PRICING IN POSITION AUCTIONS AND ONLINE ADVERTISING

MEHMET BUMIN YENMEZ

Stanford Graduate School of Business

Abstract. This paper analyzes multidimensional position auctions with general pricing rules. The preeminent example of position auctions is the "generalized second-price" (GSP) auction used by major search engines to sell online advertising. Edelman et al. (2007) establish that the unique ex-post equilibrium of the GSP auction and the dominant strategy equilibrium of the Vickrey-Clarke-Groves (VCG) mechanism have identical ex-post payoffs. However, they only analyze the GSP auction, where the price for an advertiser depends only on the bid of the next highest advertiser, with one dimensional types. This paper shows that the result still holds as long as the price for an advertiser who wins a position depends on any of the bids of advertisers who win lower positions in any "sensible" way, even if the advertisers have multidimensional types.

1. Introduction

The GSP auctions, where bidders are ranked according to their bids and pay the minimum amount necessary to preserve their rankings, have been a huge success in online advertising. For example, Google's revenue in 2008 from online advertising was \$21.13 billion, around 97 percent of Google's revenue. Likewise, the majority of Yahoo!'s \$7.21 billion revenue was from the GSP auctions.

Date: May 4, 2009, First Version: October 20,2008.

Acknowledgements: I am indebted to Alexander Frankel, Michael Ostrovsky, Dan Taylor and Robert Wilson for comments and support.

Contact: `byenmez@stanford.edu`.

Although online advertising has been the focus of position auctions literature, there are several other markets where bidders have unit demands and the same preferences over the items. For instance, the condominium market where buyers have the same ranking over condominium units and also the sales of quality-ranked raw materials. Position auctions can similarly be used in such markets.

The GSP auction is used in online advertising markets which seems to be similar to the VCG mechanism. Indeed, if there is only one item to be allocated, the GSP auction coincides with the VCG mechanism. However, Edelman et al. (2007) show that the GSP auction is not the same as the VCG mechanism when there is more than one item. In addition, they show that the GSP auction has a unique perfect Bayesian equilibrium which is ex-post payoff equivalent to the VCG mechanism. That is, although the GSP auction is not the VCG mechanism, it achieves the same ex-post allocation.

We show that the ex-post payoff equivalence is not coincidental. It still holds for a large class of position auctions. To be more precise, consider a general ascending auction of multiple items, where bidders have unit demand. Bidders share a common ordinal ranking over the items, but have different cardinal preferences. In the auction, time on the clock starts in the beginning of the auction from zero and increases continuously. The clock stops when there is only one agent remaining and the auction ends at that time. The last agent to remain in the auction wins the best item; the second to last agent wins the second best item and so on. Contrary to the usual clock auctions, the time does not show the current price. The price that winning agents pay depend on the drop out times of previous bidders. Therefore, the time increases continuously during the auction even though the price increases discretely only when a new bidder drops out.

The main result of this paper is that there is an ex-post equilibrium which is ex-post payoff equivalent to the VCG mechanism. This suggests that even if some other auction formats were used instead of the GSP auction, the revenues would have been the same for the search engines.

This generalizes the corresponding result of Edelman et al. (2007) along several aspects. First is the general pricing rule incorporated in the analysis. The price that a winner pays may depend on any of previous players' drop out times and can take many different functional forms, defining a different auction. A second generalization is that bidders have multidimensional types.

There is a growing literature on the GSP auctions in several different fields including economics and computer science (see Edelman et al. (2007), Varian (2007) and Aggarwal et al. (2006) for pioneering work). The recent literature mainly focuses on topics like budget constraints (Abrams et al. (2007) and Feldman et al. (2007)) and reserve prices (Edelman and Schwarz (2007)). However, to the best of our knowledge, there is no prior work on general pricing rules to use for position auctions.

