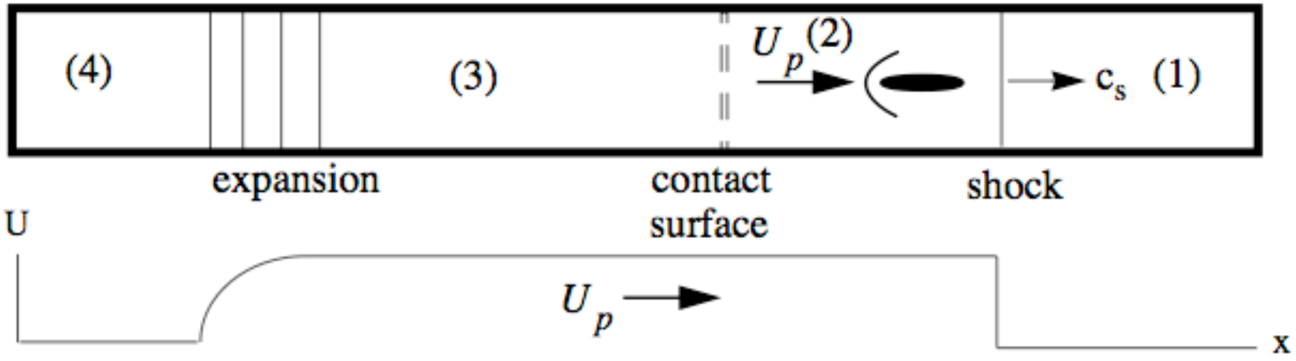
Problem 7 - A moveable piston sits in the middle of a long tube filled with Air at one atmosphere and 300°K. At time zero the piston is moved impulsively to the right at $U_p = 200m/sec$.



1) What is the pressure on the right face of the piston (region 2) in atmospheres?

2) What is the pressure on the left face of the piston (region 3) in atmospheres?

Problem 8 - One of the most versatile instruments used in the study of fluid flow is the shock tube. It can even be used as a wind tunnel as long as one is satisfied with short test times. The figure below illustrates the idea. An airfoil is placed in the shock tube and after the passage of the shock it is subject to flow of the test gas at constant velocity U_p and temperature T_2. In a real experiment the contact surface is quite turbulent and so the practical usefulness of the flow is restricted to the time after the arrival of the shock and before the arrival of the contact surface, typically a millisecond or so.

