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AA210A

Fundamentals of Compressible Flow

Chapter 10 - Gasdynamics of nozzle flow

10.1 The area-Mach number function

Quasi-1D equations of motion

$$d(\rho U) = \frac{\delta \dot{m}}{A} - \rho U \frac{dA}{A}$$

$$d(P - \tau_{xx}) + \rho U dU = -\frac{1}{2}\rho U^2\left(4C_f \frac{dx}{D}\right) + \frac{(U_{xm} - U)\delta \dot{m}}{A} - \frac{\delta F_x}{A}$$

$$d\left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right) = \delta q - \delta w + \left(h_{tm} - \left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right)\right)\frac{\delta \dot{m}}{\rho U A}$$

Assume

$$\delta \dot{m} = C_f = \delta F_x = \delta q = \delta w = 0$$

$$d(\rho U A) = 0$$

$$dP + \rho U dU = 0$$

$$C_p dT + U dU = 0$$

$$P = \rho R T$$

Equations of motion in fractional differential form

$$\frac{d\rho}{\rho} + \frac{dU^2}{2U^2} + \frac{dA}{A} = 0$$

$$\frac{dP}{P} + \frac{\gamma M^2}{2}\frac{dU^2}{U^2} = 0$$

$$\frac{dT}{T} + \frac{(\gamma - 1)M^2}{2}\frac{dU^2}{U^2} = 0$$

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

Equation of state

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

$$-\frac{\gamma M^2}{2}\frac{dU^2}{U^2} = -\frac{dU^2}{2U^2} - \frac{dA}{A} - \frac{(\gamma - 1)M^2}{2}\frac{dU^2}{U^2}$$

Effect of area change on velocity

$$\frac{dU^2}{U^2} = \left(\frac{2}{M^2 - 1}\right)\frac{dA}{A}$$

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Effect of area change on other flow variables

$$\frac{d\rho}{\rho} = -\left(\frac{M^2}{M^2 - 1}\right)\frac{dA}{A}$$

$$\frac{dP}{P} = -\left(\frac{\gamma M^2}{M^2 - 1}\right)\frac{dA}{A}$$

$$\frac{dT}{T} = -\left(\frac{(\gamma - 1)M^2}{M^2 - 1}\right)\frac{dA}{A}$$

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Effect of area change on Mach number $U^2 = \gamma R T M^2$

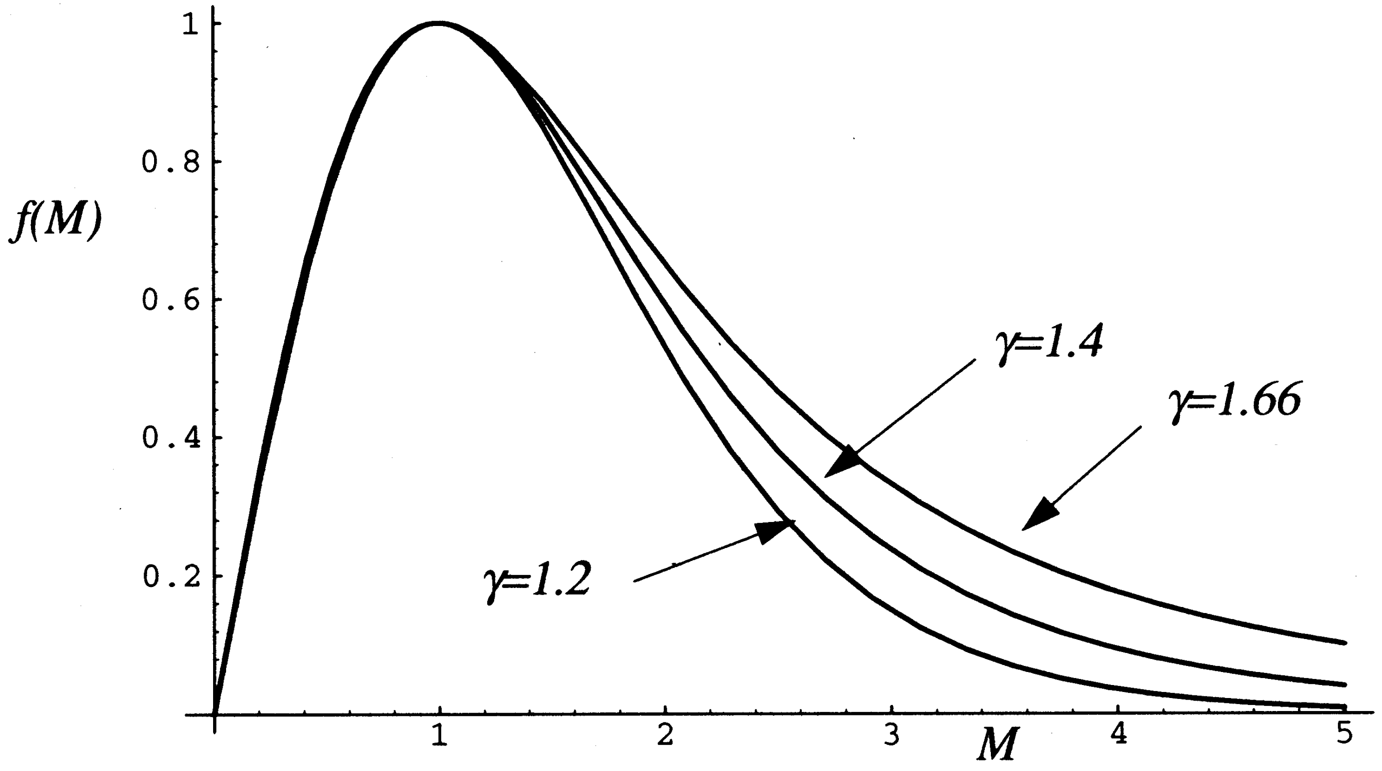
$$\frac{dU^2}{U^2} = \frac{dT}{T} + \frac{dM^2}{M^2}$$