The paper proceeds as follows: Section 2 introduces the model. Section 3 defines an equilibrium. Section 4 establishes the ex-post payoff equivalence result, and the paper ends with the conclusion on section 5.

2. The Model

There are n items to be allocated to k agents where $k > n$. Each agent has unit demand. Moreover, each agent has a value for every item such that lower numbered items have higher values. To be more specific, for each agent i there exists $v_i \in \mathbb{R}_+^n$ such that $v_{ij} > v_{ij'}$ if $j < j'$. Valuations of agents are independent of each other.

The items are to be allocated through a generalized ascending auction. In this auction, there is a clock showing the current time. The clock starts from zero and increases continuously until there is only one agent remaining in the auction. The last agent remaining in the auction gets item 1, the second to last agent gets item 2 and so forth. The first $k - n$ agents to drop out get no item and pay nothing. The j-th last agent who gets item j pays $f^j(t_k, t_{k-1}, \dots, t_{j+1})$ where t_i is the drop out time of agent i, $k \geq i$.

Suppose that f^j is continuous, non-decreasing in all coordinates and increasing in t_{j+1}. We further make the following assumption and call functions which satisfy this condition **regular**.

Assumption (Regularity). *(i)* $f^n(0,\dots,0)=0$,
(ii) $\lim_{t\to\infty} f^i(t_k,\dots,t_{i+2},t)=\infty$ *for all* i,
(iii) $f^i(t_k,\dots,t_{i+1})=f^{i-1}(t_k,\dots,t_{i+1},t_{i+1})$ *for all* i.

Condition (i) states that if the previous bidders have all dropped out at 0, then the price for bidder n must be zero. Condition (ii) says that the price a winner pays can be made arbitrarily high by increasing the previous bidder's drop out time. Condition (iii) makes the connection between f^i and f^{i-1}- that f^i is uniquely determined by looking at f^{i-1} on the domain boundary when $t_{i+2}=t_{i+1}$.

Although condition (iii) looks restrictive at first glance, it is satisfied by a large class of functions. For example, this condition is satisfied when $f^i(t_k,\dots,t_{i+1})=t_{i+1}$ which corresponds to the case of the GSP auction.[1] Other pricing functions to satisfy these conditions can be constructed very easily. Take any function $f_1:\mathbb{R}_+^{k-1}\to\mathbb{R}_+$ which is increasing in all its arguments, goes up to infinity when the last argument goes to infinity and is zero at the origin. Define $f_i:\mathbb{R}_+^{k-i}\to\mathbb{R}_+$ recursively by $f^i(t_k,\dots,t_{i+1})=f^{i-1}(t_k,\dots,t_{i+1},t_{i+1})$ for $2\le i\le n$.[2]

Since the main result states that the equilibrium constructed is an ex-post equilibrium, the distribution of values can be quite general. However, we need an assumption to guarantee that the equilibrium allocation is the efficient one. In words, what we need is that if agent i has a greater incremental value of having item $j-1$ instead of item j than agent i' for *some* j, then agent i has a greater incremental value

---

[1] In reality, search engines adjust the bids based on several factors including click-through rates. We generalize the analysis to incorporate click-through rates.

[2] To see how this construction works, consider the following example. Let $k=5$ and $n=3$. First start with a function $f_1:\mathbb{R}_+^4\to\mathbb{R}_+$ which satisfies (i) and (ii). For example, take $f_1(t_5,t_4,t_3,t_2)=e^{t_2+t_3/2}+log(t_4+1)-1$. By the recursive definition, $f_2(t_5,t_4,t_3)=f_1(t_5,t_4,t_3,t_3)=e^{3*t_3/2}+log(t_4+1)-1$. Similarly, $f_3(t_5,t_4)=f_2(t_5,t_4,t_4)=e^{3*t_4/2}+log(t_4+1)-1$.

for having item $j-1$ instead of item j than agent i' for *all* j where $2 \leq j \leq n$. Reordering the agents produces the following assumption.

