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On the mechanical energy and effective stress in saturated and unsaturated porous continua ☆

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Abstract

This paper has two objectives: (a) to formulate a mechanical theory of porous continua within the framework of strong discontinuity concept commonly employed for the analysis of strain localization in inelastic solids; and (b) to introduce an effective stress tensor for three-phase (solid–water–air) partially saturated porous continua emerging from principles of thermodynamics. To achieve the first objective, strong forms of the boundary-value problems are compared between two formulations, the first in which the velocity jump at the solid–fluid interface is treated as a strong discontinuity problem, and the second in which the strong discontinuity is smeared in the representative volume element. As for the second objective, an effective Cauchy stress tensor of the form $\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + (1 - K/K_s)\bar{p}\mathbf{1}$ emerges from the formulation, where $\boldsymbol{\sigma}$ is the total stress tensor, K and K_s are the bulk moduli of the solid matrix and solid phase, respectively, and $\bar{p}$ is the mean pore water and pore air pressures weighted according to the degree of saturation. We show from the first and second laws of thermodynamics that this effective stress tensor is power-conjugate to the solid rate of deformation tensor, and that it includes the mechanical power required to compress the solid phase. © 2005 Elsevier Ltd. All rights reserved.

1. Introduction

Porous media consist of a solid phase, usually referred to as a matrix or skeleton, as well as pores filled up with one or more types of fluids. Examples of porous materials are soils, rocks, concrete, the human bone, etc. Mechanical description of these materials is of utmost importance in many fields of engineering and science. Extensive studies have thus been conducted over the years to advance various mechanical theories of porous media for application to problems in the fields of biomechanics, geotechnical engineering,

geophysics, physical chemistry, agricultural engineering, and materials science, among others (Almeida and Spilker, 1998; Atkin and Craine, 1976; Bachmat and Bear, 1986a,b; Bear, 1972; Biot, 1955, 1956, 1967; Booker, 1974; Borja, 2004, 2005; Bowen, 1976; Carter et al., 1979; Chao et al., 1998; Coussy, 1995; Coussy et al., 1998; Cowin, 1999; de Boer, 2000; Ehlers et al., 2004; Hassanizadeh and Gray, 1979a,b, 1980, 1990; Hutter et al., 1999; Huyghe et al., 1991; Knothe-Tate, 2003; Kojic et al., 2001; Levenston et al., 1998; Lewis and Schrefler, 1987; Li et al., 2004; Moreland, 1972; Oettl et al., 2003, 2004; Oettl, 2003; Piekarski and Munro, 1977; Prevost, 1980; Slattery, 1967, 1981; Spilker et al., 1992; Suh et al., 1991; Taber, 1995; Wei and Muraleetharan, 2002a,b; Whitaker, 1986; Zhang, 2001; Zhang and Schrefler, 2004). This paper formulates a mechanical theory of porous continua within the framework of strong discontinuity concept commonly employed for the analysis of strain localization in inelastic solids. The paper also uses continuum principles of thermodynamics to develop an expression for an effective stress tensor for three-phase (solid–water–air) partially saturated porous continua. This effective stress may be used for constitutive description of the solid matrix in particular, and for mechanical modeling of the total solid–fluid mixture behavior in general. Specifically, we are interested in developing an expression for the effective stress in the regime of compressible solids, since a vast number of applications of porous media concepts deal with materials with compressible solid matrix and solid constituent.

The approach pursued in this paper is novel in that we view the velocity field in a porous medium as a dense distribution of strong discontinuity. The term 'strong discontinuity' is commonly used to describe a jump in the displacement or velocity field, represented mathematically by a Heaviside function. Such representation is commonly used for describing strain localization of inelastic solids characterized by a shear band thickness of measure zero (Borja, 2000, 2002; Borja and Regueiro, 2001), a convenient mathematical approximation of 'weak discontinuity' in which the shear band thickness is small but finite. In porous media the discontinuity occurs across the solid–fluid interface, and for low viscosity flow a strong discontinuity representation may be considered acceptable. For velocity fields exhibiting strong discontinuity we show that the governing partial differential equations and boundary conditions characterizing flow through porous media and strain localization problems are essentially the same. Apart from the constitutive descriptions, the only difference between the two problems lies in the facts that in porous continua the geometry of the surface of discontinuity is more complex and that the distribution of strong discontinuity is much more dense.

Because the discontinuity in the velocity field is extremely dense, we perform as a matter of convenience an averaging procedure that effectively smears the surfaces of discontinuity over the volume. The one-field velocity kinematics characterizing the original problem then becomes a dual field (for a two-phase solid–fluid mixture), and in the end we recover the conservation laws of mixture theory. In mixture theory the interacting phases are superposed on the same volume, and conservation laws are written in terms of the partial stress tensors obtained by multiplying the respective intrinsic stress tensors by the corresponding volume fractions. The passage from a single-field velocity formulation (strong discontinuity) to a dual field (mixture theory) is demonstrated and made possible through the notion of a representative volume element. For completeness in the presentation, we develop and compare expressions for the mechanical power emerging from the two formulations.

The first part of this paper can be differentiated from the approaches of several authors that have worked in porous media concepts in the following sense. Whereas previous works (see e.g. Atkin and Craine, 1976; Bachmat and Bear, 1986a,b; Bear, 1972; Biot, 1955, 1956, 1967; Booker, 1974; Borja, 2004, 2005; Bowen, 1976; Carter et al., 1979; Coussy, 1995; Coussy et al., 1998; de Boer, 2000; Ehlers et al., 2004; Hassanizadeh and Gray, 1979a,b, 1980, 1990; Lewis and Schrefler, 1987; Li et al., 2004; Moreland, 1972; Oettl et al., 2003, 2004; Oettl, 2003; Piekarski and Munro, 1977; Prevost, 1980) have imposed balance laws for macro-scale quantities *after* performing some form of averaging procedures, the present work imposes the balance laws right in the intrinsic phases and even *before* performing the averaging step. The averaging is then performed only as a passage to mixture theory. Therefore, we herein view the solid phase itself as a continuum within

which well-known balance laws may be written. If the solid phase is a grain of soil or a bone fiber, then continuum balance laws are written within the grain or fiber. A similar point of view is taken for the fluids as well as for other constituent phases in the mixture. In effect, we view the porous medium as a continuum (intrinsic phase) within a continuum (matrix). This approach permits a single-valued mathematical description of the velocity field, which becomes multi-valued only after performing the smearing process. Of course, the continuum-within-continuum assumption is only a point of view taken and does not apply to much finer scales, such as the nano-scale.

The second part of this paper covers the subject of partially saturated porous continua. These media consist of three phases: solid, liquid, and gas. A fundamental question that always arises is how should the effective stress be defined for such continua. As a background, the effective stress is defined as a certain portion of total stress that may be applied to the solid matrix without pore pressure to produce a given solid matrix deformation. The effective stress is always a focal point in the mechanics of porous continua since constitutive theories for the solid matrix deformation revolve around this stress. In this paper we utilize the previously derived expressions for the mechanical power to extract a stress measure that is power-conjugate to the rate of deformation of the solid matrix, which we henceforth define as the effective stress. Our formulation is general in the sense that it accounts for the compressibility of all the constituent phases (solid, liquid, and gas). The resulting expression for the effective stress is equally general and reduces to many familiar forms that have been advanced in the literature under special conditions.

2. Power density for continua with strong discontinuity

The microscopic velocity field in a porous continuum is highly discontinuous due to the relative motion occurring between adjacent interacting phases. The jump in the velocity field, or strong discontinuity, occurs at the interface between two constituent phases moving with a relative velocity. In this section we describe an averaging procedure for quantifying the mechanical power density in a multiphase continuum. For clarity in presentation we begin with a single-phase continuum exhibiting strong discontinuity across a surface and then consider the case of multiphase continua in subsequent sections. Then we specialize to a mixture consisting of solid, water and air, and present alternative forms of the power density function for this mixture.

2.1. Motivation: mechanical power in the presence of strong discontinuity

Consider a solid body at initial placement $B \subset \mathscr{B}$ and defined geometrically by a surface ∂B. We can view B as any arbitrary subregion of the entire structural domain $\mathscr{B}$. Assume that B is separated into two parts by a surface $S \subset \mathscr{S}$, and denote each part by B^+ and B^- such that $B = B^+ \cup B^-$ and $\partial B = \partial B^+ \cup \partial B^-$, see Fig. 1. Consider a shear band-type strong discontinuity on the surface S in which the velocity field $\boldsymbol{v}$ is given by the jump function

$$\boldsymbol{v} = \bar{\boldsymbol{v}} + [\![\boldsymbol{v}]\!] H_S(X), \quad H_S(X) = \begin{cases} 1 & \text{if } X \in B^+; \\ 0 & \text{if } X \in B^-, \end{cases} \tag{2.1}$$

where $\bar{\boldsymbol{v}}$ is the continuous part of $\boldsymbol{v}$ and $[\![\mathbf{v}]\!]$ is the velocity jump across S. We assume the usual balance of linear momentum along with boundary conditions. For each of the two parts, we write

$$\int_{B^-} \rho_0(\boldsymbol{G} - \boldsymbol{a})\mathrm{d}V + \int_{\partial B^-} \boldsymbol{t}_0\mathrm{d}A + \int_S \boldsymbol{P}^- \cdot \boldsymbol{N}\mathrm{d}A = \boldsymbol{0};$$
$$\int_{B^+} \rho_0(\boldsymbol{G} - \boldsymbol{a})\mathrm{d}V + \int_{\partial B^+} \boldsymbol{t}_0\mathrm{d}A - \int_S \boldsymbol{P}^+ \cdot \boldsymbol{N}\mathrm{d}A = \boldsymbol{0}, \tag{2.2}$$
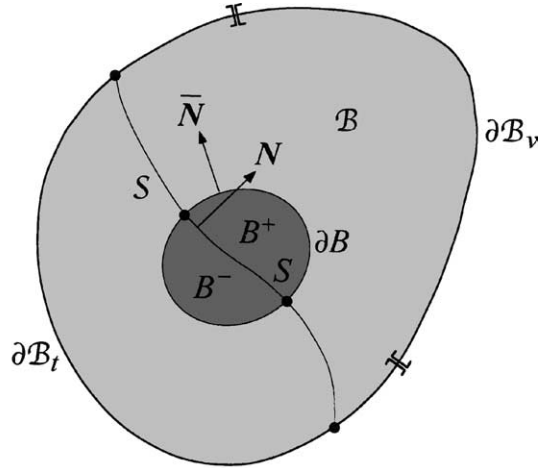
Fig. 1. Basis of volume integral for a shear band-type strong discontinuity.

where $\boldsymbol{P}$ is the non-symmetric first Piola–Kirchhoff stress tensor, $\boldsymbol{G}$ is the gravity acceleration vector, $\boldsymbol{a}$ is the material acceleration vector, ρ_0 is the reference mass density, $\boldsymbol{N}$ is the outward unit normal vector to S interpreted to belong in B^-, and $\boldsymbol{t}_0$ is the nominal traction vector on $\partial B = \partial B^+ \cup \partial B^-$. Adding the above two equations gives

$$\int_B \rho_0(\boldsymbol{G} - \boldsymbol{a})\mathrm{d}V + \int_{\partial B} \boldsymbol{t}_0\,\mathrm{d}A + \int_S [\![\boldsymbol{P}\cdot\boldsymbol{N}]\!]\mathrm{d}A = \boldsymbol{0}, \tag{2.3}$$

where $[\![\boldsymbol{P}\cdot\boldsymbol{N}]\!] = (\boldsymbol{P}^- - \boldsymbol{P}^+)\cdot\boldsymbol{N}$ denotes a jump in the traction vector across S. For equilibrium, this jump must be equal to zero point-wise on S. Using Gauss theorem we can convert the ∂B-integral into a volume integral and localize the above equation to obtain the following strong form of the boundary-value problem.