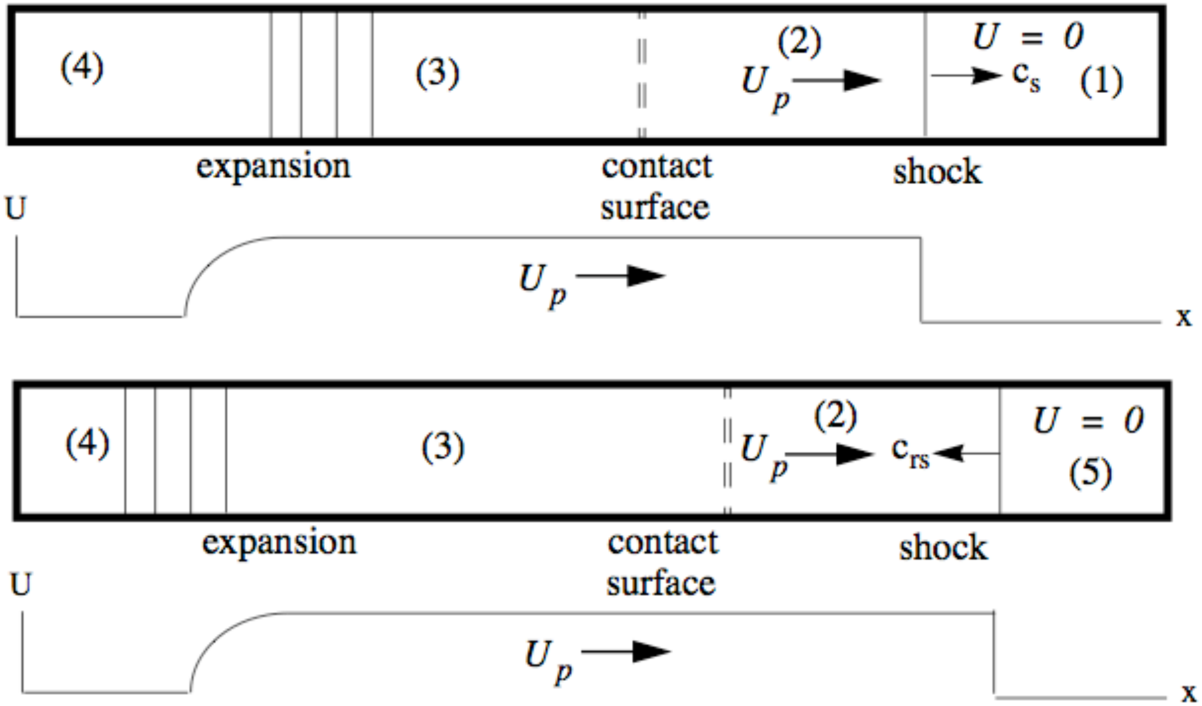


Proponents of this idea point out that if the shock Mach number is very large the flow over the body can be supersonic as suggested in the sketch above.

1) Show that this is the case.

2) For very large shock Mach number, which test gas would produce a higher Mach number over the body, helium or air. Estimate the Mach number over the body for each gas.

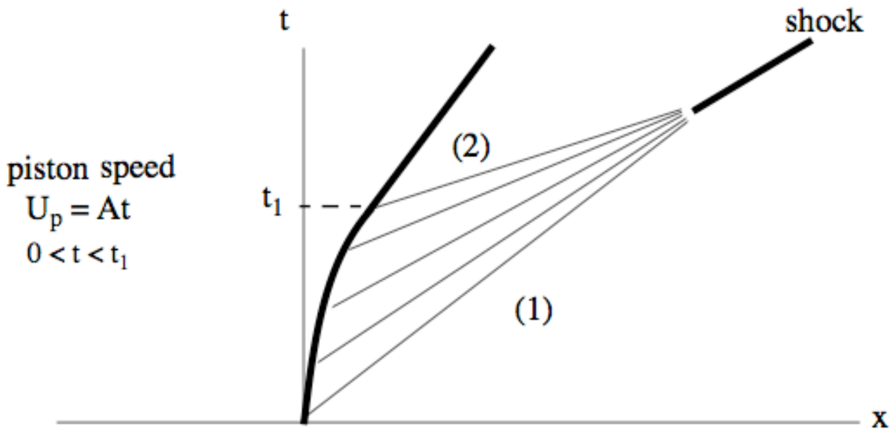
Problem 9 - The figure below shows a shock wave reflecting from the endwall of a shock tube. The reflected shock moves to the left at a constant speed c_{rs} into the gas that was compressed by the incident shock. The gas behind the reflected shock, labeled region (5), is at rest and at a substantially higher temperature and pressure than it was in state (1) before the arrival of the incident shock.

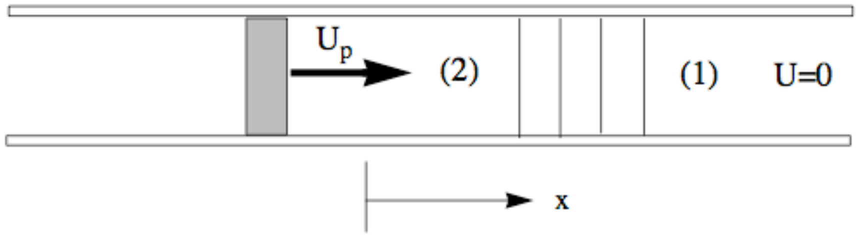


1) Suppose the gas in the driver and test sections is Helium at an initial temperature of 300°K prior to opening the diaphragm. The Mach number of the incident shock wave is 3. Determine the Mach number of the reflected shock.

2) Determine T_5/T_1.

Problem 10 - One of the key variables in the design of a shock tube is the length needed for a shock to develop from the initial compression process. Suppose a piston is used to compress a gas initially at rest in a tube. During the startup transient $0 < t < t_1$ the piston speed increases linearly with time as shown on the x-t diagram below.





In a shock tube the startup time t_1 is generally taken to be the time required for the diaphragm to open. Let the gas be Air at $T_1 = 300K$. Use $t_1 = 5 \times 10^{-3}$ sec and $A = 4 \times 10^4$ M/sec^2 to estimate the distance needed for the shock wave to form.