Assumption (**Single-crossing**). *For two agents $i < i'$ and for every $n \geq j \geq 1$ the following holds:*

$$v_{ij} - v_{i(j+1)} \geq v_{i'j} - v_{i'(j+1)}.^{3}$$

This assumption is similar to the single-crossing property used in the mechanism design literature routinely.

3. Equilibrium Analysis

In this section an equilibrium is constructed and shown to be an ex-post equilibrium.

The strategy of an agent is defined as the time to drop out given that the agent has not dropped out yet. Therefore, the dropping out strategy of agent i can be denoted by $b_i^m(v_i|t_k, t_{k-1}, \ldots, t_{m+1})$ where m is the number of agents remaining -including i- in the auction, $t_k, t_{k-1}, \ldots, t_{m+1}$ are the previous drop out times ($t_k \leq t_{k-1} \leq \ldots \leq t_{m+1}$) and v_i is the value vector of agent i. Since the auction stops when there is only one agent remaining in the auction, m is at least 2. If no bidder has dropped out yet denote the strategy of agent i by $b_i^k(v_i|\emptyset)$.

The intuition of the equilibrium is as follows. Each agent, at the time they drop out, should be indifferent to 1) dropping out then and getting the corresponding item at the current price or 2) getting the next item with one bidder dropping out at that time. Therefore, $b_i^m(v_i|t_k, t_{k-1}, \ldots, t_{m+1})$ is given by an indifference equation for each m. These conditions can be written as follows:

---

[3] Here, let $v_{i(n+1)} = 0$ for all i.

$$\begin{aligned} v_{in} - f^n(b_i^k(v_i|\emptyset), \ldots, b_i^k(v_i|\emptyset)) &= 0, \\ v_{in} - f^n(t_k, b_i^{k-1}(v_i|t_k), \ldots, b_i^{k-1}(v_i|t_k)) &= 0, \\ &\vdots \\ v_{in} - f^n(t_k, \ldots, t_{n+2}, b_i^{n+1}(v_i|t_k, \ldots, t_{n+2})) &= 0, \end{aligned}$$

$$\begin{aligned} v_{in} - f^n(t_k, \ldots, t_{n+1}) &= v_{i(n-1)} - f^{n-1}(t_k, \ldots, t_{n+1}, b_i^n(v_i|t_k, \ldots, t_{n+1})), \\ v_{i(n-1)} - f^{n-1}(t_k, \ldots, t_n) &= v_{i(n-2)} - f^{n-2}(t_k, \ldots, t_n, b_i^{n-1}(v_i|t_k, \ldots, t_n)), \\ &\vdots \\ v_{i2} - f^2(t_k, \ldots, t_3) &= v_{i1} - f^1(t_k, \ldots, t_3, b_i^2(v_i|t_k, \ldots, t_3)). \end{aligned}$$

For general f and v, there might not be any drop out strategies to satisfy the indifference conditions. Lemma 1 proves that our assumptionss guarantee the existence of such strategies.

Lemma 1. *If $\{v_i\}_{i=1}^k$ satisfies the single-crossing assumption and $\{f^i\}_{i=1}^n$ is regular then there exist $b_i^m(v_i|t_k, t_{k-1}, \ldots, t_{m+1})$ satisfying the indifference conditions for $i \leq m$ and $t_k \leq t_{k-1} \leq \ldots \leq t_3$:*

Proof. In the proof, the existence of b_i^k for all i, b_i^{k-1} for $i < k$ and b_i^n for $i < n+1$ are proven. The existence of the remaining bidding strategies can be shown exactly like one of these proofs.

(Existence of b_i^k): Since $f^n(0, \ldots, 0) = 0$ (condition (i)) and $\lim_{t\to\infty} f^n(t, \ldots, t) = \infty$ (this is implied by (ii)) then there exists $b_i^k(v_i|\emptyset)$ such that $f^n(b_i^k(v_i|\emptyset), \ldots, b_i^k(v_i|\emptyset)) = v_{in}$ by continuity of f^n.