Given $\rho_0 : \mathcal{B} \to R^1$, $\boldsymbol{v}_0 : \partial \mathcal{B}_v \to R^{n_{\mathrm{sd}}}$, and $\boldsymbol{t}_0 : \partial \mathcal{B}_t \to R^{n_{\mathrm{sd}}}$, find the velocity field $\boldsymbol{v} : \mathcal{B} \to R^{n_{\mathrm{sd}}}$ such that

$$\begin{aligned}
&\rho_0(\boldsymbol{G} - \boldsymbol{a}) + \mathrm{DIV}\,\boldsymbol{P} = \boldsymbol{0} \quad \text{in } \mathcal{B} \\
&\boldsymbol{P}\cdot\overline{\boldsymbol{N}} = \boldsymbol{t}_0 \quad \text{on } \partial\mathcal{B}_t \\
&\boldsymbol{v} = \boldsymbol{v}_0 \quad \text{on } \partial\mathcal{B}_u \\
&[\![\boldsymbol{P}\cdot\boldsymbol{N}]\!] = \boldsymbol{0} \quad \text{on } \mathscr{S},
\end{aligned} \tag{2.4}$$

where $\overline{\boldsymbol{N}}$ is the outward unit normal vector to $\partial\mathcal{B}$, n_{sd} is the number of spatial dimensions, DIV is the material divergence operator, and where $\overline{\partial\mathcal{B}_v \cup \partial\mathcal{B}_t} = \partial\mathcal{B}$ and $\partial\mathcal{B}_v \cap \partial\mathcal{B}_t = \emptyset$.

Under the setting described above, the mechanical power expended by the body force vector $\rho_0\boldsymbol{G}$ and nominal traction vector $\boldsymbol{t}_0$ in the same arbitrary region $B \subset \mathcal{B}$ *taken as a whole* is given by

$$P(B) = \int_B \rho_0\boldsymbol{G}\cdot\boldsymbol{v}\mathrm{d}V + \int_{\partial B} \boldsymbol{t}_0\cdot\boldsymbol{v}\mathrm{d}A. \tag{2.5}$$

On the other hand, the total mechanical power expended by these same forces *taking B^+ and B^- individually* is given by

$$\overline{P}(B) = P(B^+) + P(B^-) = \int_B \rho_0\boldsymbol{G}\cdot\boldsymbol{v}\mathrm{d}V + \int_{\partial B} \boldsymbol{t}_0\cdot\boldsymbol{v}\mathrm{d}A - \int_S \boldsymbol{t}_S\cdot[\![\boldsymbol{v}]\!]\mathrm{d}A, \tag{2.6}$$

where $\boldsymbol{t}_S = \boldsymbol{P}^- \cdot \boldsymbol{N} = \boldsymbol{P}^+ \cdot \boldsymbol{N}$ is the traction vector generated on the surface of discontinuity $S \subset \mathscr{S}$. Note that in the presence of strong discontinuity the mechanical power in a body taken as a whole is not the same as that obtained from taking the sum of its individual parts.

Whether B is taken as a whole or as a two-part system, its total kinetic energy is given by

$$K = \frac{1}{2} \int_B \rho_0 \boldsymbol{v} \cdot \boldsymbol{v} \, \mathrm{d}V. \tag{2.7}$$

The rate of change of internal energy $\dot{E}$ is then the difference between the mechanical power and the rate of change of kinetic energy. If B is taken as a whole, we get (see Appendix A)

$$\dot{E} = P(B) - \dot{K} = \int_B \boldsymbol{P} : \dot{\boldsymbol{F}} \, \mathrm{d}V + \int_S \boldsymbol{t}_S \cdot [\![\boldsymbol{v}]\!] \, \mathrm{d}A, \tag{2.8}$$

where $\dot{\boldsymbol{F}} = \partial \boldsymbol{v} / \partial \boldsymbol{X} = \boldsymbol{l} \cdot \boldsymbol{F}$, $\boldsymbol{l}$ is the velocity gradient, and $\boldsymbol{F}$ is the deformation gradient. In this case, $\dot{E}$ represents the mechanical power of the stresses in deforming the entire body plus the power produced by the traction vector in realizing the velocity jump on S. If B is taken as an independent two-part system, then we get

$$\dot{\overline{E}} = \overline{P}(B) - \dot{K} = \int_B \boldsymbol{P} : \dot{\boldsymbol{F}} \, \mathrm{d}V \tag{2.9}$$

and $\dot{\overline{E}}$ simply represents the mechanical power expended by the stresses in deforming B^+ and B^-. In taking the body as a whole, one generates the traction term in the internal power $\dot{E}$ that can be used, for example, to formulate a decohesion constitutive law on the surface of discontinuity using the standard Coleman relation (see Ortiz and Pandolfi, 1999).

2.2. Mechanical power in two-phase solid–fluid continua

In two-phase solid–fluid continua the strong discontinuity occurs on a more complex surface distributed within a volume. Fig. 2 shows two superposed but non-overlapping solid and fluid matrices following their own respective motions. The strong discontinuity occurs across the interface of the solid and fluid matrices
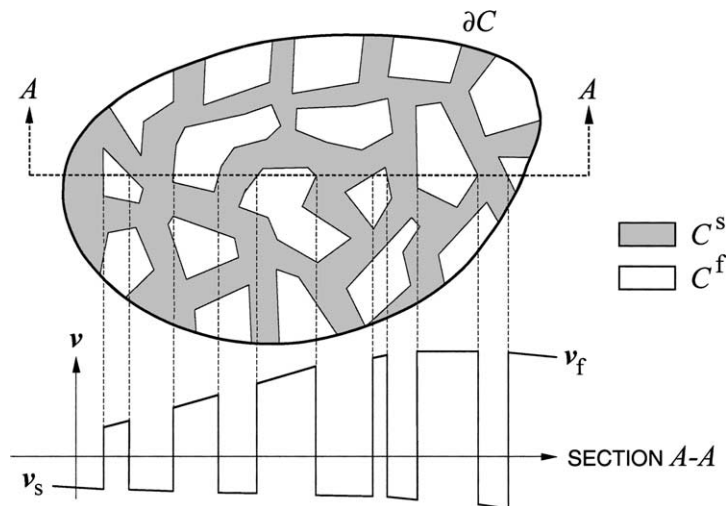

Fig. 2. Solid–fluid mixture and single-valued velocity field exhibiting multiple strong discontinuities.
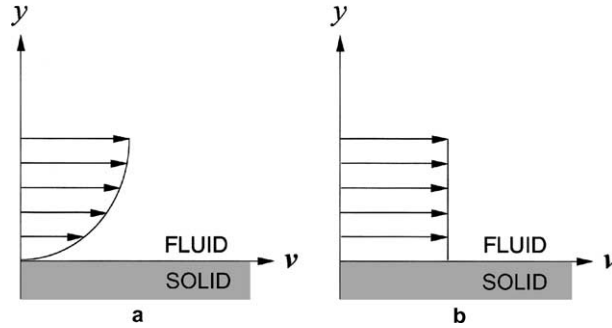
Fig. 3. Relative velocity near solid–fluid interface: (a) weak discontinuity (jump in velocity gradient); (b) strong discontinuity (jump in velocity).

in the presence of a relative motion. Here we take both the solid and fluid matrices to be continuous so that no element of S is disjoint; otherwise, we have either an isolated solid or an isolated void. Note in Fig. 2 that the fluid elements appear disjoint but they could be connected at the throat of the void into the page. Also, when dealing with viscous fluids the strong discontinuity could in fact resemble a weak discontinuity where the jump occurs in the velocity gradient (see Fig. 3). At the interface between the solid and fluid, shown in Fig. 3(a), shear stresses develop as the solid attempts to slow down the fluid flowing over it. This shear stress gives rise to a skin friction drag proportional to the fluid viscosity and the gradient of the velocity profile across the boundary layer. At low viscosity this velocity gradient could be very large, and the profile resembles a jump in the velocity itself, shown in Fig. 3(b).

Since S is a distributed surface within the volume, the solid and fluid regions are also distributed within this same volume. It is generally impractical to describe the fluid motion in Lagrangian form unless the pullback is made with respect to the solid matrix reference configuration, so here we describe the composite mixture in Eulerian form. Let $C^s \subset \mathscr{C}^s$ and $C^f \subset \mathscr{C}^f$ denote any arbitrary non-overlapping regions defined by the solid and fluid phases in the current configuration, respectively, and let $S \subset \mathscr{S}$ denote the distributed surface of strong discontinuity separating C^s from C^f, also given in the current configuration. The volumes and surface boundaries of the mixture are denoted as follows: $C = C^s \cup C^f$ and $\mathscr{C} = \mathscr{C}^s \cup \mathscr{C}^f$, where $C \subset \mathscr{C}$; $\partial C = \partial C^s \cup \partial C^f$ and $\partial \mathscr{C} = \partial \mathscr{C}^s \cup \partial \mathscr{C}^f$, where $\partial C \subset \partial \mathscr{C}$, see Fig. 2.

The intrinsic velocity field is given by

$$v = \begin{cases} v_{(s)} & \text{if } x \in C^s; \\ v_{(f)} & \text{if } x \in C^f, \end{cases} \quad v_{(f)} = v_{(s)} + [\![v]\!] \quad \text{on } S, \tag{2.10}$$

where x is the spatial position vector. Note that v is a single-valued velocity field since $C^s \cap C^f = \emptyset$ by definition of non-overlapping regions. Balance of linear momentum along with boundary conditions in any arbitrary volume C can then be written for each phase as

$$\int_{C^s} \rho_s [g - a_{(s)}] dV^s + \int_{\partial C^s} t_s \, dA^s + \int_S \sigma_s \cdot n \, dA = 0;$$
$$\int_{C^f} \rho_f [g - a_{(f)}] dV^f + \int_{\partial C^f} t_f \, dA^f - \int_S \sigma_f \cdot n \, dA = 0, \tag{2.11}$$

where $g = G$, n is the outward unit normal vector to S interpreted to belong in the solid region C^s, ρ_s and ρ_f are intrinsic mass densities of the solid and fluid, respectively (mass of solid/fluid over the volume of solid/fluid), t_s and t_f are the intrinsic Cauchy traction vectors, and σ_s and σ_f are the corresponding intrinsic Cauchy stress tensors (force in solid/fluid per unit area of solid/fluid).

Taking the velocities $v_{(s)}$ and $v_{(f)}$ as Eulerian variables, the material accelerations are given by

$$a_{(s)} = \frac{\partial v_{(s)}}{\partial t} + \operatorname{grad} v_{(s)} \cdot v_{(s)}, \quad a_{(f)} = \frac{\partial v_{(f)}}{\partial t} + \operatorname{grad} v_{(f)} \cdot v_{(f)}. \tag{2.12}$$

Adding the equations in (2.11) gives

$$\int_{C^s} \rho_s [g - a_{(s)}] dV^s + \int_{C^f} \rho_f [g - a_{(f)}] dV^f + \int_{\partial C^s} t_s \, dA^s + \int_{\partial C^f} t_f \, dA^f + \int_S [\![\sigma \cdot n]\!] dA = 0, \tag{2.13}$$

where $[\![\sigma \cdot n]\!] = (\sigma_s - \sigma_f) \cdot n$ is the jump in the traction vector across S. Again, this jump must be point-wise equal to zero for equilibrium. Alternatively, we can write (2.13) in a more compact form

$$\int_C \rho(g - a) dV + \int_{\partial C} t \, dA + \int_S [\![\sigma \cdot n]\!] dA = 0, \tag{2.14}$$

where

$$\rho = \begin{cases} \rho_s & \text{if } x \in C^s; \\ \rho_f & \text{if } x \in C^f, \end{cases} \quad a = \begin{cases} a_{(s)} & \text{if } x \in C^s; \\ a_{(f)} & \text{if } x \in C^f, \end{cases} \quad t = \begin{cases} t_s & \text{if } x \in C^s; \\ t_f & \text{if } x \in C^f. \end{cases} \tag{2.15}$$

Localizing for any arbitrary volume C, the strong form of the boundary-value problem can be written as follows.