$$\left(\frac{2}{M^2-1}\right)\frac{dA}{A} = -\frac{(\gamma-1)M^2}{M^2-1}\frac{dA}{A} + \frac{dM^2}{M^2}$$

$$\frac{dA}{A} = \frac{M^2-1}{2\left(1+\left(\frac{\gamma-1}{2}\right)M^2\right)}\frac{dM^2}{M^2}$$

$$\int_{M^2}^{1}\frac{M^2-1}{2\left(1+\left(\frac{\gamma-1}{2}\right)M^2\right)}\frac{dM^2}{M^2} = \int_{A}^{A^*}\frac{dA}{A}$$

$$f(M) = \frac{A^*}{A} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{M}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}}.$$



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Mass Conservation

$$\dot{m} = \rho U A$$

$$\dot{m} = \rho U A = \frac{P}{RT}(\gamma R T)^{1/2} M A$$

$$\frac{T_t}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

$$\frac{P_t}{P} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

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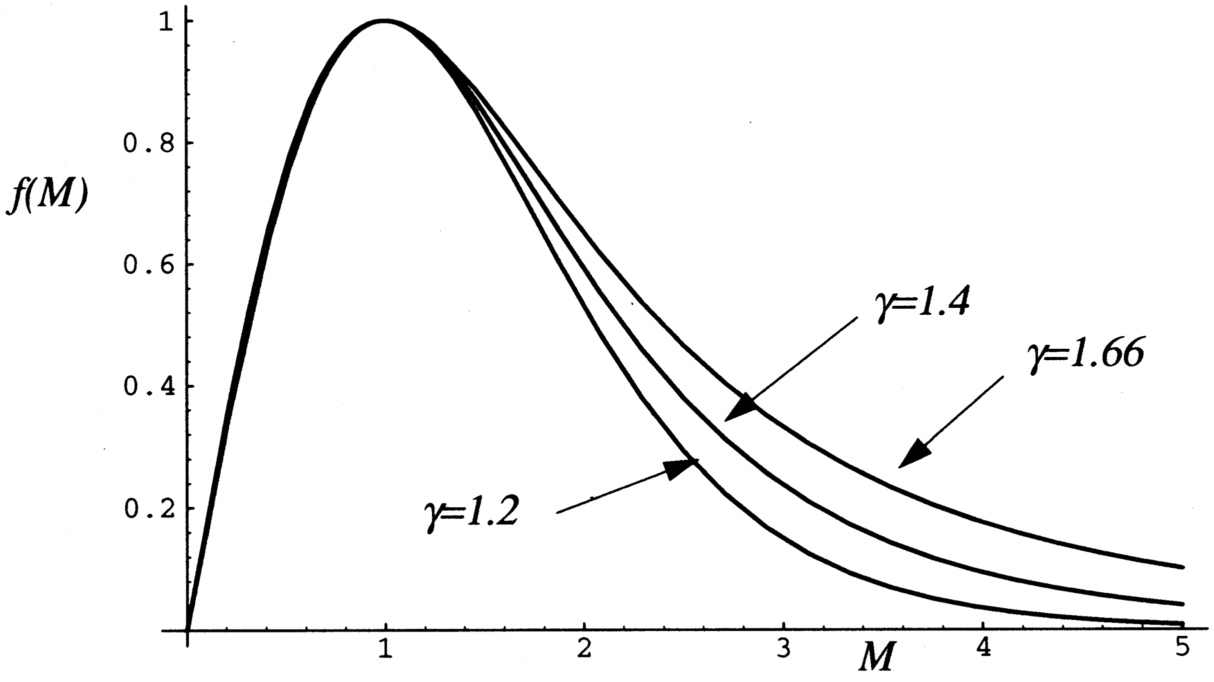
$$\dot{m} = \rho U A = \frac{\gamma}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \left(\frac{P_t A}{\sqrt{\gamma R T_t}}\right) f\left(M\right)$$



Between any two points in a channel with zero mass addition

$$\dot{m}_1 = \dot{m}_2$$

$$\frac{P_{t1} A_1}{\sqrt{T_{t1}}} f\left(M_1\right) = \frac{P_{t2} A_2}{\sqrt{T_{t2}}} f\left(M_2\right)$$

