Finite deformation formulation for embedded frictional crack with the extended finite element method

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SUMMARY

We reformulate an extended finite element (FE) framework for embedded frictional cracks in elastoplastic solids to accommodate finite deformation, including finite stretching and rotation. For the FE representation, we consider a Galerkin approximation in which both the trial and weighting functions adapt to the current contact configuration. Contact and frictional constraints employ two Kuhn–Tucker conditions, a contact/separation constraint nesting over a stick/slip constraint for the case when the crack faces are in frictional sliding mode. We integrate finite deformation bulk plasticity into the formulation using the multiplicative decomposition technique of nonlinear continuum mechanics. We then present plane strain simulations demonstrating various aspects of the extended FE solutions. The mechanisms considered include combined opening and frictional sliding in initially straight, curved, and S-shaped cracks, with and without bulk plasticity. To gain further insight into the extended FE solutions, we perform mesh convergence studies focusing on both the global and the local responses of structures with cracks, including the distribution of the normal component of traction on the crack faces. Copyright © 2009 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Frictional sliding occurs in a variety of problems and is the core of the science and technology of interacting surfaces called tribology [1–3]. Frictional sliding can take place over large planar or curved surfaces on the scale of several kilometers, such as in geologic faults [4, 5]. It can also occur over very small, micron-scale contact surfaces encountered, for example, in powder assembly [6–8]. The surfaces of frictional contact are generally well defined in most boundary-value problems, although they could experience significant alteration from damage and erosion produced by continued rubbing and sliding of the surfaces [9]. In the case of high-pressure,

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high-velocity sliding encountered during seismic faulting, the contact surfaces could erode from frictional melting [10–13].

There exists a large class of problems where a crack, or fault, nucleates in an initially undamaged solid and propagates in an unspecified direction. In this case, the sliding surfaces are not initially well defined and the sides of the finite elements (FEs) cannot be aligned to these sliding surfaces to allow the use of conventional nonlinear contact mechanics techniques. Apart from mesh adaptivity that is traditionally used to simulate an evolving interface produced by an advancing crack, the extended FE method is now commonly used for crack propagation problems. The idea behind this method is that as the crack propagates anywhere through a fixed FE grid, the FEs sliced by the crack only need to be enriched with additional nodal degrees of freedom to accommodate the geometry of the advancing crack. The enrichments are determined along with the standard displacement degrees of freedom in one simultaneous global solution process.

Advances in the extended FE solution of crack propagation problems have focused largely on opening/closing modes, see e.g. [14–16]. By comparison, very little attention has been paid to addressing the problem of frictional sliding [17, 18]. Frictional contact has enormous implications in many areas of science and engineering. Frictional crack propagation generates additional complexity wherein the solution needs to satisfy two nested Kuhn–Tucker constraints: an outer constraint describing the contact/separation mode, and an inner constraint (for contact mode) describing the stick/slip condition. The stick/slip condition is not relevant for problems involving simple opening/closing modes, but it could complicate the frictional crack problem when the deformation is very large.

To embed a frictional contact into a solid, nonlinear contact mechanics algorithms can be used with the extended FE method. However, instead of working with node-to-segment contacts, the contact condition should now be tailored for FEs sliced by the crack. Dolbow *et al.* [18] proposed a solution strategy using a variational equation based on the total displacements on each face of the crack. They assumed the crack to be initially open, and iteratively ‘closed’ it to satisfy the unilateral contact and frictional constraints. Unfortunately, their solution is hampered by the very slow convergence rate of the iteration. Very recently, Borja [19] and Liu and Borja [20, 21] reformulated a variational equation in terms of the relative displacement of the crack faces and showed that improved efficiency can be achieved with Newton iteration. However, their work has been limited to infinitesimal deformation. The objective of the present paper is to address frictional crack problems in the context of large deformation.