Given $\rho : \mathscr{C} \to R^1$, $v_{(0)} : \partial \mathscr{C}_v \to R^{n_{sd}}$, and $t_0 : \partial \mathscr{C}_t \to R^{n_{sd}}$, find the velocity field $v : \mathscr{C} \to R^{n_{sd}}$ such that

$$\begin{aligned} \rho(g - a) + \operatorname{div} \sigma &= 0 \quad \text{in } \mathscr{C} \\ \sigma \cdot \bar{n} &= t \quad \text{on } \partial \mathscr{C}_t \\ v &= v_{(0)} \quad \text{on } \partial \mathscr{C}_v \\ [\![\sigma \cdot n]\!] &= 0 \quad \text{on } \mathscr{S}, \end{aligned} \tag{2.16}$$

where $\bar{n}$ is the outward unit normal vector to $\partial \mathscr{C}$.

Remark 1. The above problem determines a single-valued velocity field v since for every position vector $x \notin \mathscr{S}$ there exists a corresponding unique velocity v. To uniquely define the velocity field for a point on $\mathscr{S}$, one needs to prescribe whether the point in question is interpreted to belong in the solid domain $\mathscr{C}^s$ or in the fluid domain $\mathscr{C}^f$, see (2.10). Note that the velocity field is highly irregular and contains a large number of "jumps" (Fig. 2). Accordingly, the Dirichlet and Neumann boundaries must be so defined that the two phases are non-overlapping in the sense $\partial \mathscr{C}_v^s \cap \partial \mathscr{C}_v^f = \partial \mathscr{C}_t^s \cap \partial \mathscr{C}_t^f = \partial \mathscr{C}_v^s \cap \partial \mathscr{C}_t^s = \partial \mathscr{C}_v^f \cap \partial \mathscr{C}_t^f = \emptyset$. Finally, it is possible that the solid itself could experience a strong discontinuity (for example, if the solid matrix is a chain of particles that could slip at their points of contact), but in the above strong form the surface $\mathscr{S}$ is understood as simply the boundary separating the solid and fluid phases of the mixture.

Under the setting described above, the mechanical power expended by the body force and Cauchy traction vectors in the arbitrary region C *taken as a whole* is given by

$$P(C) = \int_C \rho g \cdot v \, dV + \int_{\partial C} t \cdot v \, dA. \tag{2.17}$$

On the other hand, the total mechanical power expended by these same forces *taking C^s and C^f individually* is given by

$$\overline{P}(C) = P(C^s) + P(C^f) = \int_C \rho g \cdot v \, dV + \int_{\partial C} t \cdot v \, dA - \int_S t_S \cdot [\![v]\!] dA, \tag{2.18}$$

where $t_S = \sigma_s \cdot n = \sigma_f \cdot n$ is the Cauchy traction vector on S.

The total kinetic energy of the solid–fluid system is given by

$$K = \frac{1}{2} \int_C \rho \boldsymbol{v} \cdot \boldsymbol{v} \, \mathrm{d}V. \tag{2.19}$$

The rate of change of internal energy taking C as a whole is

$$\dot{E} = P(C) - \dot{K} = \int_C \boldsymbol{\sigma} : \boldsymbol{d} \, \mathrm{d}V + \int_S \boldsymbol{t}_S \cdot [\![\boldsymbol{v}]\!] \mathrm{d}A, \tag{2.20}$$

where

$$\boldsymbol{d} = \mathrm{sym}(\boldsymbol{l}), \quad \boldsymbol{l} = \mathrm{grad}\,\boldsymbol{v}. \tag{2.21}$$

Thus, $\dot{E}$ represents the mechanical power of the stresses in deforming the solid and fluid phases, plus the power produced by the traction vector in realizing the velocity jump on the solid–fluid interface. If C is taken as an independent two-phase system, then we get

$$\bar{\dot{E}} = \bar{P}(C) - \dot{K} = \int_C \boldsymbol{\sigma} : \boldsymbol{d} \, \mathrm{d}V, \tag{2.22}$$

and $\bar{\dot{E}}$ is simply the power performed by the solid and fluid stresses in deforming their respective domains. Similar to the result derived in Section 2.1, taking the body as a whole generates the traction term $\boldsymbol{t}_S$ that can be used to formulate a thermodynamically consistent interface constitutive law describing the interaction of the solid and fluid phases.

The similarity in form of all the results developed in this section with those presented in Section 2.1 for a shear band-type strong discontinuity is noteworthy. The governing partial differential equations for the two problems are the same if one only notes that C^s takes the role of B^- and C^f is equivalent to B^+. However, there is a greater challenge with the porous media problem definition since the geometry of the surface of strong discontinuity is much more complex, and its distribution within the body is immensely dense.

2.3. Dense distribution of strong discontinuity: passage to mixture theory

In porous materials such as soil, rock, ceramics and bones, the distribution of strong discontinuity is very dense, as shown in Fig. 4. Thus, it would be impractical to delineate the geometry of the surface S. A more practical approach of quantifying a dense distribution of strong discontinuity is to replace in (2.11) the resultant traction vector on S with the following volume integrals:

$$\int_{S \in C} \boldsymbol{\sigma}_s \cdot \boldsymbol{n} \, \mathrm{d}A = \int_C \boldsymbol{h}^s \, \mathrm{d}V;$$
$$-\int_{S \in C} \boldsymbol{\sigma}_f \cdot \boldsymbol{n} \, \mathrm{d}A = \int_C \boldsymbol{h}^f \, \mathrm{d}V, \tag{2.23}$$

where S/C is the specific surface area, defined as the average total surface area of strong discontinuity contained per unit volume of the solid–fluid mixture. Note that the above conversion of the area integral into a volume integral results from the smearing of S over C and does not emanate from the use of Gauss theorem since S is not a closed surface boundary of C but rather is a dense distribution that lies in the interior of C. However, the problem with the above equations is that they do not alone uniquely define the equivalent body force density vectors $\boldsymbol{h}^s$ and $\boldsymbol{h}^f$ since one can construct an infinite number of such vectors satisfying (2.23) for any given region of integration C.

Unique definitions of body force density vectors $\boldsymbol{h}^s$ and $\boldsymbol{h}^f$ can be obtained if the region of integration is taken as the representative volume element (RVE), defined in principle as an element that is large enough that statistical variations of $\boldsymbol{h}^s$ and $\boldsymbol{h}^f$ are insignificant within this element but small enough that
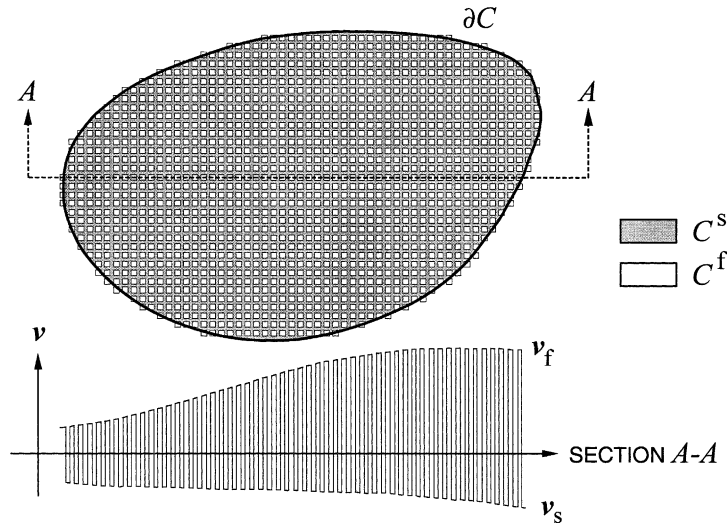
Fig. 4. Dense distribution of strong discontinuities: single-valued velocity field becomes a dual field as the density of velocity jumps approaches infinity.

macroscopic variations of these density functions are properly represented in the neighboring RVE. The subject of averaging procedure itself is outside the scope of this paper (the readers may want to consult de Boer, 2000; Hassanizadeh and Gray, 1979a) for an exhaustive treatment of this topic); our objective here is to simply use this technique to recover the equations of mixture theory. Let $\overline{C} \subset C$ be any arbitrary RVE and evaluate (2.23) over this volume. By definition of an RVE, $\boldsymbol{h}^{\mathrm{s}}$ and $\boldsymbol{h}^{\mathrm{f}}$ are statistically constant within this region, and so

$$\boldsymbol{h}^{\mathrm{s}} = \frac{1}{\overline{C}} \int_{S \in \overline{C}} \boldsymbol{\sigma}_{\mathrm{s}} \cdot \boldsymbol{n} \, \mathrm{d}A, \quad \boldsymbol{h}^{\mathrm{f}} = -\frac{1}{\overline{C}} \int_{S \in \overline{C}} \boldsymbol{\sigma}_{\mathrm{f}} \cdot \boldsymbol{n} \, \mathrm{d}A. \tag{2.24}$$

Adding these last two equations and imposing the traction jump condition gives

$$\boldsymbol{h}^{\mathrm{s}} + \boldsymbol{h}^{\mathrm{f}} = \frac{1}{\overline{C}} \int_{S \in \overline{C}} [\![\boldsymbol{\sigma} \cdot \boldsymbol{n}]\!] \, \mathrm{d}A = \boldsymbol{0}, \tag{2.25}$$

since $[\![\boldsymbol{\sigma} \cdot \boldsymbol{n}]\!] = \boldsymbol{0}$ point-wise on $\mathscr{S}$. Thus, the density force vectors $\boldsymbol{h}^{\alpha}$ must be self-equilibrating.

Next, consider the following equivalent integral representations:

$$\int_{C^{\alpha}} \xi_{\alpha} \, \mathrm{d}V^{\alpha} = \int_{C} \xi^{\alpha} \, \mathrm{d}V, \quad \xi^{\alpha} = \xi_{\alpha} \phi^{\alpha},$$
$$\int_{\partial C^{\alpha}} \eta_{\alpha} \, \mathrm{d}A^{\alpha} = \int_{\partial C} \eta^{\alpha} \mathrm{d}A, \quad \eta^{\alpha} = \eta_{\alpha} \phi^{\alpha}, \tag{2.26}$$

where

$$\phi^{\alpha} = \frac{1}{\overline{C}} \int_{\overline{C}} \chi^{\alpha} \mathrm{d}V, \quad \chi^{\alpha} = \begin{cases} 1 & \text{if } \boldsymbol{x} \in \overline{C}^{\alpha}, \\ 0 & \text{otherwise} \end{cases} \tag{2.27}$$

for $\alpha = \mathrm{s}, \mathrm{f}$. Here, χ^{α} is a Boolean indicator function, ϕ^{α} is the volume fraction for constituent α, and $(\xi_{\alpha}, \eta_{\alpha})$ are any scalar, vector, or tensor-valued intrinsic density functions associated with constituent α. The above equations take the area and volume fractions to be statistically the same, which is true for a dense distri-

bution of strong discontinuity (see proof in Appendix B). Now, by definition of an RVE, ξ^α is statistically constant in $\overline{C}$ and η^α is statistically constant on $\partial\overline{C}$. Evaluating (2.26) in RVE gives

$$\xi^\alpha = \frac{1}{\overline{C}} \int_{\overline{C}^\alpha} \xi_\alpha \, dV^\alpha, \quad \eta^\alpha = \frac{1}{\partial\overline{C}} \int_{\partial\overline{C}^\alpha} \eta_\alpha \, dA^\alpha, \tag{2.28}$$

thus providing for unique definitions of such density functions. As a general notation, phase designations in superscript form pertain to the 'smeared' partial density functions, and in subscript form to intrinsic or true density functions.