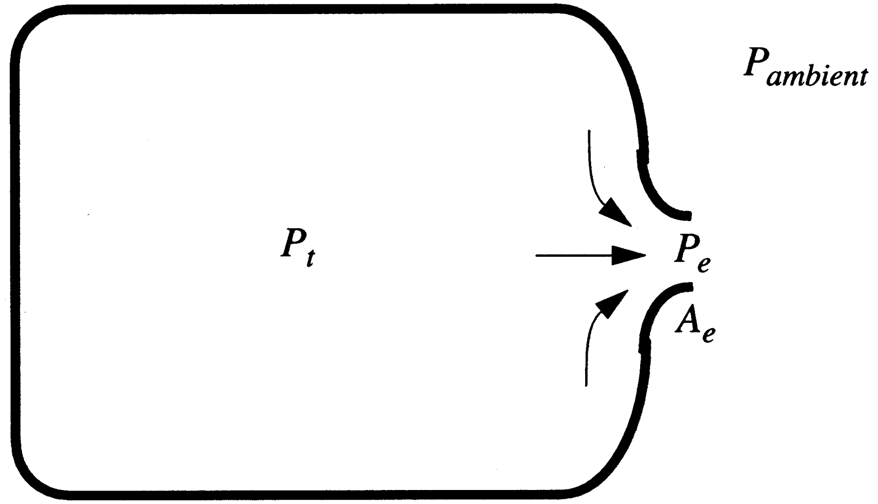
If the flow is adiabatic

$$P_{t1} A_1 f\left(M_1\right) = P_{t2} A_2 f\left(M_2\right)$$

If the flow is adiabatic and isentropic

$$A_1 f\left(M_1\right) = A_2 f\left(M_2\right)$$

10.2 A Simple convergent nozzle



If the flow is subsonic

$$P_e = P_{ambient}$$

For subsonic flow the exit Mach number can be determined from

$$\frac{P_t}{P_e} = \left(1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\right)^{\frac{\gamma}{\gamma - 1}}$$

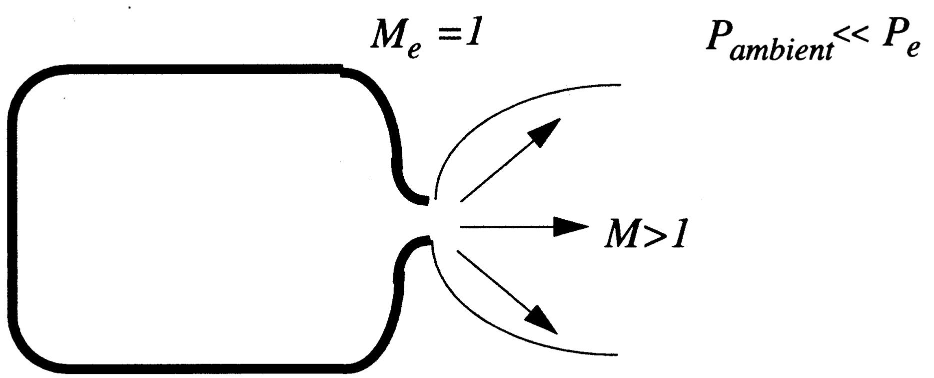
Thus

$$M_e = \left(\frac{2}{\gamma - 1}\right)^{\frac{1}{2}}\left(\left(\frac{P_t}{P_{ambient}}\right)^{\frac{\gamma - 1}{\gamma}} - 1\right)^{\frac{1}{2}}$$

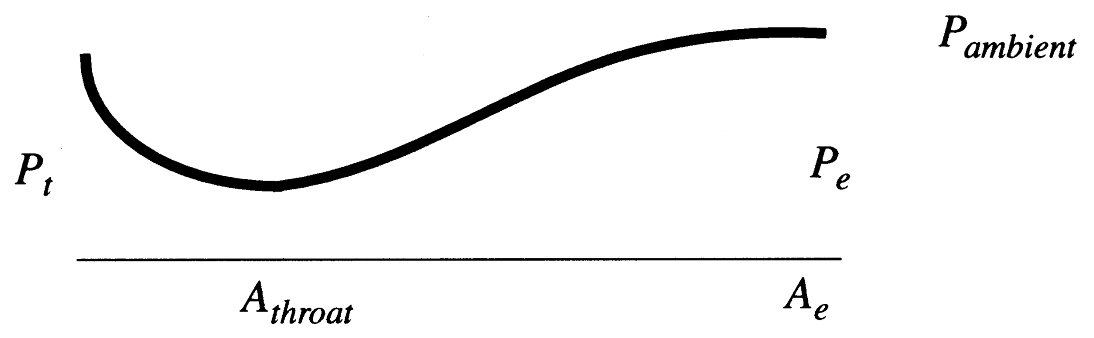
The exit Mach number reaches one when

$$\frac{P_t}{P_{ambient}} = \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma}{\gamma - 1}}$$

If the pressure ratio is very high the flow from the nozzle will spread rapidly.



10.3 Converging-diverging nozzle



Determine two critical exit Mach numbers from

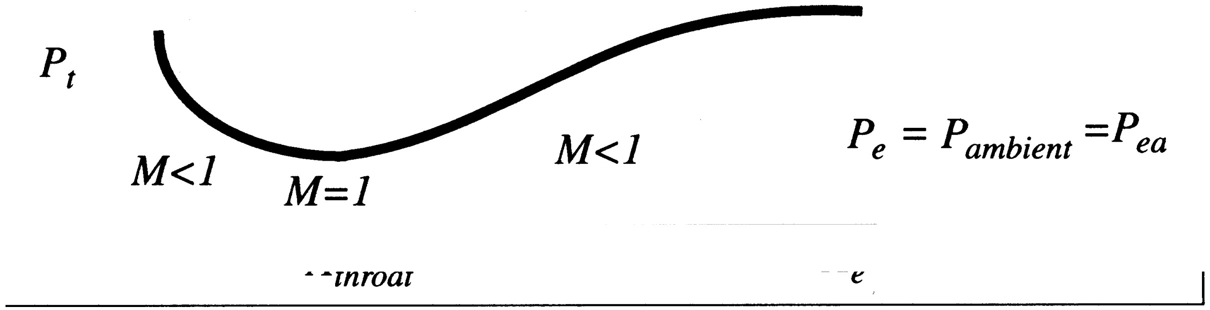
$$\frac{A_t}{A_e} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{M_e}{\left(1+\frac{\gamma-1}{2}M_e^{\,2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}.$$