Finite deformation theory is central to problems involving large stretching and rotation. Subsequent loading on geometrically deformed domains could influence not only the geometry of an existing crack but also the nucleation and propagation directions. Splay faulting in subduction zones is influenced by the stresses induced by tectonic compression and the kilometer-scale deformation occurring in these zones [22–25]. Large deformation effects could also explain the anomalous patterns of compaction bands observed in many geologic formations [26–28], as well as provide insights into the shape of surface folds observed in these formations [29]. In engineering applications, large deformation is a central issue in delamination [30], metal and polymer extrusions [31–33], and forming of ductile metal sheets [34, 35].

A coherent finite deformation theory for frictional crack propagation is yet to emerge in the context of the extended FE method. Wells *et al.* [30] formulated a finite deformation theory for delamination, which applies to opening mode (traction-free) cracks. Following the formulation in [30], Gasser and Holzaphel [36] formulated a finite deformation theory for cracking in concrete, including contact between the crack faces and tangential sliding, using three configurations: one

for each of the two sides of a crack separated by a strong discontinuity, and a third, ‘fictitious’ current crack configuration midway between the two initially overlapping crack faces where they enforced the material decohesion law. In so doing, they avoided a contact search, but their resulting algorithm is limited to small slips. For large frictional slips, the fictitious (midway) configuration is not physically meaningful in the sense that a meter-scale enhanced element, for example, may continue to interact at this fictitious configuration even though they may be kilometers apart in the current configuration. The law of friction states that a point on a crack interacts with the material it is currently in contact with, and not with the one it was previously attached to. It thus appears that a robust extended FE formulation for frictional crack necessitates a contact search in the finite deformation regime to account for the evolving geometry of contact.

In this paper we develop a finite deformation theory for frictional crack propagation in the framework of the extended FE method. The formulation follows essentially the same line of development as in our previous work. However, we now formulate the variational equation in Lagrangian description. By Lagrangian description, we mean that the crack is uniquely defined in some reference configuration (not necessarily undeformed), and its geometry in the current configuration is inferred from a push-forward transformation induced by the deformation gradient. We note, however, that the two Kuhn–Tucker conditions are imposed in the current configuration, which means that the two faces of a crack may not overlap and the shear traction may not exceed a maximum value in the current configuration. In contrast to the approach of Gasser and Holzaphel, we propose a formulation involving a contact search that allows us to capture slip due to frictional sliding that could be many times greater than the element dimensions. To this end, we employ the advances of nonlinear contact mechanics in the finite deformation regime, but now allow the crack to traverse the interior of the FEs. To achieve this, it is important to abandon the traditional notion of splitting the displacement field into continuous and discontinuous parts as used, for example, in local enrichment strategies such as the assumed enhanced strain method.

As large deformation typically generates an inelastic constitutive response, we also include bulk plasticity in the mathematical formulation. To this end, we use the multiplicative decomposition technique of nonlinear continuum mechanics that accommodates many popular plasticity models for natural and engineered materials [37–41]. Sources of nonlinearities handled by the model include: (a) geometric nonlinearity; (b) nonlinearity in the bulk material response; (c) nonlinearity in the frictional contact response; and (d) nonlinearity with unknown active constraints.

Finally, we present plane strain simulations demonstrating the various aspects of the formulation. The mechanisms considered include combined opening and frictional sliding in initially straight, curved, and S-shaped cracks, with and without bulk plasticity. In the first set of simulations, we consider a rectangular body with an edge crack subjected to vertical compression and perform mesh convergence studies. The second set of simulations involves crack propagation in hyperelastic solids and localized plasticity near the crack tip. In the third set of simulations, we consider an S-shaped frictional crack embedded in an elastoplastic solid subjected to very large deformation. In all of the simulations the cumulative slip is many times greater than the element dimensions.

2. PROBLEM STATEMENT

Following the developments in [19, 20, 30, 42], we consider a body $\mathcal{B}$ bounded externally by surface $\partial\mathcal{B}$ and internally by a pair of crack faces $\mathcal{S}_-$ and $\mathcal{S}_+$ in the reference configuration (the choice of $\mathcal{S}_-$ and $\mathcal{S}_+$ is *irrelevant* in the present case, as shown later with a numerical example).