Specifically, for two-phase solid–fluid continua the relevant volume fractions satisfy the closure condition

$$\phi^s + \phi^f = 1. \tag{2.29}$$

Defining the partial mass densities as

$$\rho^s = \phi^s \rho_s, \quad \rho^f = \phi^f \rho_f, \quad \rho = \rho^s + \rho^f \tag{2.30}$$

and the partial traction vectors as

$$\boldsymbol{t}^s = \phi^s \boldsymbol{t}_s, \quad \boldsymbol{t}^f = \phi^f \boldsymbol{t}_f, \quad \boldsymbol{t} = \boldsymbol{t}^s + \boldsymbol{t}^f \tag{2.31}$$

the linear momentum balance (2.11) along with the integral representations (2.26) yield the pair of equations

$$\int_C \rho^s(\boldsymbol{g} - \boldsymbol{a}_s) \, dV + \int_{\partial C} \boldsymbol{t}^s \, dA + \int_C \boldsymbol{h}^s \, dV = \boldsymbol{0};$$
$$\int_C \rho^f(\boldsymbol{g} - \boldsymbol{a}_f) \, dV + \int_{\partial C} \boldsymbol{t}^f \, dA + \int_C \boldsymbol{h}^f \, dV = \boldsymbol{0}, \tag{2.32}$$

where $\boldsymbol{a}_s$ and $\boldsymbol{a}_f$ are the material accelerations of the solid and fluid *matrices*, respectively. These two-field acceleration vectors must be distinguished from the single-valued intrinsic material accelerations $\boldsymbol{a}_{(s)}$ and $\boldsymbol{a}_{(f)}$ of the individual constituent phases defined in (2.12). Also, note that $\boldsymbol{a}_s$ and $\boldsymbol{a}_f$ are not density functions, but instead are kinematical variables describing the motion of the respective matrices or constituent skeletons, and so their indices are not raised.

Adding the two equations above gives

$$\int_C \rho(\boldsymbol{g} - \overline{\boldsymbol{a}}) \, dV + \int_{\partial C} \boldsymbol{t} \, dA + \int_C (\boldsymbol{h}^s + \boldsymbol{h}^f) \, dV = \boldsymbol{0}, \tag{2.33}$$

where

$$\overline{\boldsymbol{a}} = \frac{1}{\rho}(\rho^s \boldsymbol{a}_s + \rho^f \boldsymbol{a}_f) \tag{2.34}$$

is the mean material acceleration of the solid–fluid mixture. Though similar in form to (2.14), Eq. (2.33) has a different meaning since it provides a two-field description of the mixture. The above result is the foundation of mixture theory in which the medium is viewed as composed of two or more interacting materials superposed spatially in the same region (Atkin and Craine, 1976; Bowen, 1976; de Boer, 2000).

Since ∂C is a closed boundary defining the volume C, Gauss theorem may be used to convert the surface integral over ∂C into a volume integral over C. Localization of (2.32) then leads to the following strong form of the boundary-value problem.

Given $\rho^s : \mathscr{C} \to R^1$, $\rho^f : \mathscr{C} \to R^1$, $\boldsymbol{v}_{0s} : \partial \mathscr{C}_v^s \to R^{n_{sd}}$, $\boldsymbol{v}_{0f} : \partial \mathscr{C}_v^f \to R^{n_{sd}}$, $\boldsymbol{t} : \partial \mathscr{C}_t^s \to R^{n_{sd}}$, and $p_0^f : \partial \mathscr{C}_t^f \to R^1$, find the velocity fields $\boldsymbol{v}_s \times \boldsymbol{v}_f : \mathscr{C} \to R^{2n_{sd}}$ such that

$$\rho^s(\boldsymbol{g} - \boldsymbol{a}_s) + \operatorname{div} \boldsymbol{\sigma}^s + \boldsymbol{h}^s = \boldsymbol{0} \quad \text{in } \mathscr{C}$$

$$\rho^f(\boldsymbol{g} - \boldsymbol{a}_f) + \operatorname{div} \boldsymbol{\sigma}^f + \boldsymbol{h}^f = \boldsymbol{0} \quad \text{in } \mathscr{C}$$

$$\boldsymbol{\sigma} \cdot \bar{\boldsymbol{n}} = \boldsymbol{t} \quad \text{on } \partial \mathscr{C}_t^s$$

$$-\boldsymbol{\sigma}^f \cdot \bar{\boldsymbol{n}} = p_0^f \bar{\boldsymbol{n}} \quad \text{on } \partial \mathscr{C}_t^f \tag{2.35}$$

$$\boldsymbol{v}_s = \boldsymbol{v}_{0s} \quad \text{on } \partial \mathscr{C}_v^s$$

$$\boldsymbol{v}_f = \boldsymbol{v}_{0f} \quad \text{on } \partial \mathscr{C}_v^f$$

$$\boldsymbol{h}^s + \boldsymbol{h}^f = \boldsymbol{0} \quad \text{in } \mathscr{C},$$

where

$$\boldsymbol{\sigma}^s = \phi^s \boldsymbol{\sigma}_s, \quad \boldsymbol{\sigma}^f = \phi^f \boldsymbol{\sigma}_f, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^s + \boldsymbol{\sigma}^f \tag{2.36}$$

are the partial Cauchy stress tensors.

Remark 2. In contrast to the strong form (2.14), which yields a single-valued velocity field $\boldsymbol{v}$, the above form yields superposed dual velocity fields $\boldsymbol{v}_s \times \boldsymbol{v}_f : \mathscr{C} \to R^{2n_{sd}}$ of the solid and fluid skeletons for every macroscopic point $\boldsymbol{x}$. The density functions $\boldsymbol{h}^s \times \boldsymbol{h}^f : \mathscr{C} \to R^{2n_{sd}}$ and the partial stress tensors $\boldsymbol{\sigma}^s \times \boldsymbol{\sigma}^f : \mathscr{C} \to 2M$, where M is the space of symmetric second-order tensors, are also dual fields. Furthermore, under the present setting the boundaries are required to be non-overlapping in the sense $\partial \mathscr{C}_v^s \cap \partial \mathscr{C}_t^s = \partial \mathscr{C}_v^f \cap \partial \mathscr{C}_t^f = \emptyset$, but $\partial \mathscr{C}_v^s$ could overlap with $\partial \mathscr{C}_v^f$, and $\partial \mathscr{C}_t^s$ could overlap with $\partial \mathscr{C}_t^f$ (cf. Remark 1).

For the problem described above, the mechanical power expended by the body force and Cauchy traction vectors in any arbitrary volume $C \subset \mathscr{C}$ *taken as a whole* is now given by

$$P(C) = \int_C \boldsymbol{g} \cdot (\rho^s \boldsymbol{v}_s + \rho^f \boldsymbol{v}_f) \mathrm{d}V + \int_{\partial C} (\boldsymbol{t}^s \cdot \boldsymbol{v}_s + \boldsymbol{t}^f \cdot \boldsymbol{v}_f) \mathrm{d}A. \tag{2.37}$$

On the other hand, the total mechanical power expended by these same forces *taking C^s and C^f individually* is given by

$$\overline{P}(C) = P(C^s) + P(C^f) = \int_C \boldsymbol{g} \cdot (\rho^s \boldsymbol{v}_s + \rho^f \boldsymbol{v}_f) \mathrm{d}V + \int_{\partial C} (\boldsymbol{t}^s \cdot \boldsymbol{v}_s + \boldsymbol{t}^f \cdot \boldsymbol{v}_f) \mathrm{d}A - \int_C \boldsymbol{h}^s \cdot \tilde{\boldsymbol{v}}_f \mathrm{d}V, \tag{2.38}$$

where $\tilde{\boldsymbol{v}}_f = \boldsymbol{v}_f - \boldsymbol{v}_s$ is the relative velocity of fluid to solid. Note in the last equation the additional integral representing the mechanical power of the body force vector $\boldsymbol{h}^s$ produced by the relative velocity of fluid to solid. As before, the total kinetic energy of the solid–fluid system is given by

$$K = \frac{1}{2} \int_C (\rho^s \boldsymbol{v}_s \cdot \boldsymbol{v}_s + \rho^f \boldsymbol{v}_f \cdot \boldsymbol{v}_f) \mathrm{d}V. \tag{2.39}$$

Thus, the rate of change of internal energy taking C as a whole is

$$\dot{E} = P(C) - \dot{K} = \int_C (\boldsymbol{\sigma}^s : \boldsymbol{d}_s + \boldsymbol{\sigma}^f : \boldsymbol{d}_f) \mathrm{d}V - \int_C \boldsymbol{h}^s \cdot \tilde{\boldsymbol{v}}_f \mathrm{d}V, \tag{2.40}$$

whereas the rate of change is

$$\dot{\overline{E}} = \overline{P}(C) - \dot{K} = \int_C (\boldsymbol{\sigma}^s : \boldsymbol{d}_s + \boldsymbol{\sigma}^f : \boldsymbol{d}_f) \mathrm{d}V \tag{2.41}$$

if C is taken as an independent two-phase system.

3. Effective stress in a porous continuum

The effective stress is a certain portion of total stress that may be applied to the solid matrix of a porous continuum without pore pressure to produce a given solid matrix deformation (see Bishop, 1959; Bishop and Blight, 1963; Loret and Khalili, 2002; Nur and Byerlee, 1971; Schrefler, 1984; Skempton, 1961; Terzaghi, 1943). The effective and total stresses are the same when the pore pressure is zero, but when the pore contains fluids that also carry part of the applied load the effective stress is different from the total stress. Constitutive theory for granular materials is usually expressed in terms of the effective stress, so defining its mathematical form as a function of the total stress and pore pressure is crucial for constitutive modeling. Because the effective stress is assumed to be fully responsible for deforming the solid matrix, its definition must reflect a mechanical energy or power conjugate relationship with the solid deformation or the solid rate of deformation. In this section we describe a complete form of effective stress for saturated and unsaturated porous continua based on the expression for mechanical power developed in the previous section. The formulation given below revolves around the smeared strong discontinuity model presented in Section 2.3.

3.1. Conservation laws for three-phase porous continua

We extend the results of the previous section to a three-phase mixture of solid, liquid (e.g., water) and gas (e.g., air), comprising an unsaturated or partially saturated porous medium. In the limit of perfect saturation, the formulation reduces to a two-phase solid–liquid mixture. In the interest of brevity, we assume that there are no mass exchanges among the constituent phases. We also assume a condition of barotropic flow, in which a functional is postulated to exist between the intrinsic pore pressure and intrinsic mass density. This latter assumption not only facilitates the introduction of the intrinsic bulk moduli of the constituent phases but also provides a closed-form analytical expression for the so-called bulk stiffness of the porous solid skeleton.