(Existence of b_i^{k-1}): To show that $b_i^{k-1}(v_i|t_k)$ exists, first note that t_k is such that $v_{kn} = f^n(t_k, \ldots, t_k) \leq v_{in}$. This holds because otherwise player i ($i < k$) would have dropped-out before player k by the single-crossing assumption.

Let $g(t) = v_{in} - f^n(t_k, t, \ldots, t)$. Now, $g(t_k) = v_{in} - f^n(t_k, t_k, \ldots, t_k) = (v_{in} - v_{kn}) + (v_{kn} - f^n(t_k, t_k, \ldots, t_k)) = v_{in} - v_{kn} \geq 0$ for $i < k$ where the definition of t_k is used. Hence, $g(t_k) \geq 0$. Now, since $\lim_{t\to\infty} g(t) =$

$-\infty$ by monotonicity of f^n and (ii), there exists $b_i^{k-1}(v_i|t_k)$ such that $g(b_i^{k-1}(v_i|t_k)) = 0$ by continuity of f^n which implies that $v_{in} - f^n(t_k, b_i^{k-1}(v_i|t_k), \ldots, b_i^{k-1}(v_i|t_k)) = 0$.

(Existence of b_i^n): To show that $b_i^n(v_i|t_k, \ldots, t_{n+1})$ exists, note that $t_k, \ldots, t_{n+1}$ are such that $v_{(n+1)n} - f^n(t_k, \ldots, t_{n+2}, t_{n+1}) = 0$.

Let $g'(t) = v_{in} - f^n(t_k, \ldots, t_{n+1}) - v_{i(n-1)} + f^{n-1}(t_k, \ldots, t_{n+1}, t)$. Now, $g'(t_{n+1}) = v_{in} - f^n(t_k, \ldots, t_{n+1}) - v_{i(n-1)} + f^{n-1}(t_k, \ldots, t_{n+1}, t_{n+1}) = v_{in} - v_{i(n-1)} \leq 0$ by (iii). Now, $\lim_{t\to\infty} g'(t) = \infty$ by (ii) point which implies that there exists $b_i^n(v_i|t_k, \ldots, t_{n+1})$ such that $g'(b_i^n(v_i|t_k, \ldots, t_{n+1})) = 0$ by continuity of f^n. By definition of g', this is equivalent to $v_{in} - f^n(t_k, \ldots, t_{n+1}) = v_{i(n-1)} - f^{n-1}(t_k, \ldots, t_{n+1}, b_i^n(v_i|t_k, \ldots, t_{n+1}))$. ☐

Remark 1. *In order to accommodate click through rates used in online advertising, we generalize the previous lemma in the following way. Let* $\{\alpha_i\}_{i=1}^n$ *be a sequence of non-increasing real numbers. The pricing rule and the value functions can be defined using a vector* v *which satisfies the single-crossing condition and* f *which is regular. The value that agent* i *assigns to item* j *is* $v'_{ij} = \alpha_j v_{ij}$. *The price function for item* j *is* $f'^j(t_k, t_{k+1}, \ldots, t_{j+1}) = \alpha_j f^j(t_k, t_{k+1}, \ldots, t_{j+1})$.