The corresponding critical exit pressures are determined from

$$\frac{P_t}{P_{ea}} = \left(1 + \left(\frac{\gamma - 1}{2}\right)M_{ea}^2\right)^{\frac{\gamma}{\gamma - 1}}$$

$$\frac{P_t}{P_{eb}} = \left(1 + \left(\frac{\gamma - 1}{2}\right)M_{eb}^2\right)^{\frac{\gamma}{\gamma - 1}}$$

10.3.1 Case 1 - Isentropic, subsonic flow in the nozzle



$$1 < P_t/P_{ambient} < P_t/P_{ea}$$

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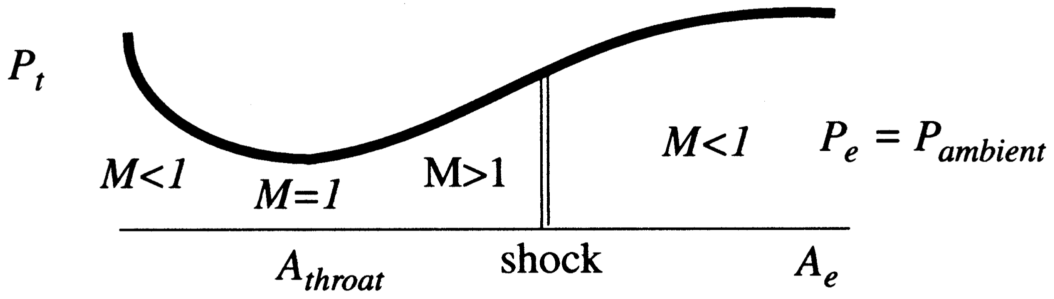
10.3.2 Case 2 - Non-isentropic flow - shock in the nozzle



$$\dot{m}_{throat} = \dot{m}_{exit}$$

$$P_t A_{throat} = P_{te} A_e f(M_e)$$

The exit flow is subsonic and so the exit pressure matches the ambient pressure.

$$P_{te} = P_e\left(1 + \left(\frac{\gamma-1}{2}\right)M_e^2\right)^{\frac{\gamma}{\gamma-1}} = P_{ambient}\left(1 + \left(\frac{\gamma-1}{2}\right)M_e^2\right)^{\frac{\gamma}{\gamma-1}}.$$

Solve for the exit mach number

$$\left(\frac{P_t}{P_{ambient}}\right)\left(\frac{A_{throat}}{A_e}\right) = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} M_e\left(1 + \left(\frac{\gamma-1}{2}\right)M_e^2\right)^{\frac{1}{2}}$$

Now determine the stagnation pressure ratio across the nozzle.

$$\frac{P_{te}}{P_t} = \frac{A_{throat}}{A_e}\frac{1}{f(M_e)} < 1$$

The shock Mach number is now determined from

$$\frac{P_{te}}{P_t} = \left(\frac{\left(\frac{\gamma + 1}{2}\right) M_{shock}^2}{1 + \frac{\gamma - 1}{2} M_{shock}^2} \right)^{\frac{\gamma}{\gamma - 1}} \left(\frac{\frac{\gamma + 1}{2}}{\gamma M_{shock}^2 - \left(\frac{\gamma - 1}{2}\right)} \right)^{\frac{1}{\gamma - 1}}$$

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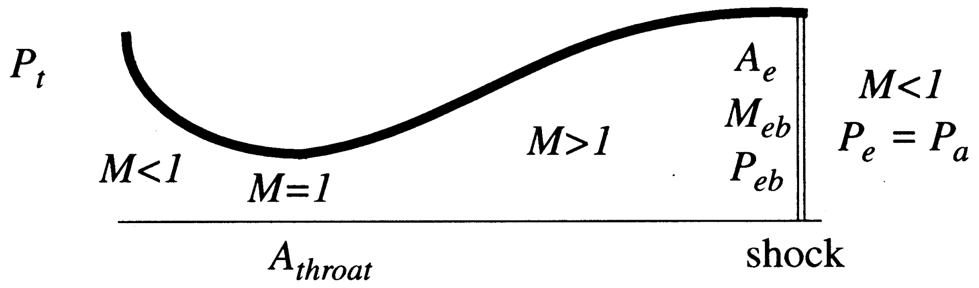
As the nozzle pressure ratio is increased the shock moves downstream until it sits at the nozzle exit.