Balance of mass for a three-phase solid–water–air mixture is written as (Borja, 2005)

$$\frac{d\rho^s}{dt} + \rho^s \operatorname{div} \boldsymbol{v} = 0,$$
$$\frac{d\rho^\alpha}{dt} + \rho^\alpha \operatorname{div} \boldsymbol{v} = -\operatorname{div} \boldsymbol{w}^\alpha, \quad \alpha = \mathrm{w, a},$$

(3.1)

where $\boldsymbol{w}^\alpha = \rho^\alpha \widetilde{\boldsymbol{v}}_\alpha$ (for $\alpha = \mathrm{w, a}$) is the Eulerian relative flow vector of the α phase with respect to the solid matrix, $\widetilde{\boldsymbol{v}}_\alpha = \boldsymbol{v}_\alpha - \boldsymbol{v}$ is the relative velocity of fluid α to the solid phase, and $\boldsymbol{v} \equiv \boldsymbol{v}_\mathrm{s}$. Barotropic flows satisfy functional relations of the form (Malvern, 1969)

$$f_\alpha(p_\alpha, \rho_\alpha) = 0, \quad \alpha = \mathrm{s, w, a},$$

(3.2)

where $p_\alpha = -\operatorname{tr}(\boldsymbol{\sigma}_\alpha)/3$ denotes the intrinsic pressure equal to the actual force per unit actual area acting on the α phase, and ρ_α is the intrinsic mass density of the α phase. The above functional form is a particular case of equation of state independent of temperature and suggests that there exists a one-on-one relation between the pressure p_α and mass density ρ_α for a given constituent phase α.

The material time derivative following the solid phase trajectory is written as:

$$\frac{d\rho_\alpha}{dt} = \rho_\alpha'(p_\alpha) \frac{dp_\alpha}{dt}, \quad \alpha = \mathrm{s, w, a},$$

(3.3)

where the prime denotes ordinary differentiation of the functional relation (3.2). Thus

$$\frac{d\rho^\alpha}{dt} = \frac{d(\phi^\alpha \rho_\alpha)}{dt} = \phi^\alpha \rho_\alpha'(p_\alpha) \frac{dp_\alpha}{dt} + \rho_\alpha \frac{d\phi^\alpha}{dt}.$$

(3.4)

Denoting the bulk modulus of the α phase by

$$K_\alpha = \rho_\alpha p'_\alpha(\rho_\alpha), \quad \alpha = \mathrm{s, w, a}, \tag{3.5}$$

balance of mass for the solid and fluid then becomes

$$\frac{\mathrm{d}\phi^{\mathrm{s}}}{\mathrm{d}t} + \frac{\phi^{\mathrm{s}}}{K_{\mathrm{s}}}\frac{\mathrm{d}p_{\mathrm{s}}}{\mathrm{d}t} + \phi^{\mathrm{s}}\operatorname{div}\boldsymbol{v} = 0,$$
$$\frac{\mathrm{d}\phi^{\alpha}}{\mathrm{d}t} + \frac{\phi^{\alpha}}{K_{\alpha}}\frac{\mathrm{d}p_{\alpha}}{\mathrm{d}t} + \phi^{\alpha}\operatorname{div}\boldsymbol{v} = -\frac{1}{\rho_{\alpha}}\operatorname{div}\boldsymbol{w}^{\alpha}, \quad \alpha = \mathrm{w, a}. \tag{3.6}$$

Adding the above equations gives balance of mass for the mixture

$$\frac{\phi^{\mathrm{s}}}{K_{\mathrm{s}}}\frac{\mathrm{d}p_{\mathrm{s}}}{\mathrm{d}t} + \sum_{\alpha=\mathrm{w,a}}\frac{\phi^{\alpha}}{K_{\alpha}}\frac{\mathrm{d}p_{\alpha}}{\mathrm{d}t} + \operatorname{div}\boldsymbol{v} = -\sum_{\alpha=\mathrm{w,a}}\frac{1}{\rho_{\alpha}}\operatorname{div}\boldsymbol{w}^{\alpha}. \tag{3.7}$$

Note that the mass of the solid phase is conserved in $\mathrm{d}\rho_{\mathrm{s}}/\mathrm{d}t$ but not the mass of the fluid phase in the material time derivative $\mathrm{d}\rho_{\alpha}/\mathrm{d}t$. That a tangent bulk modulus can be defined for the fluid even if the solid motion does not follow the fluid trajectory is due to the assumed equation of state, $f_{\alpha}(p_{\alpha}, \rho_{\alpha}) = 0$, suggesting a one-on-one relation between p_{α} and ρ_{α}.

We now introduce void fractions ψ^{w} and ψ^{a}, defined as the respective ratios between the volumes of the water and air phases in the void to the volume of the void itself

$$\psi^{\mathrm{w}} = \frac{\mathrm{d}V_{\mathrm{w}}}{\mathrm{d}V_{\mathrm{w}} + \mathrm{d}V_{\mathrm{a}}} = \frac{\phi^{\mathrm{w}}}{1 - \phi^{\mathrm{s}}}, \quad \psi^{\mathrm{a}} = \frac{\mathrm{d}V_{\mathrm{a}}}{\mathrm{d}V_{\mathrm{w}} + \mathrm{d}V_{\mathrm{a}}} = \frac{\phi^{\mathrm{a}}}{1 - \phi^{\mathrm{s}}}. \tag{3.8}$$

The term ψ^{w} is often called the degree of saturation, commonly denoted by the symbol S_{r} in the literature, whereas $1 - \phi^{\mathrm{s}}$ is the porosity. The saturation condition is

$$\psi^{\mathrm{w}} + \psi^{\mathrm{a}} = 1. \tag{3.9}$$

Fully dry and fully saturated mixtures are then given by conditions $\psi^{\mathrm{w}} = 0$ and $\psi^{\mathrm{w}} = 1$, respectively.

Taking the material time derivative of (3.8) in the direction of the solid phase motion gives

$$\frac{\mathrm{d}\phi^{\alpha}}{\mathrm{d}t} = (1 - \phi^{\mathrm{s}})\frac{\mathrm{d}\psi^{\alpha}}{\mathrm{d}t} - \psi^{\alpha}\frac{\mathrm{d}\phi^{\mathrm{s}}}{\mathrm{d}t} = (1 - \phi^{\mathrm{s}})\frac{\mathrm{d}\psi^{\alpha}}{\mathrm{d}t} + \psi^{\alpha}\left(\frac{\phi^{\mathrm{s}}}{K_{\mathrm{s}}}\frac{\mathrm{d}p_{\mathrm{s}}}{\mathrm{d}t} + \phi^{\mathrm{s}}\operatorname{div}\boldsymbol{v}\right), \quad \alpha = \mathrm{w, a}, \tag{3.10}$$

where the second line follows from the first line of (3.6). Substituting this result into the second line of (3.6) gives

$$(1 - \phi^{\mathrm{s}})\frac{\mathrm{d}\psi^{\alpha}}{\mathrm{d}t} + \frac{\phi^{\alpha}}{K_{\alpha}}\frac{\mathrm{d}p_{\alpha}}{\mathrm{d}t} + \frac{\psi^{\alpha}\phi^{\mathrm{s}}}{K_{\mathrm{s}}}\frac{\mathrm{d}p_{\mathrm{s}}}{\mathrm{d}t} + \psi^{\alpha}\operatorname{div}\boldsymbol{v} = -\frac{1}{\rho_{\alpha}}\operatorname{div}\boldsymbol{w}^{\alpha}, \tag{3.11}$$

for $\alpha = \mathrm{w,a}$. This is an equivalent expression of combined balance of mass for the solid and fluid α.

Balance of linear momentum for a three-phase solid–water–air mixture is written as

$$\rho^{\mathrm{s}}(\boldsymbol{g} - \boldsymbol{a}) + \operatorname{div}\boldsymbol{\sigma}^{\mathrm{s}} + \boldsymbol{h}^{\mathrm{s}} = \boldsymbol{0},$$
$$\rho^{\alpha}(\boldsymbol{g} - \boldsymbol{a}_{\alpha}) + \operatorname{div}\boldsymbol{\sigma}^{\alpha} + \boldsymbol{h}^{\alpha} = \boldsymbol{0}, \quad \alpha = \mathrm{w, a}, \tag{3.12}$$

where $\boldsymbol{a} \equiv \boldsymbol{a}_{\mathrm{s}}$. The total Cauchy stress tensor $\boldsymbol{\sigma}$ can be obtained from the sum of the partial stresses,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{s}} + \boldsymbol{\sigma}^{\mathrm{w}} + \boldsymbol{\sigma}^{\mathrm{a}}. \tag{3.13}$$

Thus, balance of linear momentum for the entire mixture becomes

$$\left(\rho\boldsymbol{g} - \rho^{\mathrm{s}}\boldsymbol{a} - \sum_{\alpha=\mathrm{w,a}}\rho^{\alpha}\boldsymbol{a}_{\alpha}\right) + \operatorname{div}\boldsymbol{\sigma} = \left[\rho(\boldsymbol{g} - \boldsymbol{a}) - \sum_{\alpha=\mathrm{w,a}}\rho^{\alpha}\widetilde{\boldsymbol{a}}_{\alpha}\right] + \operatorname{div}\boldsymbol{\sigma} = \boldsymbol{0}, \tag{3.14}$$

where $\rho = \rho^s + \rho^w + \rho^a$ is the total mass density of the mixture and $\widetilde{a}_\alpha = a_\alpha - a$ is the relative material acceleration of fluid α to that of the solid phase.

Balance of energy for a three-phase solid–water–air mixture yields the following expression for the material time derivative of internal energy

$$\rho\dot{\overline{e}} = \boldsymbol{\sigma}^s : \boldsymbol{d} + \sum_{\alpha=w,a} \boldsymbol{\sigma}^\alpha : \boldsymbol{d}_\alpha + r - \operatorname{div}\boldsymbol{q}, \tag{3.15}$$

where $\dot{\overline{e}}$ is the rate of change of internal energy per unit total mass of the mixture, r is the heat supply per unit volume of the mixture, $\boldsymbol{q}$ is the heat flux vector, $\boldsymbol{d}_\alpha$ is the rate of deformation for the α-phase, and $\boldsymbol{d} \equiv \boldsymbol{d}_s$. In the above equation, we have taken the mixture as an independent three-phase system, and thus $\dot{\overline{e}}$ is the mass density associated with the rate of change of total internal energy, $\overline{E}$. If we assume that $\boldsymbol{\sigma}^w$ and $\boldsymbol{\sigma}^a$ are isotropic tensors, then we can write

$$\boldsymbol{\sigma}^w = -\phi^w p_w \mathbf{1}, \quad \boldsymbol{\sigma}^a = -\phi^a p_a \mathbf{1}, \tag{3.16}$$

where p_w and p_a are the same intrinsic pore water and pore air pressures defined previously, and $\mathbf{1}$ is the second-order identity tensor. In terms of the pore pressures, the expression for the material time derivative of internal energy becomes

$$\rho\dot{\overline{e}} = \boldsymbol{\sigma}^s : \boldsymbol{d} - \sum_{\alpha=w,a} \phi^\alpha p_\alpha \operatorname{div}\boldsymbol{v}_\alpha + r - \operatorname{div}\boldsymbol{q}. \tag{3.17}$$

3.2. A definition of effective stress

Consider once again the energy balance equation (3.15). Adding the null scalar product $(\boldsymbol{\sigma} - \boldsymbol{\sigma}^s - \boldsymbol{\sigma}^w - \boldsymbol{\sigma}^a) : \boldsymbol{d}$ to this equation gives

$$\rho\dot{\overline{e}} = \boldsymbol{\sigma} : \boldsymbol{d} + \sum_{\alpha=w,a} \boldsymbol{\sigma}^\alpha : \widetilde{\boldsymbol{d}}_\alpha + r - \operatorname{div}\boldsymbol{q}, \quad \widetilde{\boldsymbol{d}}_\alpha = \boldsymbol{d}_\alpha - \boldsymbol{d} \tag{3.18}$$

or, equivalently,

$$\rho\dot{\overline{e}} = \boldsymbol{\sigma} : \boldsymbol{d} - \sum_{\alpha=w,a} \phi^\alpha p_\alpha \operatorname{div}\widetilde{\boldsymbol{v}}_\alpha + r - \operatorname{div}\boldsymbol{q}, \quad \widetilde{\boldsymbol{v}}_\alpha = \boldsymbol{v}_\alpha - \boldsymbol{v}. \tag{3.19}$$

This suggests that the mechanical power of the partial Cauchy stress tensors is equal to the mechanical power of the total Cauchy stress tensor in deforming the solid skeleton volume plus the mechanical power of the pore pressures in either injecting or extracting fluids into or from the solid matrix. This energy expression agrees with that presented by Biot (1973) for a two-phase solid–liquid mixture. In arriving at his result, Biot effectively took the saturated solid–liquid mixture, not as a whole, but as the sum of its individual parts, and so his energy equation does not contain the contributions from the interactive forces $\boldsymbol{h}^s$ and $\boldsymbol{h}^f$. In contrast, Houlsby (1997, 1979) presented an energy equation that takes the mixture as a whole.