We claim that Lemma 1 still holds for $\{v'_i\}_{i=1}^k$ *and* $\{f'^i\}_{i=1}^n$. *Note that* $\{v'_i\}_{i=1}^k$ *also satisfies the single-crossing condition. The proof is still the same with the following change in the existence of* b_i^n. *Let,* $g'(t) = v'_{in} - f'^n(t_k, \ldots, t_{n+1}) - v'_{i(n-1)} + f'^{n-1}(t_k, \ldots, t_{n+1}, t)$. *Now,* $g'(t_{n+1}) = v'_{in} - f'^n(t_k, \ldots, t_{n+1}) - v'_{i(n-1)} + f'^{n-1}(t_k, \ldots, t_{n+1}, t_{n+1}) = \alpha_n(v_{in} - f^n(t_k, \ldots, t_{n+1})) - \alpha_{n-1}(v_{i(n-1)} + f^{n-1}(t_k, \ldots, t_{n+1}, t_{n+1})) \leq \alpha_n(v_{in} - f^n(t_k, \ldots, t_{n+1})) - \alpha_n(v_{i(n-1)} + f^{n-1}(t_k, \ldots, t_{n+1}, t_{n+1})) \leq \alpha_n(v_{in} - v_{i(n-1)}) \leq 0$. *Now,* $\lim_{t\to\infty} g'(t) = \infty$ *by (ii) point which implies that there exists* $b_i^n(v_i|t_k, \ldots, t_{n+1})$ *such that* $g'(b_i^n(v_i|t_k, \ldots, t_{n+1})) = 0$ *by continuity of* f^n. *By definition of* g', *this is equivalent to* $v'_{in} - f'^n(t_k, \ldots, t_{n+1}) = v'_{i(n-1)} - f'^{n-1}(t_k, \ldots, t_{n+1}, b_i^n(v_i|t_k, \ldots, t_{n+1}))$.

The first result is that the drop out strategy profiles defined in this section constitute an equilibrium.

Theorem 1. *The strategy profile in which every agent* i *adopts* $b_i^m(v_i|\cdot)$ *is an ex-post Nash equilibrium.*

Proof. Let t_i, $i \geq 2$ denote the drop out time of agent i (agent 1 is the last one to remain in the auction, so agent 1 does not have a drop out time). By construction $t_2 \geq t_3 \geq \ldots \geq t_k$. With this outcome agents $n+1$ to k get no item and pay nothing. Agent i with $1 \leq i \leq n$ gets item i and pays $f^i(t_k, \ldots, t_{i+1})$. The proof shows that agent i cannot benefit by dropping out for another item $j \leq n$. The analysis can be split into two cases depending on the ranking of i and j. The two cases can be analyzed analogously, so only the case where $i > j$ is provided.

If agent i drops out for a better item j, the drop out times of agents k to $i+1$ do not change, however agents $i-1$ to $j+1$ no longer have the same drop out times. However, the new drop out times can be calculated using their strategies. Let $t'_{i-1}, \ldots, t'_{j+1}$ be the new drop out times.

The proof is done by induction on $h = i - j$. The following inequality is needed:

$$v_{ii} - f^i(t_k, \ldots, t_{i+1}) \geq v_{i(i-h)} - f^{i-h}(t_k, \ldots, t_{i+1}, t'_{i-1}, \ldots, t'_{i-h}).$$

Base Case ($h = 1$)

The indifference condition for agent $i-1$'s drop out time t'_{i-1} is:

$$v_{(i-1)i} - f^i(t_k, \ldots, t_{i+1}) = v_{(i-1)(i-1)} - f^{i-1}(t_k, \ldots, t_{i+1}, t'_{i-1}).$$

Rearrange this to get:

$$f^{i-1}(t_k, \ldots, t_{i+1}, t'_{i-1}) - f^i(t_k, \ldots, t_{i+1}) = v_{(i-1)(i-1)} - v_{(i-1)i}$$

By single-crossing assumption, $v_{(i-1)(i-1)} - v_{(i-1)i} \geq v_{i(i-1)} - v_{ii}$. Therefore, replace the right hand side of the above equality by $v_{i(i-1)} - v_{ii}$ to get:

$$f^{i-1}(t_k, \ldots, t_{i+1}, t'_{i-1}) - f^i(t_k, \ldots, t_{i+1}) \geq v_{i(i-1)} - v_{ii}.$$

Rearranging this gives the desired inequality for $h = 1$.