The Mach number behind the shock is

$$M^2_{e(behind\ shock)} = \frac{1 + \frac{\gamma - 1}{2} M^2_{eb}}{\gamma M^2_{eb} - \left(\frac{\gamma - 1}{2}\right)}$$

This condition is reached when

$$\left.\left(\frac{P_t}{P_{ambient}}\right)\right|_{exit \perp shock} =$$

$$\left(\frac{A_e}{A_{throat}}\right)\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} M_{e(behind\ shock)}\left(1+\left(\frac{\gamma-1}{2}\right)M^2_{e(behind\ shock)}\right)^{\frac{1}{2}}$$

In summary, the shock-in-nozzle case occurs over the range

$$\frac{P_t}{P_{ea}} < \frac{P_t}{P_{ambient}} < \left.\left(\frac{P_t}{P_{ambient}}\right)\right|_{exit \perp shock}$$

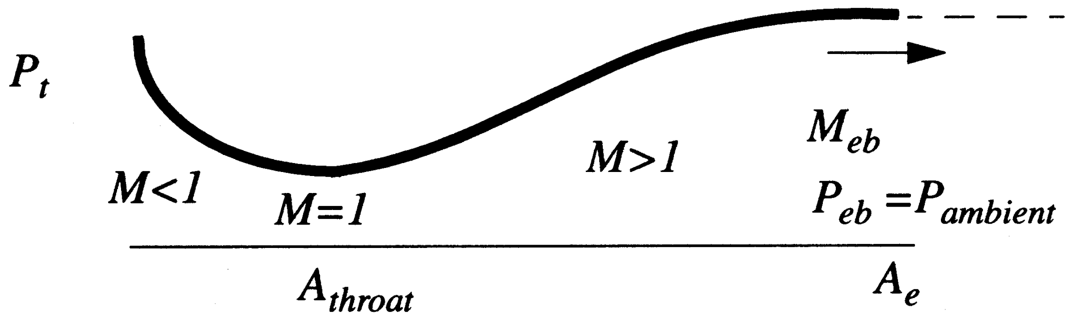
10.3.3 Case 3 - Isentropic supersonic flow in the nozzle

i) Over expanded flow

$$\left.\left(\frac{P_t}{P_{ambient}}\right)\right|_{exit\ \perp\ shock} < P_t/P_{ambient} < P_t/P_{eb}$$

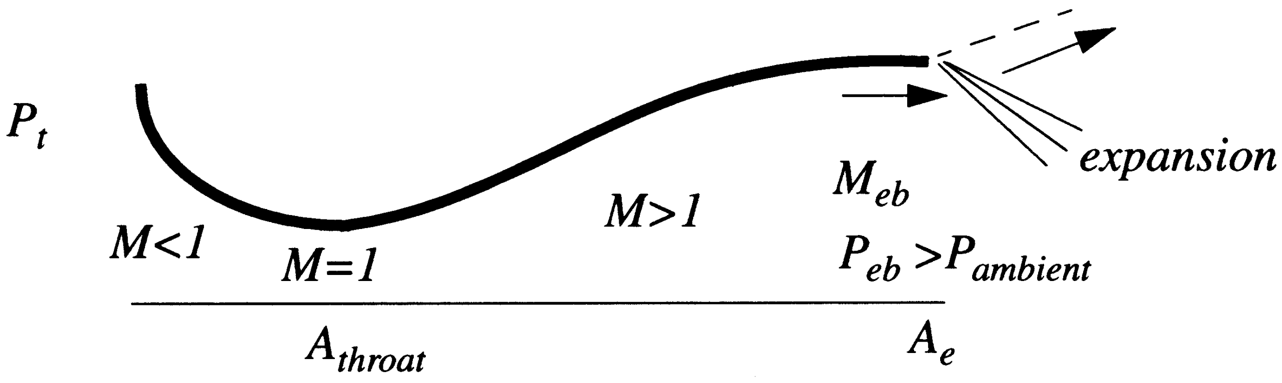
ii) Fully expanded flow

$$P_t/P_{ambient} = P_t/P_{eb}$$



iii) Under expanded flow

$$P_t/P_{ambient} > P_t/P_{eb}$$



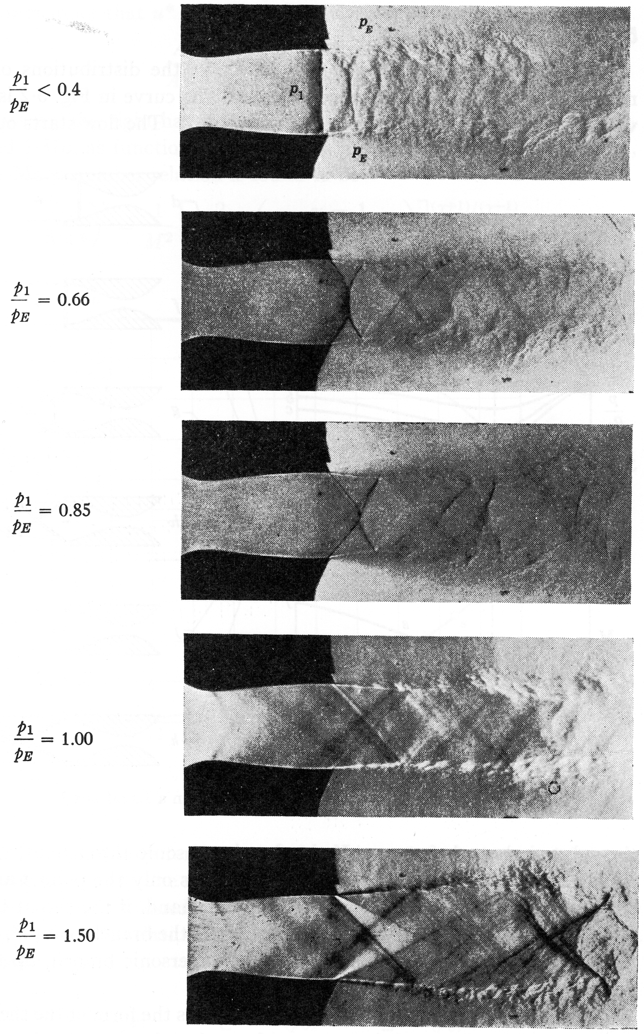
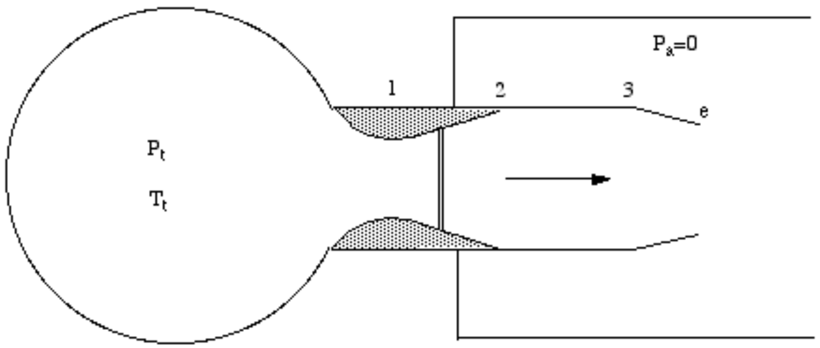