Next, we use the product rule of differentiation to evaluate the divergence of the Eulerian relative flow vector,

$$\operatorname{div}\boldsymbol{w}^\alpha = \operatorname{div}(\phi^\alpha \rho_\alpha \widetilde{\boldsymbol{v}}_\alpha) = \phi^\alpha \rho_\alpha \operatorname{div}\widetilde{\boldsymbol{v}}_\alpha + \widetilde{\boldsymbol{v}}_\alpha \cdot \operatorname{grad}(\phi^\alpha \rho_\alpha). \tag{3.20}$$

Substituting this result into the right-hand side of (3.11) and re-arranging gives

$$-\phi^\alpha \operatorname{div}\widetilde{\boldsymbol{v}}_\alpha = \frac{1}{\rho_\alpha} \widetilde{\boldsymbol{v}}_\alpha \cdot \operatorname{grad}(\phi^\alpha \rho_\alpha) + (1 - \phi^s)\frac{d\psi^\alpha}{dt} + \frac{\phi^\alpha}{K_\alpha}\frac{dp_\alpha}{dt} + \frac{\psi^\alpha \phi^s}{K_s}\frac{dp_s}{dt} + \psi^\alpha \operatorname{div}\boldsymbol{v}. \tag{3.21}$$

Inserting this expression into (3.19) and using the identity $\operatorname{div} \boldsymbol{v} = \mathbf{1} : \boldsymbol{d}$ gives

$$\rho \dot{\bar{e}} = \left(\boldsymbol{\sigma} + \sum_{\alpha=w,a} \psi^{\alpha} p_{\alpha} \mathbf{1} \right) : \boldsymbol{d} + \sum_{\alpha=w,a} \left[\frac{1}{\rho_{\alpha}} \widetilde{\boldsymbol{v}}_{\alpha} \cdot \operatorname{grad}(\phi^{\alpha} \rho_{\alpha}) + (1 - \phi^{s}) \frac{d\psi^{\alpha}}{dt} + \frac{\phi^{\alpha}}{K_{\alpha}} \frac{dp_{\alpha}}{dt} + \frac{\psi^{\alpha} \phi^{s}}{K_{s}} \frac{dp_{s}}{dt} \right] p_{\alpha}$$
$$+ r - \operatorname{div} \boldsymbol{q}. \tag{3.22}$$

We can now define a Cauchy effective stress tensor as

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + \bar{p} \mathbf{1}, \tag{3.23}$$

where

$$\bar{p} = \sum_{\alpha=w,a} \psi^{\alpha} p_{\alpha} = \psi^{w} p_{w} + (1 - \psi^{w}) p_{a} \tag{3.24}$$

is the mean neutral stress, or pore pressure. The effective stress tensor $\boldsymbol{\sigma}'$ is power-conjugate to the rate of deformation tensor $\boldsymbol{d}$, while the mean neutral stress $\bar{p}$ represents the relative contributions of the intrinsic pore water and pore air pressures, weighted according to the degree of saturation.

The above definition of an effective stress is analogous to that used by Bishop (1959) for partially saturated soils except that the degree of saturation ψ^{w} is now used as the weighting factor. This particular form utilizing the degree of saturation as the weighting factor appears to have been first proposed by Schrefler (1984) and used subsequently by other authors. The above definition of effective stress has an interesting feature in that in the fully saturated regime, $\psi^{a} = \phi^{a} = 0$, $\psi^{w} = 1$, and $\phi^{w} + \phi^{s} = 1$. In this case the Cauchy effective stress tensor reduces to the simple form

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p_{w} \mathbf{1}, \tag{3.25}$$

which we readily recognize as the effective stress proposed by Terzaghi (1943) for fully saturated soils.

In addition to the mechanical power of the effective stress tensor $\boldsymbol{\sigma}'$, other mechanical powers appear in (3.22). The first term of the summation

$$\sum_{\alpha=w,a} \left[\frac{1}{\rho_{\alpha}} \widetilde{\boldsymbol{v}}_{\alpha} \cdot \operatorname{grad}(\phi^{\alpha} \rho_{\alpha}) \right] p_{\alpha} = \sum_{\alpha=w,a} \left(\frac{1}{\rho_{\alpha}} \operatorname{div} \boldsymbol{w}^{\alpha} - \phi^{\alpha} \operatorname{div} \widetilde{\boldsymbol{v}}_{\alpha} \right) p_{\alpha}, \tag{3.26}$$

arises from the product rule of differentiation, (3.20), and represents the mechanical power produced by the relative flow. Hence, this term vanishes when $\boldsymbol{w}^{\alpha} = \rho^{\alpha} \widetilde{\boldsymbol{v}}_{\alpha} = \mathbf{0}$. The second term

$$(1 - \phi^{s}) \sum_{\alpha=w,a} \frac{d\psi^{\alpha}}{dt} p_{\alpha} = -s(1 - \phi^{s}) \frac{d\psi^{w}}{dt}, \quad s = p_{a} - p_{w} \tag{3.27}$$

is the mechanical power of the pore pressure difference s, usually referred to in the literature as the suction stress. This power vanishes for perfectly saturated flow. The third and fourth terms are powers of the intrinsic solid stresses and pore pressures in compressing the solids and fluids, and if we revert back to the original definitions of the intrinsic bulk moduli we get

$$\sum_{\alpha=w,a} \left(\frac{\phi^{\alpha}}{K_{\alpha}} \frac{dp_{\alpha}}{dt} + \frac{\psi^{\alpha} \phi^{s}}{K_{s}} \frac{dp_{s}}{dt} \right) p_{\alpha} = \sum_{\alpha=w,a} \left(\frac{\phi^{\alpha}}{\rho_{\alpha}} \frac{d\rho_{\alpha}}{dt} + \frac{\psi^{\alpha} \phi^{s}}{\rho_{s}} \frac{d\rho_{s}}{dt} \right) p_{\alpha}. \tag{3.28}$$

Using these alternative expressions, we can thus rewrite (3.22) as

$$\rho \dot{\bar{e}} = \boldsymbol{\sigma}' : \boldsymbol{d} + \sum_{\alpha=w,a} \left(\frac{1}{\rho_{\alpha}} \operatorname{div} \boldsymbol{w}^{\alpha} - \phi^{\alpha} \operatorname{div} \widetilde{\boldsymbol{v}}_{\alpha} \right) p_{\alpha} - s(1 - \phi^{s}) \frac{d\psi^{w}}{dt} + \sum_{\alpha=w,a} \left(\frac{\phi^{\alpha}}{K_{\alpha}} \frac{dp_{\alpha}}{dt} + \frac{\psi^{\alpha} \phi^{s}}{K_{s}} \frac{dp_{s}}{dt} \right) p_{\alpha}$$
$$+ r - \operatorname{div} \boldsymbol{q}. \tag{3.29}$$

3.3. An alternative definition of effective stress

The mechanical power of the Cauchy stress tensor $\boldsymbol{\sigma}'$ does not include the component required to compress the intrinsic solid volume itself. The latter power is given by the term

$$\sum_{\alpha=w,a} \left(\frac{\psi^\alpha \phi^s}{K_s} \frac{\mathrm{d}p_s}{\mathrm{d}t} \right) p_\alpha$$

appearing on the right-hand side of (3.29). In the discussion below we show that this term is also a function of $\mathrm{div}\,\boldsymbol{v}$, and thus also of the solid rate of deformation $\boldsymbol{d}$. To demonstrate this latter point, consider a function $\widetilde{f}_s = 0$ relating the partial solid pressure $p^s = \phi^s p_s$ to partial solid mass density $\rho^s = \phi^s \rho_s$

$$\widetilde{f}_s(p^s, \rho^s) = \widetilde{f}_s(\phi^s p_s, \rho^s) = 0. \tag{3.30}$$

Such a one-on-one relationship implies an elastic solid matrix. From the function $\widetilde{f}_s = 0$ we can write an equivalent constitutive relation

$$p_s = \widetilde{p}_s(\rho^s, \phi^s). \tag{3.31}$$

Taking the material time derivative in the direction of the solid phase motion gives

$$\frac{\mathrm{d}p_s}{\mathrm{d}t} = \frac{\partial \widetilde{p}_s}{\partial \rho^s} \frac{\mathrm{d}\rho^s}{\mathrm{d}t} + \frac{\partial \widetilde{p}_s}{\partial \phi^s} \frac{\mathrm{d}\phi^s}{\mathrm{d}t}. \tag{3.32}$$

Now, consider the identity

$$\frac{1}{\rho^s} \frac{\mathrm{d}\rho^s}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\ln\left(\frac{\rho^s}{\rho_0^s}\right) \right] = \frac{\mathrm{d}}{\mathrm{d}t}(\ln J^{-1}) = -\mathrm{div}\,\boldsymbol{v}, \tag{3.33}$$

where $\rho_0^s = J\rho^s$ is the pull-back solid partial mass density. Furthermore, from balance of mass for the solid phase (3.6), we write

$$\frac{\mathrm{d}\phi^s}{\mathrm{d}t} = -\frac{\phi^s}{K_s} \frac{\mathrm{d}p_s}{\mathrm{d}t} - \phi^s \mathrm{div}\,\boldsymbol{v}. \tag{3.34}$$

Substituting these last two equations into (3.32) gives

$$\left(1 + \frac{\partial \widetilde{p}_s}{\partial \phi^s} \frac{\phi^s}{K_s} \right) \frac{\mathrm{d}p_s}{\mathrm{d}t} = -\left(\rho^s \frac{\partial \widetilde{p}_s}{\partial \rho^s} + \phi^s \frac{\partial \widetilde{p}_s}{\partial \phi^s} \right) \mathrm{div}\,\boldsymbol{v} \tag{3.35}$$

or

$$\phi^s \frac{\mathrm{d}p_s}{\mathrm{d}t} = -K \mathrm{div}\,\boldsymbol{v}, \tag{3.36}$$

where

$$K = \phi^s \left(\rho^s \frac{\partial \widetilde{p}_s}{\partial \rho^s} + \phi^s \frac{\partial \widetilde{p}_s}{\partial \phi^s} \right) \bigg/ \left(1 + \frac{\partial \widetilde{p}_s}{\partial \phi^s} \frac{\phi^s}{K_s} \right) \tag{3.37}$$

is the bulk modulus of the solid skeleton. The bulk modulus K represents the volumetric stiffness of the solid matrix and must not be confused with the intrinsic bulk modulus K_s representing the volumetric stiffness of the solid itself. Furthermore, the bulk modulus K is the slope relating $\phi^s \mathrm{d}p_s/\mathrm{d}t$ (and not $\mathrm{d}p^s/\mathrm{d}t$) with $\mathrm{div}\,\boldsymbol{v}$. If the solid matrix is not elastic, then K may be taken as the instantaneous tangent bulk modulus for a given loading path.