Figure 1. Reference (left) and current (right) configurations for a solid with an embedded crack. Motion ϕ brings faces $\tilde{\mathcal{S}}_+$ and $\tilde{\mathcal{S}}_-$ of the crack to full frictional contact at $\phi(\tilde{\mathcal{S}}_+) = \phi(\tilde{\mathcal{S}}_-)$, while the remaining crack faces are subjected to opening mode.

Without loss of generality, we assume that the crack is initially closed so that $\mathcal{S}_- = \mathcal{S}_+ \equiv \mathcal{S}$. Furthermore, we denote the subdomains on the negative and positive sides of the crack by $\mathcal{B}_-$ and $\mathcal{B}_+$, respectively. We now consider a motion ϕ that brings the body $\mathcal{B}$ to the current configuration $\phi(\mathcal{B})$. This same motion brings segments $\tilde{\mathcal{S}}_- \subset \mathcal{S}_-$ and $\tilde{\mathcal{S}}_+ \subset \mathcal{S}_+$ of the crack faces to the frictional contact at $\phi(\tilde{\mathcal{S}}_-) = \phi(\tilde{\mathcal{S}}_+)$, see Figure 1. For simplicity we assume that the remainder of the contact faces, $\phi(\mathcal{S}_- \setminus \tilde{\mathcal{S}}_-)$ and $\phi(\mathcal{S}_+ \setminus \tilde{\mathcal{S}}_+)$, experience an opening mode and are free of traction.

The displacement field $\mathbf{u}(\mathbf{X})$ in the current configuration is conventionally decomposed as [43–45]

$$\mathbf{u}(\mathbf{X}) = \bar{\mathbf{u}}(\mathbf{X}) + M_{\mathcal{S}}(\mathbf{X})\tilde{\mathbf{u}}(\mathbf{X}), \quad (1)$$

where $\bar{\mathbf{u}}(\mathbf{X})$ and $M_{\mathcal{S}}(\mathbf{X})\tilde{\mathbf{u}}(\mathbf{X})$ represent the continuous and discontinuous parts of $\mathbf{u}(\mathbf{X})$, respectively, and $\mathbf{X}$ is the position vector of any material point $\mathbf{X} \in \mathcal{B}$. The scalar function $M_{\mathcal{S}}(\mathbf{X})$ is given by the equation

$$M_{\mathcal{S}}(\mathbf{X}) = H_{\mathcal{S}}(\mathbf{X}) - f^h(\mathbf{X}), \quad (2)$$

where $H_{\mathcal{S}}(\mathbf{X})$ is the Heaviside function defined by

$$H_{\mathcal{S}}(\mathbf{X}) = \begin{cases} 1, & \mathbf{X} \in \mathcal{B}_+ \\ 0, & \mathbf{X} \in \mathcal{B}_- \end{cases}, \quad (3)$$

and $f^h(\mathbf{X})$, called blending function in [19, 20], is any arbitrary smooth function that satisfies the requirements $f^h = 0$ in $\mathcal{B}_- \setminus \mathcal{B}_-^h$ and $f^h = 1$ in $\mathcal{B}_+ \setminus \mathcal{B}_+^h$. We also define the support of $M_{\mathcal{S}}$ as

$$\text{supp}(M_{\mathcal{S}}(\mathbf{X})) = \mathcal{B}^h = \mathcal{B}_-^h \cup \mathcal{B}_+^h. \quad (4)$$

The ‘jump,’ or discontinuity, in the displacement at $\mathbf{X}$ is

$$[[\mathbf{u}(\mathbf{X})]] = [[M_{\mathcal{S}}(\mathbf{X})\tilde{\mathbf{u}}(\mathbf{X})]] = \tilde{\mathbf{u}}(\mathbf{X}). \quad (5)$$

In the context of enriched FE methods, this kinematical description may be sufficient when the slip on the surface of discontinuity does not exceed the element dimensions. Later (in Section 3) we shall recognize that this may not be sufficient to describe the kinematics of deformation for very large slips, and thus we will propose an alternative decomposition appropriate for such conditions.