FIG. 5·4 Schlieren photographs of flow from a supersonic nozzle at different back pressures. The photographs, from top to bottom, may be compared with Fig. 5·3, sketches *d, g, h, j, k,* respectively. Reproduced from: L. Howarth (ed.), *Modern Developments in Fluid Dynamics, High Speed Flow,* Oxford, 1953.

Figure 5.4 from Liepmann and Roshko

10.4.3 Gasdynamics of a double throat - Supersonic Wind Tunnel Start and Unstart



$$A_2/A_1 = 6 \qquad A_3 = A_2 \qquad A_e/A_1 = 2$$

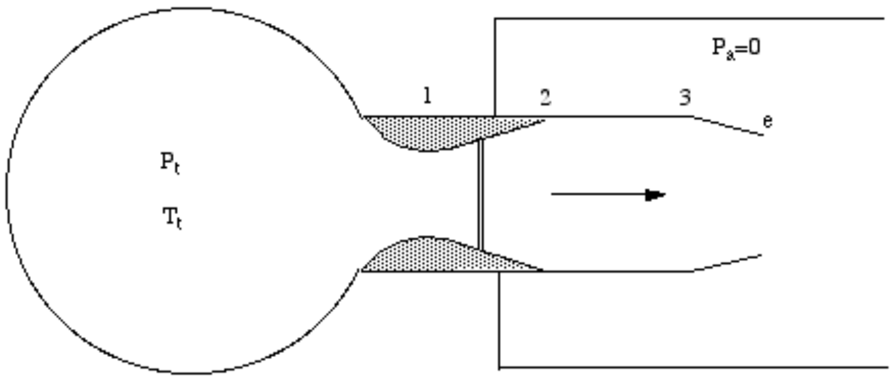
1) Determine P_{te}/P_t.

$$\dot{m}_1 = \dot{m}_e$$

$$\left(\frac{P_{t1}A_1}{\sqrt{\gamma R T_{t1}}}\right) f(M_1) = \left(\frac{P_{te}A_e}{\sqrt{\gamma R T_{te}}}\right) f(M_e) \tag{10.53}$$

The flow exits to vacuum and so the large pressure ratio across the system essentially guarantees that both throats must be choked, $M_1 = 1$ and $M_e = 1$.

$$P_{te}/P_t = A_1/A_e = 0.5. \tag{10.54}$$



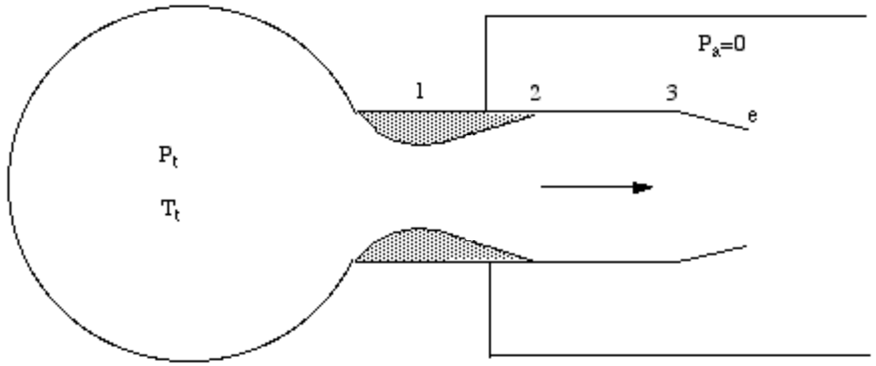
2) Determine the shock Mach number

From the relations for shock wave flow, the shock Mach number that reduces the stagnation pressure by half for a gas with $\gamma = 1.4$ is $M_s = 2.5$.