For the bulk modulus K to be well defined, neither the numerator nor the denominator in Eq. (3.37) may vanish. The numerator is zero when $\widetilde{p}_s$ depends on ρ^s and ϕ^s through the intrinsic solid mass density $\rho_s = \rho^s/\phi^s$, i.e., when

$$\widetilde{p}_s = \widetilde{p}_s(\rho^s, \phi^s) = \widetilde{p}_s(\rho^s/\phi^s) = \widetilde{p}_s(\rho_s). \tag{3.38}$$

If this is the case, then

$$\rho^s \frac{\partial \widetilde{p}_s}{\partial \rho^s} + \phi^s \frac{\partial \widetilde{p}_s}{\partial \phi^s} = \rho^s \frac{\partial \widetilde{p}_s}{\partial \rho_s} \frac{1}{\phi^s} + \phi^s \frac{\partial \widetilde{p}_s}{\partial \rho_s} \left(-\frac{\rho^s}{\phi^{s2}} \right) \equiv 0. \tag{3.39}$$

This result makes perfect sense since, in this case, $\widetilde{f}_s$ is identical to f_s and therefore is a redundant function. Also, the denominator is zero when

$$\frac{\partial \widetilde{p}_s}{\partial \phi^s} = -\frac{K_s}{\phi^s} = -\frac{\rho_s}{\phi^s} p_s'(\rho_s) \quad \Longleftrightarrow \quad \phi^s \frac{\partial \widetilde{p}_s}{\partial \phi^s} + \rho_s p_s'(\rho_s) = 0, \tag{3.40}$$

where $p_s'(\rho_s)$ is the derivative obtained from the functional $f_s(p_s, \rho_s) = 0$. Again, the denominator vanishes when f_s and $\widetilde{f}_s$ are the same function since this yields $p_s'(\rho_s) = \partial \widetilde{p}_s / \partial \rho_s$. Thus, for K to be well defined, f_s and $\widetilde{f}_s$ must reflect two different constitutive functions, one representing the behavior of the solid phase and the other representing the behavior of the solid matrix.

We now insert (3.36) into (3.29) and collect terms to obtain the following alternative expression for the rate of change of internal energy:

$$\rho \dot{e} = \overline{\boldsymbol{\sigma}} : \boldsymbol{d} + \sum_{\alpha=\text{w,a}} \left(\frac{1}{\rho_\alpha} \operatorname{div} \boldsymbol{w}^\alpha - \phi^\alpha \operatorname{div} \widetilde{\boldsymbol{v}}_\alpha \right) p_\alpha - s(1 - \phi^s) \frac{\mathrm{d}\psi^w}{\mathrm{d}t} + \sum_{\alpha=\text{w,a}} \frac{\phi^\alpha}{K_\alpha} \frac{\mathrm{d}p_\alpha}{\mathrm{d}t} p_\alpha + r - \operatorname{div} \boldsymbol{q}, \tag{3.41}$$

where

$$\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \left(1 - \frac{K}{K_s} \right) \sum_{\alpha=\text{w,a}} \psi^\alpha p_\alpha \mathbf{1} = \boldsymbol{\sigma} + \left(1 - \frac{K}{K_s} \right) \bar{p} \mathbf{1} \tag{3.42}$$

is now a new measure of effective stress, with $\bar{p}$ being the same mean neutral stress defined in (3.24). The effective stress $\overline{\boldsymbol{\sigma}}$ has a form remarkably similar to that presented by Skempton (1961) for partially saturated porous media, except that Skempton used Bishop's (1959, 1963) parameter χ in place of the degree of saturation ψ^w to determine the mean neutral stress $\bar{p}$. Continuum theory of thermodynamics is unable to support the validity of the Skempton stress, though, obviously, experimental evidence has supported such definition. To the knowledge of the author, this is the first time that the effective stress tensor $\overline{\boldsymbol{\sigma}}$, which reflects the effects of solid compressibility and partial saturation, has emerged from continuum principles of thermodynamics.

The ratio K/K_s is typically in the range 0.1–0.5 for saturated rocks and concrete (Skempton, 1961), as well as in other porous materials such as the human bone and cornea, and therefore may not be ignored in general. In soils, however, the ratio K/K_s is typically very small and is usually ignored. If K/K_s is ignored, then $\overline{\boldsymbol{\sigma}}$ reduces to the effective stress $\boldsymbol{\sigma}'$ introduced in Section 3.2. Furthermore, in the fully saturated case the more general expression (3.42) reduces to

$$\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \left(1 - \frac{K}{K_s} \right) p_w \mathbf{1}. \tag{3.43}$$

This expression for the effective stress was also presented by Skempton (1961) for fully saturated materials, and was shown by Nur and Byerlee (1971) to be theoretically exact for fully saturated elastic porous media. Finally, Terzaghi's (1943) effective stress is recovered from (3.43) when $K/K_s \approx 0$. Thus, $\overline{\boldsymbol{\sigma}}$ is the most complete definition of effective stress currently supported by continuum principles of thermodynamics, from which the Schrefler, Skempton, Nur–Byerlee, and Terzaghi stresses may be derived under special conditions.

3.4. Implications to the second law

We assume a free energy density Ψ per unit current volume of the mixture. For the sake of simplicity we only consider mechanical power in the following discussion. The Clausius–Duhem inequality yields the local dissipation function

$$\mathscr{D} = \sum_{\alpha=\mathrm{s,w,a}} \boldsymbol{\sigma}^\alpha : \boldsymbol{d}_\alpha - \dot{\Psi} \geqslant 0. \tag{3.44}$$

Below we compare the mathematical structures of this dissipation function for the two effective stress tensors $\boldsymbol{\sigma}'$ and $\overline{\boldsymbol{\sigma}}$. Our objective here is to highlight a problem with enforcing the dissipation inequality using the effective stress tensor $\boldsymbol{\sigma}'$ in the regime of compressible solids. To this end, it suffices to use the infinitesimal theory.

In the infinitesimal theory the reduced dissipation inequality is written as

$$\mathscr{D} = \sum_{\alpha=\mathrm{s,w,a}} \boldsymbol{\sigma}^\alpha : \dot{\boldsymbol{\epsilon}}_\alpha - \dot{\Psi} \geqslant 0, \tag{3.45}$$

where $\dot{\boldsymbol{\epsilon}}_\alpha$ is the infinitesimal strain rate tensor for phase α. Let $\dot{\vartheta}^\alpha = \phi^\alpha \dot{p}_\alpha / K_\alpha$ for $\alpha = \mathrm{s, w, a}$. Ignoring nonmechanical powers, the local dissipation function using the effective stress tensor $\boldsymbol{\sigma}'$ is written as

$$\mathscr{D} = \boldsymbol{\sigma}' : \dot{\boldsymbol{\epsilon}} + \sum_{\alpha=\mathrm{w,a}} \left[\frac{1}{\rho_\alpha} \widetilde{\boldsymbol{v}}_\alpha \cdot \mathrm{grad}(\phi^\alpha \rho_\alpha) \right] p_\alpha - s(1-\phi^\mathrm{s})\dot{\psi}^\mathrm{w} + \sum_{\alpha=\mathrm{w,a}} (\dot{\vartheta}^\alpha + \psi^\alpha \dot{\vartheta}^\mathrm{s}) p_\alpha - \dot{\Psi} \geqslant 0. \tag{3.46}$$

Following plasticity theory, we assume a standard additive decomposition of the solid matrix strain rate, $\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^\mathrm{e} + \dot{\boldsymbol{\epsilon}}^\mathrm{p}$. A similar additive decomposition may be made for the volumetric strain rates $\dot{\vartheta}^\alpha$ (see Borja, 2004).

We now write the free energy function in the form

$$\Psi = \Psi(\boldsymbol{\epsilon}^\mathrm{e}, \widetilde{\boldsymbol{u}}_\mathrm{w}, \widetilde{\boldsymbol{u}}_\mathrm{a}, \vartheta^\mathrm{s}, \vartheta^\mathrm{w}, \vartheta^\mathrm{a}, \boldsymbol{\xi}), \tag{3.47}$$

where $\dot{\widetilde{\boldsymbol{u}}}_\alpha = \widetilde{\boldsymbol{v}}_\alpha$ for $\alpha = \mathrm{w, a}$; and $\boldsymbol{\xi}$ is a vector of strain-like plastic internal variables. This form of the free energy is motivated by the mathematical structure of the local dissipation function and allows extraction of the relevant constitutive expressions using the standard Coleman arguments. The material time derivative of the free energy is then obtained from the chain rule as

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \boldsymbol{\epsilon}^\mathrm{e}} : \dot{\boldsymbol{\epsilon}}^\mathrm{e} + \sum_{\alpha=\mathrm{w,a}} \frac{\partial \Psi}{\partial \widetilde{\boldsymbol{u}}_\alpha} \cdot \widetilde{\boldsymbol{v}}_\alpha + \sum_{\alpha=\mathrm{s,w,a}} \frac{\partial \Psi}{\partial \vartheta^\alpha} \dot{\vartheta}^\alpha + \frac{\partial \Psi}{\partial \boldsymbol{\xi}} \cdot \dot{\boldsymbol{\xi}}. \tag{3.48}$$

Next, we substitute (3.48) into (3.46) and collect terms to obtain

$$\mathscr{D} = \left(\boldsymbol{\sigma}' - \frac{\partial \Psi}{\partial \boldsymbol{\epsilon}^\mathrm{e}} \right) : \dot{\boldsymbol{\epsilon}}^\mathrm{e} + \sum_{\alpha=\mathrm{w,a}} \left(\boldsymbol{g}_\alpha - \frac{\partial \Psi}{\partial \widetilde{\boldsymbol{u}}_\alpha} \right) \cdot \widetilde{\boldsymbol{v}}_\alpha + \sum_{\alpha=\mathrm{w,a}} \left(p_\alpha - \frac{\partial \Psi}{\partial \vartheta^\alpha} \right) \dot{\vartheta}^\alpha$$
$$+ \left(\overline{p} - \frac{\partial \Psi}{\partial \vartheta^\mathrm{s}} \right) \dot{\vartheta}^\mathrm{s} + \boldsymbol{\sigma}' : \dot{\boldsymbol{\epsilon}}^\mathrm{p} - s(1-\phi^\mathrm{s})\dot{\psi}^\mathrm{w} + \boldsymbol{q} \cdot \dot{\boldsymbol{\xi}} \geqslant 0, \tag{3.49}$$

where $\boldsymbol{g}_\alpha = p_\alpha \mathrm{grad}(\rho^\alpha)/\rho_\alpha$ is the Gibbs potential for fluid α per unit current volume of the mixture, and $\boldsymbol{q} = -\partial \Psi/\partial \boldsymbol{\xi}$ is a vector of stress-like plastic internal variables conjugate to $\boldsymbol{\xi}$. The Coleman relation then yields the constitutive expressions

$$\boldsymbol{\sigma}' = \frac{\partial \Psi}{\partial \boldsymbol{\epsilon}^\mathrm{e}}, \quad \boldsymbol{g}_\alpha = \frac{\partial \Psi}{\partial \widetilde{\boldsymbol{u}}_\alpha}\bigg|_{\alpha=\mathrm{w,a}}, \quad p_\alpha = \frac{\partial \Psi}{\partial \vartheta^\alpha}\bigg|_{\alpha=\mathrm{w,a}}, \quad \overline{p} = \frac{\partial \Psi}{\partial \vartheta^\mathrm{s}}. \tag{3.50}$$

Note that the expressions for p_α and $\bar{p}$ are not redundant since $\bar{p}$ contains a third independent term, the degree of saturation ψ^w, as shown in (3.24). Hence, by prescribing a constitutive expression for $\bar{p}$ we are effectively prescribing a constitutive expression for ψ^w, and in fact the latter constitutive expression is given by

$$\psi^w = \left(\frac{\partial \Psi}{\partial \vartheta^a} - \frac{\partial \Psi}{\partial \vartheta^s} \right) \Bigg/ \left(\frac{\partial \Psi}{\partial \vartheta^a} - \frac{\partial \Psi}{\partial \vartheta^w} \right). \tag{3.51}$$

Substituting (3.50) into the local dissipation function gives

$$\mathscr{D} = \boldsymbol{\sigma}' : \dot{\boldsymbol{\epsilon}}^p - s(1 - \phi^s)\dot{\psi}^w + \boldsymbol{q} \cdot \dot{\boldsymbol{\xi}} \geqslant 0. \tag{3.52}$$

However, this inequality will not close since ψ^w is constrained to follow the constitutive relation (3.51), and, accordingly, it is not possible to freely postulate a flow rule for $\dot{\psi}^w$, which could then lead to a violation of the dissipation inequality. The problem may be traced from the fact that the formulation is over-prescribed in that ϑ^s and $\boldsymbol{\epsilon}$ have been treated as independent kinematical variables, yet they both result from the same solid motion. This affirms that $\boldsymbol{\sigma}'$ is an incomplete effective stress tensor in the regime of compressible solids.