For the assumed form of displacement decomposition, the deformation map $\phi(X)$ is given by

$$\phi(X) := \mathbf{x}(X) = \mathbf{X} + \bar{\mathbf{u}}(X) + M_{\mathcal{S}}(X)\tilde{\mathbf{u}}(X). \quad (6)$$

For any material point $X \notin \mathcal{S}$, the deformation gradient is given by

$$\mathbf{F}(X, t) = \frac{\partial \phi}{\partial \mathbf{X}} = \mathbf{1} + \frac{\partial \bar{\mathbf{u}}(X)}{\partial \mathbf{X}} - \frac{\partial(f^h(X)\tilde{\mathbf{u}}(X))}{\partial \mathbf{X}} + H_{\mathcal{S}}(X) \frac{\partial \tilde{\mathbf{u}}(X)}{\partial \mathbf{X}}, \quad (7)$$

where $\mathbf{1}$ is the Kronecker delta tensor. The above expression holds even for material points on the crack faces, i.e. for $X_- \in \tilde{\mathcal{S}}_-$ and $X_+ \in \tilde{\mathcal{S}}_+$, provided that one evaluates the Heaviside function appropriately. The component with the Heaviside function represents the discontinuity, or ‘jump,’ of the deformation gradient on each side of the crack.

For completeness, we recall Nanson’s formula for the transformation of infinitesimal areas given by the relation

$$\mathbf{n} da = J \mathbf{F}^{-T} \cdot \mathbf{N} dA, \quad (8)$$

where $dA \in \mathcal{S}$ is an infinitesimal area with unit normal vector $\mathbf{N}$ in the reference configuration, $J = \det(\mathbf{F})$ is the jacobian determinant, and da and $\mathbf{n}$ are the respective push-forwards of dA and $\mathbf{N}$ in the current configuration. As noted earlier from (7), the deformation gradient $\mathbf{F}$ is discontinuous on each side of the crack. This implies that if the crack is closed in the reference configuration and either opens or displaces tangentially in the current configuration, the reference unit vector $\mathbf{N}$ and the associated differential area dA may be mapped to two different normal vectors $\mathbf{n}$ and differential areas da in the current configuration.

We assume quasi-static loading and write the governing equilibrium equations as follows:

$$\text{DIV}(\mathbf{P}) + \rho_0 \mathbf{G} = \mathbf{0} \quad \text{in } \mathcal{B} \setminus \mathcal{S}, \quad (9)$$

$$\mathbf{P} \cdot \mathbf{V} = \mathbf{t}_0 \quad \text{on } \partial \mathcal{B}_t, \quad (10)$$

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \partial \mathcal{B}_u, \quad (11)$$

where $\mathbf{P} = J \boldsymbol{\sigma} \cdot \mathbf{F}^{-T}$ is the unsymmetric first Piola–Kirchhoff stress tensor, $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor, ρ_0 is the mass density in the reference configuration, $\mathbf{G}$ is the vector of gravity accelerations, $\mathbf{V}$ is the outward unit vector to $\partial \mathcal{B}_t$ in the undeformed configuration, and $\mathbf{t}_0$ is the nominal traction vector acting on the Neumann boundary $\partial \mathcal{B}_t \subset \partial \mathcal{B}$. As usual, we assume that the remainder of the boundary, $\partial \mathcal{B}_u \subset \partial \mathcal{B}$, is of Dirichlet type where the surface displacements are given, and that $\overline{\partial \mathcal{B}_u} \cup \overline{\partial \mathcal{B}_t} = \overline{\partial \mathcal{B}}$ and $\overline{\partial \mathcal{B}_u} \cap \overline{\partial \mathcal{B}_t} = \emptyset$.