3) Determine the Mach numbers at stations 2 and 3

$$A_e / A_3 = 1/3 \Rightarrow M_3 = 0.195 \qquad (10.55)$$

$$M_2 = 0.195$$



4) Suppose A_e is reduced to the point where $A_e = A_1$. What happens to the shock?

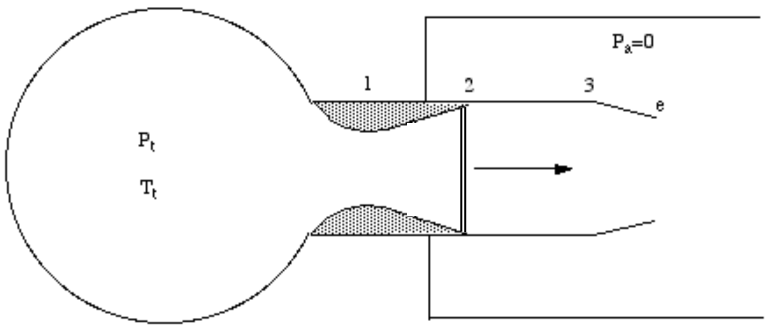
$$\dot{m}_1 = \dot{m}_e$$

$$\left(\frac{P_{t1}A_1}{\sqrt{\gamma R T_{t1}}}\right) f(M_1) = \left(\frac{P_{te}A_e}{\sqrt{\gamma R T_{te}}}\right) f(M_e) \qquad (10.53)$$

$$P_{te}/P_t = A_1/A_e = 1.0 \qquad (10.56)$$

5) Suppose A_e is made smaller than A_1, what happens?

$$f(M_1) = A_e/A_1$$



6) Suppose A_e is increased above $A_e/A_1 = 2$. What happens to the shock?

The shock moves downstream until a point is reached where it sits just upstream of station 2.

$$M_2 = 3.368 \qquad P_{te}/P_t = 0.2388$$

$$A_e/A_1 = P_t/P_{te} = 4.188 \tag{10.58}$$

Throughout this process the exit is at $M_e = 1$ and the flow in the test section is subsonic due to the presence of the shock. In fact the Mach number in the test section from station 2 to 3 would be the Mach number behind a Mach 3.368 shock which is 0.4566. Note that this is consistent with the area ratio $A_3/A_e = 6/4.188 = 1.433$ for which the subsonic solution of (10.16) is 0.4566.

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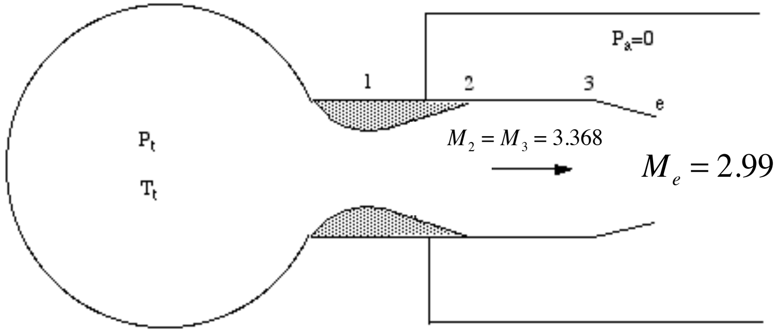

7) Now suppose A_e/A_1 is increased just slightly above 4.188, what happens?

All supersonic flow is established in the wind tunnel - the tunnel is said to have started.

$$(P_{t1}A_1) = (P_{te}A_e)f(M_e) \qquad (10.59)$$

$$f(M_e) = A_1/A_e = 1/4.188 \qquad (10.60)$$

The Mach number at the exit throat is now the supersonic root of (10.60), $M_e = 2.99$. If A_e/A_1 is increased further the exit Mach number increases according to Equation (10.60). If A_e/A_1 is reduced below 4.188 the exit Mach number reduces below 2.99 until it approaches one from above as $A_e/A_1 \to 1 + \varepsilon$. If A_e/A_1 is reduced below one the wind tunnel *unstarts* and the flow between 1 and the exit is all subsonic with $M_1 = M_e = 1$.

10.5 Problems

Problem 1 - Consider the expression ρU^n ; n=1 corresponds to the mass flux, n=2 corresponds to the momentum flux and n=3 corresponds to the energy flux of a compressible gas. Use the momentum equation

$$dP + \rho U dU = 0 \tag{9.61}$$

to determine the Mach number (as a function of n) at which ρU^n is a maximum in steady flow.

Problem 2 In the double-throat example above the flow exhausts into a vacuum chamber. Suppose the pressure P_a is not zero. What is the minimum pressure ratio P_t/P_a that would be required for the supersonic tunnel to *start* as described in the example?

Problem 3 - In the double-throat example above suppose the effect of wall friction is included. How would the answers to the problem change? Would the various values calculated in the problem increase, decrease or remain the same and why?