We now repeat the formulation for the reduced dissipation inequality, this time using the alternative effective stress tensor $\bar{\boldsymbol{\sigma}}$. Writing the local dissipation function in terms of this tensor, we get

$$\mathscr{D} = \bar{\boldsymbol{\sigma}} : \dot{\boldsymbol{\epsilon}} + \sum_{\alpha=w,a} \left[\frac{1}{\rho_\alpha} \widetilde{\boldsymbol{v}}_\alpha \cdot \mathrm{grad}(\phi^\alpha \rho_\alpha) \right] p_\alpha - s(1 - \phi^s)\dot{\psi}^w + \sum_{\alpha=w,a} (\dot{\vartheta}^\alpha p_\alpha) - \dot{\Psi} \geqslant 0. \tag{3.53}$$

Now, consider a new free energy function of the form

$$\Psi = \Psi(\boldsymbol{\epsilon}^e, \widetilde{\boldsymbol{u}}_w, \widetilde{\boldsymbol{u}}_a, \vartheta^w, \vartheta^a, \boldsymbol{\xi}). \tag{3.54}$$

Note that this free energy no longer contains the term ϑ^s since we recognize that the compression of the solid phase is now included in the solid matrix strain tensor $\boldsymbol{\epsilon}$. Taking the material time derivative and using the standard Coleman relation once again yields the constitutive expressions

$$\bar{\boldsymbol{\sigma}} = \frac{\partial \Psi}{\partial \boldsymbol{\epsilon}^e}, \quad \boldsymbol{g}_\alpha = \frac{\partial \Psi}{\partial \widetilde{\boldsymbol{u}}_\alpha}\bigg|_{\alpha=w,a}, \quad p_\alpha = \frac{\partial \Psi}{\partial \vartheta^\alpha}\bigg|_{\alpha=w,a}. \tag{3.55}$$

The local dissipation function then reduces to

$$\mathscr{D} = \bar{\boldsymbol{\sigma}} : \dot{\boldsymbol{\epsilon}}^p - s(1 - \phi^s)\dot{\psi}^w + \boldsymbol{q} \cdot \dot{\boldsymbol{\xi}} \geqslant 0. \tag{3.56}$$

Note that the formulation no longer imposes a constitutive constraint on the degree of saturation ψ^w. Thus, we may now postulate appropriate flow rules for the rate variables $\dot{\boldsymbol{\epsilon}}^p$, $\dot{\psi}^w$, and $\dot{\boldsymbol{\xi}}$ as in classical plasticity theory.

It is clear from the first law that a number of alternative effective stress measures may emerge if the sole criterion is that these stresses must be power-conjugate to the rate of deformation tensor $\boldsymbol{d}$. However, the second law uniquely identifies a complete effective stress tensor $\bar{\boldsymbol{\sigma}}$ necessary to satisfy the entropy production inequality. Using $\bar{\boldsymbol{\sigma}}$ as the basis of phenomenological plasticity formulation, for example, we now postulate a yield function for the solid matrix of the form

$$F(\bar{\boldsymbol{\sigma}}, s, \boldsymbol{q}) = 0. \tag{3.57}$$

In unsaturated porous media, note that the yield function now depends explicitly on the suction stress s as required by the final form of the reduced dissipation inequality (3.56). Maximum plastic dissipation is recovered from the normality conditions

$$\dot{\boldsymbol{\epsilon}}^p = \dot{\lambda} \frac{\partial F}{\partial \bar{\boldsymbol{\sigma}}}, \quad -\dot{\psi}^w = \dot{\lambda} \frac{\partial F}{\partial s}, \quad \dot{\boldsymbol{\xi}} = \dot{\lambda} \frac{\partial F}{\partial \boldsymbol{q}}, \tag{3.58}$$

where $\dot{\lambda} \geqslant 0$ is a non-negative plastic multiplier. The first term pertains to associative plastic flow; the second term denotes associative saturation; and the last term denotes associative hardening. Satisfaction of (3.58) guarantees convexity of the elastic region. Of course, reality hardly justifies such ideal normality conditions. For example, there is substantial experimental evidence supporting a non-associative (empirical) relation between $\dot{\psi}^{w}$ and s for unsaturated soils (van Genuchten, 1980) that could even lead to a non-convex yield function (Borja, 2004; Wheeler and Sivakumar, 1995).

4. Summary and conclusions

We have formulated a mechanical theory of porous continua as a problem of strong discontinuity characterized by jumps in the velocity field at the interfaces of two contacting constituents. The jump condition requires continuity of the traction vector, similar to that employed in the analysis of strain localization in inelastic solids. Porous continua are then viewed as a dense distribution of strong discontinuity that can be smeared throughout the representative volume element. In the limit of insignificant statistical variations within this volume, we recover the equations of classical mixture theory. As a byproduct of the energy formulation, two expressions for the mechanical power have been advanced, one in which the mixture is taken as a whole, and a second in which the mixture is taken to be composed of individual parts. Taking the mixture as a whole introduces an additional term in the expression for the mechanical power that can be used to extract interface constitutive laws using the standard Coleman relations.

We have utilized the results of the energy formulation to analyze the problem of partially saturated porous continua. An effective stress tensor for a three-phase (solid–water–air) continuum of the following form has emerged from the formulation: $\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + (1 - K/K_{s})\bar{p}\mathbf{1}$, where $\boldsymbol{\sigma}$ is the total stress tensor, K and K_{s} are bulk moduli of the solid matrix and solid phase, respectively, and $\bar{p} = \psi^{w}p_{w} + (1 - \psi^{w})p_{a}$ is the mean pore water and pore air pressures weighted according to the degree of saturation ψ^{w}. We show from the first and second laws of thermodynamics that this effective stress is power-conjugate to the solid rate of deformation, and that it can be used in the regime of compressible solids. Under special cases, this effective stress reduces to Schrefler's stress in the regime of incompressible solids, to the Skempton and Nur–Byerlee stress in the regime of full saturation, and to Terzaghi's stress in the regime of incompressible solids and full saturation. As a byproduct of the formulation, we have presented closed-form expressions for the bulk moduli of the solid phase and the solid matrix in terms of an equation of state under a condition of barotropic flow.

Acknowledgements

Appendix A

Consider the surface integral

$$\int_{\partial B} \boldsymbol{t}_0 \cdot \boldsymbol{v}\, dA = \int_{\partial B^+} \boldsymbol{t}_0 \cdot \boldsymbol{v}\, dA + \int_{\partial B^-} \boldsymbol{t}_0 \cdot \boldsymbol{v}\, dA = \int_{\partial B^+} \boldsymbol{t}_0 \cdot \boldsymbol{v}\, dA + \int_{\partial B^-} \boldsymbol{t}_0 \cdot \boldsymbol{v}\, dA + \int_S [\![\boldsymbol{t}_S]\!] \cdot \bar{\boldsymbol{v}}\, dA, \tag{A.1}$$

where $t_S = P^- \cdot N = P^+ \cdot N$, since the traction jump $[\![t_S]\!] = \mathbf{0}$ point-wise across S. Noting that $v = \bar{v}$ on S^- and $v = \bar{v} + [\![v]\!]$ on S^+, where S^- (alternatively, S^+) is the surface of discontinuity interpreted to belong in B^- (alternatively, B^+), we get

$$\int_S [\![t_S]\!] \cdot \bar{v} \, dA = \int_{S^-} t_S \cdot v \, dA + \int_{S^+} (-t_S) \cdot v \, dA + \int_S t_S \cdot [\![v]\!] \, dA. \tag{A.2}$$

Substituting in (A.1) gives

$$\int_{\partial B} t_0 \cdot v \, dA = \left[\int_{\partial B^+} t_0 \cdot v \, dA + \int_{S^+} (-t_S) \cdot v \, dA \right] + \left[\int_{\partial B^-} t_0 \cdot v \, dA + \int_{S^-} t_S \cdot v \, dA \right] $$
$$+ \int_S t_S \cdot [\![v]\!] \, dA. \tag{A.3}$$

The quantities inside the big brackets are closed surface integrals, so we can use Gauss theorem to write

$$\int_{\partial B} t_0 \cdot v \, dA = \int_{B^+} \mathrm{DIV}(P \cdot v) \, dV + \int_{B^-} \mathrm{DIV}(P \cdot v) \, dV + \int_S t_S \cdot [\![v]\!] \, dA$$
$$= \int_{B^+} (P \cdot \dot{F} + v \cdot \mathrm{DIV}\, P) \, dV + \int_{B^-} (P \cdot \dot{F} + v \cdot \mathrm{DIV}\, P) \, dV + \int_S t_S \cdot [\![v]\!] \, dA. \tag{A.4}$$

Substituting into the expression for P in (2.5), using Reynold's transport theorem on the kinetic energy to get

$$\dot{K} = \int_B \rho_0 a \cdot v \, dV \tag{A.5}$$

and imposing balance of momentum, we then obtain (2.8).

Appendix B

Consider a rectangular RVE and align the coordinate axes with the sides of this volume such that it lies in the first octant with one corner at the origin. Let ψ_1^s, ψ_2^s, and ψ_3^s be the area fractions of the solid on the faces with normals parallel to the x_1, x_2, and x_3 directions, respectively. Formally,

$$\psi_i^s = \frac{1}{\partial \overline{C}_i} \int_{\partial \overline{C}_i} \widetilde{\chi}^s \, dA, \tag{B.1}$$

where $\widetilde{\chi}^s$ is an area indicator function defined by the relation

$$\widetilde{\chi}^s = \begin{cases} 1 & \text{if } x \in \partial \overline{C}^s, \\ 0 & \text{otherwise} \end{cases} \tag{B.2}$$

and $\partial \overline{C}_i$ is the face being integrated over. Since we assume there is little spatial variation of the properties, we can integrate over either the positive or negative face, or any cross-section of the RVE parallel to them, and obtain the same value of ψ_i^s. Hence, if the length of parallel to side x_i is Δx_i,

$$\psi_i^s = \frac{1}{\partial \overline{C}_i \Delta x_i} \int_0^{\Delta x_i} \int_{\partial \overline{C}_i} \widetilde{\chi}^s \, dA \, dx_i = \frac{1}{\overline{C}} \int_{\overline{C}} \chi^s \, dV = \phi^s, \tag{B.3}$$

where χ^s is a volume indicator function defined by

$$\chi^s = \begin{cases} 1 & \text{if } \boldsymbol{x} \in \overline{C}^s, \\ 0 & \text{otherwise} \end{cases} \tag{B.4}$$

and ϕ^s is the volume fraction of the solid phase. This holds for any direction; hence, all the area fractions must be equal to the volume fraction for an RVE. A similar argument may be used for the fluid phase (cf. Moreland, 1972).

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