Suppose that the inequality holds for $h = s - 1$, that is:

(1)
$$v_{ii} - f^i(t_k, \ldots, t_{i+1}) \geq v_{i(i-s+1)} - f^{i-s+1}(t_k, \ldots, t_{i+1}, t'_{i-1}, \ldots, t'_{i-s+1}).$$

Now, let us show the case when $h = s$.

General Case ($h = s$)

The indifference condition for agent $i - s$'s drop out time t'_{i-s} is:

$$v_{(i-s)(i-s+1)} - f^{i-s+1}(t_k, \ldots, t_{i+1}, t'_{i-1}, \ldots, t'_{i-s+1}) = \\ v_{(i-s)(i-s)} - f^{i-s}(t_k, \ldots, t_{i+1}, t'_{i-1}, \ldots, t'_{i-s}).$$

Rearrange this to get:

$$f^{i-s}(t_k, \ldots, t_{i+1}, t'_{i-1}, \ldots, t'_{i-s}) - f^{i-s+1}(t_k, \ldots, t_{i+1}, t'_{i-1}, \ldots, t'_{i-s+1}) = \\ v_{(i-s)(i-s)} - v_{(i-s)(i-s+1)}.$$

By single-crossing assumption, $v_{(i-s)(i-s)} - v_{(i-s)(i-s+1)} \geq v_{i(i-s)} - v_{i(i-s+1)}$. Therefore we can replace the right hand side of the above equality by $v_{i(i-s)} - v_{i(i-s+1)}$ to get:

$$f^{i-s}(t_k, \ldots, t_{i+1}, t'_{i-1}, \ldots, t'_{i-s}) - f^{i-s+1}(t_k, \ldots, t_{i+1}, t'_{i-1}, \ldots, t'_{i-s+1}) \geq \\ v_{i(i-s)} - v_{i(i-s+1)}.$$

Rearranging gives:

$$v_{i(i-s+1)} - f^{i-s+1}(t_k, \ldots, t_{i+1}, t'_{i-1}, \ldots, t'_{i-s+1}) \geq \\ v_{i(i-s)} - f^{i-s}(t_k, \ldots, t_{i+1}, t'_{i-1}, \ldots, t'_{i-s}).$$

The last inequality and Inequality 1 imply that:

$$v_{ii} - f^{i}(t_k, \ldots, t_{i+1}) \geq v_{i(i-s)} - f^{i-s}(t_k, \ldots, t_{i+1}, t'_{i-1}, \ldots, t'_{i-s}).$$

This finishes the argument for the general case. □

4. Ex-post Equivalence to VCG mechanism

This section shows that the equilibrium payoffs are ex-post equivalent to the VCG mechanism payoffs. Note that the usual revenue equivalence results (see Myerson (1981) and Riley and Samuelson (1981)) are for interim payoffs.

The VCG mechanism allocates items efficiently and the side payments are such that everybody pays the allocational externality that they cause to the rest of the agents. Note that by single-crossing assumption the assortative matching that we get in the equilibrium is the efficient one. The payments must be compared to show ex-post equivalence of the payoffs.

Theorem 2. *The equilibrium defined in the previous section is ex-post equivalent to the VCG mechanism outcome.*

Proof. Since ex-post allocations are the same in both mechanisms, payments must be compared to show ex-post equivalence. Agents $n+1$ to k pay 0 in both cases.

For agent i with $n \geq i \geq 1$ consider the VCG mechanism payment. Agent i pays the allocational externality that they cause. If agent i were absent from the auction the total allocational utility would be:

$$v_{11} + v_{22} + \ldots + v_{(i-1)(i-1)} + v_{(i+1)i} + \ldots + v_{(n+1)n}.$$

With agent i present in the auction the allocational utility for the rest of the agents is:

$$v_{11} + \ldots + v_{(i-1)(i-1)} + v_{(i+1)(i+1)} + \ldots + v_{nn}.$$

Therefore, agent i's payment in the VCG mechanism is:

$$VCG_i = (v_{(i+1)i} - v_{(i+1)(i+1)}) + \ldots + (v_{n(n-1)} - v_{nn}) + v_{(n+1)n}.$$

Now, look at agent i's payment in the equilibrium defined. Let $t_k, \ldots, t_2$ be the drop out times of agents $k, \ldots, 2$.