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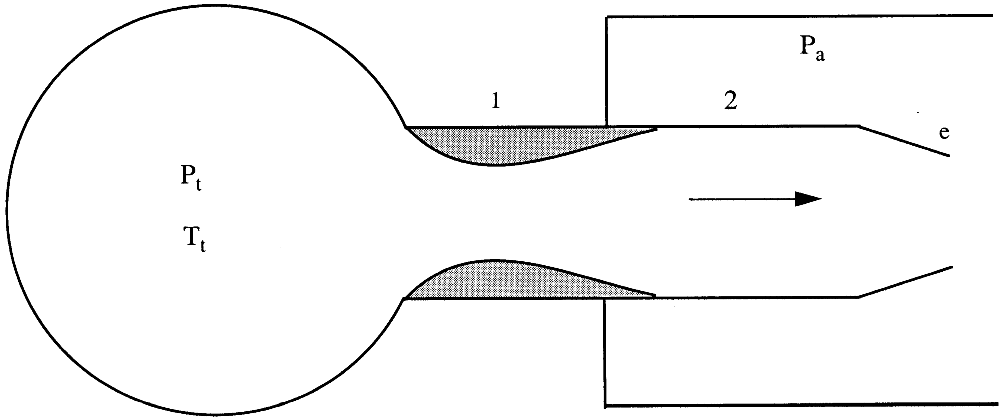
Problem 4 - The figure below shows a supersonic wind tunnel which uses Helium as a working gas. A very large plenum contains the gas at constant stagnation pressure and temperature P_t, T_t. Supersonic flow is established in the test section and the flow exhausts to a large tank at pressure P_a.



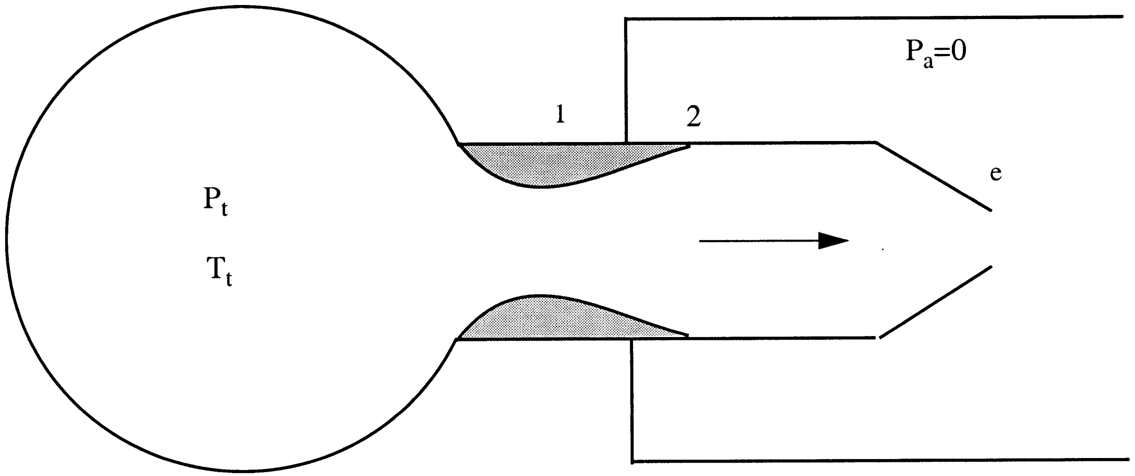
The exit area A_e can be varied in order to change the flow conditions in the tunnel Initially $A_2/A_e = 4$, and $A_2/A_1 = 8$. The gas temperature in the plenum is $T_t = 300K$. Neglect wall friction. Let $P_t/P_0 = 40$.

1) Determine the Mach numbers at A_e, A_1 and A_2

2) Determine the velocity U_e and pressure ratio P_e/P_0.

3) Suppose A_e is reduced. Determine the value of A_e/A_2 which would cause the Mach number at A_e to approach one (from above). Suppose A_e is reduced slightly below this value - what happens to the supersonic flow in the tunnel? Determine P_{te}/P_t and the Mach numbers at A_1, A_2 and A_e for this case.

Problem 5 - The figure below shows a supersonic wind tunnel which uses Air as a working gas. A very large plenum contains the gas at constant stagnation pressure and temperature, P_t, T_t. The flow exhausts to a large tank which is maintained at vacuum $P_a = 0$. The upstream nozzle area ratio is $A_2/A_1 = 3$. The downstream throat area A_e can be varied in order to change the flow conditions in the tunnel. Initially, $A_e = 0$. Neglect wall friction. Assign numerical values where appropriate.



1) Suppose A_e/A_1 is slowly increased from zero. Plot P_{te}/P_t as a function of A_e/A_1 for the range $0 \le A_e/A_1 \le 3$.

2) Now with $A_e/A_1 = 3$ initially, let A_e be decreased back to zero. Plot P_{te}/P_t as a function of A_e/A_1 for this process.

Problem 6 - In Chapter 2 we looked at the blowdown through a small nozzle of a calorically perfect gas from a large adiabatic pressure vessel at initial pressure P_i and temperature T_i to the surroundings at pressure P_a and temperature T_a. I would like you to reconsider that problem from the point of view of the conservation equations for mass and energy. Use a control volume analysis to determine the relationship between the pressure, density and temperature in the vessel as the mass is expelled. Show that the final temperature derived from a control volume analysis is the same as that predicted by integrating the Gibbs equation.

Problem 7 - Consider the inverse of Problem 6. A highly evacuated, thermally insulated flask is placed in a room with air temperature T_a. The air is allowed to enter the flask through a slightly opened stopcock until the pressure inside equals the pressure in the room. Assume the air to be calorically perfect. State any other assumptions needed to solve the problem.

(i)Use a control volume analysis to determine the relationship between the pressure, density and temperature in the vessel as mass enters the vessel.

(ii)Determine the entropy change per unit mass during the process for the gas that enters the vessel.

(iii)Determine the final temperature of the gas in the vessel.