The indifference condition of agent $n+1$ is $v_{(n+1)n} - f^n(t_k, \ldots, t_{n+1}) = 0$. This implies:

$$f^n(t_k, \ldots, t_{n+1}) = v_{(n+1)n}. \tag{2}$$

Agent n's equilibrium payment is $f^n(t_k, \ldots, t_{n+1})$ and equals to VCG_n.

Similarly, the indifference condition of agent n is $v_{nn} - f^n(t_k, \ldots, t_{n+1}) = v_{n(n-1)} - f^{n-1}(t_k, \ldots, t_n)$. Use Equality 2 to substitute $v_{(n+1)n}$ for $f^n(t_k, \ldots, t_{n+1})$ and rewrite to get:

$$f^{n-1}(t_k, \ldots, t_n) = (v_{n(n-1)} - v_{nn}) + v_{(n+1)n}$$

This is agent $n-1$'s payment in the equilibrium and equal to VCG_{n-1}. It is easy to see by induction that

$$f^i(t_k, \ldots, t_{i+1})) = (v_{(i+1)i} - v_{(i+1)(i+1)}) + \ldots + (v_{n(n-1)} - v_{nn}) + v_{(n+1)n}.$$

which is agent i's payment and is equal to VCG_i for $n \geq i \geq 1$. □

5. Conclusion

This paper introduced position auctions with general pricing mechanisms and showed that the ex-post payoffs are equal to those of the VCG mechanism when bidders have unit demand. The general pricing rules include that of the GSP auctions used by the major search engines.

The GSP auctions were first analyzed by Edelman et al. (2007) where agents have one dimensional types. In our analysis, we allow agents to have multidimensional types satisfying the single-crossing property. Similarly, we consider general auctions which allow the price for an advertiser to depend on any of the bids of advertisers who win lower positions.

However, in our analysis, only one equilibrium was constructed and analyzed. As in Edelman et al. (2007), it might be the case that this equilibrium is the unique perfect Bayesian equilibrium under some technical conditions. This remains an open question for future research.

References

Abrams, Z., Mendelevitch, O., Tomlin, J., 2007. Optimal delivery of sponsored search advertisements subject to budget constraints. In: EC '07: Proceedings of the 8th ACM conference on Electronic commerce. ACM, New York, NY, USA, pp. 272–278.

Aggarwal, G., Goel, A., Motwani, R., 2006. Truthful auctions for pricing search keywords. In: EC '06: Proceedings of the 7th ACM conference on Electronic commerce. ACM, New York, NY, USA, pp. 1–7.

Edelman, B., Ostrovsky, M., Schwarz, M., March 2007. Internet advertising and the generalized second-price auction: Selling billions of dollars worth of keywords. American Economic Review 97 (1), 242–259.

Edelman, B., Schwarz, M., 2007. Optimal auction design in a multi-unit environment: The case of sponsored search auctions, unpublished Manuscript.

Feldman, J., Muthukrishnan, S., Pal, M., Stein, C., 2007. Budget optimization in search-based advertising auctions. In: EC '07: Proceedings of the 8th ACM conference on Electronic commerce. ACM, New York, NY, USA, pp. 40–49.

Myerson, R. B., February 1981. Optimal auction design. Mathematics of Operations Research 6 (1), 58–73.

Riley, J. G., Samuelson, W. F., June 1981. Optimal auctions. American Economic Review 71 (3), 381–392.

Varian, H. R., December 2007. Position auctions. International Journal of Industrial Organization 25 (6), 1163–1178.