We augment the above equations with the following conditions in the presence of a crack. Recalling that $\tilde{\mathcal{S}}_- \subset \mathcal{S}_-$ and $\tilde{\mathcal{S}}_+ \subset \mathcal{S}_+$ are assumed to be in contact mode in the current configuration so that $\phi(\tilde{\mathcal{S}}_-) = \phi(\tilde{\mathcal{S}}_+)$, then we have

$$\mathbf{P}_- \cdot \mathbf{N}_- dA_- = \begin{cases} d\mathbf{f}_{\mathcal{S}_-} & \text{on } \tilde{\mathcal{S}}_- \\ \mathbf{0} & \text{on } \mathcal{S}_- \setminus \tilde{\mathcal{S}}_- \end{cases} \quad (12)$$

$$\mathbf{P}_+ \cdot \mathbf{N}_+ dA_+ = \begin{cases} d\mathbf{f}_{\mathcal{S}_+} & \text{on } \tilde{\mathcal{S}}_+ \\ \mathbf{0} & \text{on } \mathcal{S}_+ \setminus \tilde{\mathcal{S}}_+ \end{cases}, \quad (13)$$

and $\mathrm{d}\mathbf{f}_{\mathcal{S}_-} = -\mathrm{d}\mathbf{f}_{\mathcal{S}_+}$ at a common contact point. In the above equations, reference differential areas $N_- \mathrm{d}A_-$ and $N_+ \mathrm{d}A_+$ associated with the reference points $\mathbf{X}_- \neq \mathbf{X}_+$ are assumed to map to a common contact differential area $\pm \mathbf{n} \mathrm{d}a$ at point $\mathbf{x}$ in the current configuration. Equivalently, one can express the constraint on the contact surface in the current configuration using the Cauchy stress tensor $\boldsymbol{\sigma}$ as follows:

$$\mathbf{n} \cdot \boldsymbol{\sigma}_- = \begin{cases} \mathbf{t}_{\mathcal{S}_-} & \text{on } \phi(\tilde{\mathcal{S}}_-) \\ \mathbf{0} & \text{on } \phi(\mathcal{S}_- \setminus \tilde{\mathcal{S}}_-) \end{cases} \quad (14)$$

$$-\mathbf{n} \cdot \boldsymbol{\sigma}_+ = \begin{cases} \mathbf{t}_{\mathcal{S}_+} & \text{on } \phi(\tilde{\mathcal{S}}_+) \\ \mathbf{0} & \text{on } \phi(\mathcal{S}_+ \setminus \tilde{\mathcal{S}}_+) \end{cases}. \quad (15)$$

Continuity of the Cauchy traction vector then requires that $\mathbf{t}_{\mathcal{S}_-} = -\mathbf{t}_{\mathcal{S}_+}$, where $\mathbf{n}$ and $-\mathbf{n}$ are the push-forwards of N_- and N_+ , respectively, and $\boldsymbol{\sigma}_-$ and $\boldsymbol{\sigma}_+$ are the Cauchy stress tensors on each face of the crack. Note that while the nominal traction vector $\mathbf{t}_{\mathcal{S}_0}$ is discontinuous, the Cauchy traction vector $\mathbf{t}_{\mathcal{S}}$ is continuous across the surface of the crack.

3. VARIATIONAL FORMULATION

In developing the variational equation for the finite deformation frictional contact problem, it is important to recognize that the connectivity of the domain changes with the configuration. This requires that the incremental displacement field be defined with respect to the prevailing contact configuration. Consider then a domain in the current configuration $\phi(\mathcal{B} \setminus \mathcal{S})$ bounded externally by the surface $\phi(\partial \mathcal{B})$ and by the faces $\phi(\mathcal{S}_-)$ and $\phi(\mathcal{S}_+)$ of a crack. We define the set of trial functions $\mathcal{U}$ by

$$\mathcal{U} = \{u_i : \phi(\mathcal{B} \setminus \mathcal{S}) | u_i \in H^1, u_i = u_{0i} \text{ on } \phi(\partial \mathcal{B}_{u_i})\}, \quad (16)$$

and the set of variations $\mathcal{V}$ by

$$\mathcal{V} = \{\eta_i : \phi(\mathcal{B} \setminus \mathcal{S}) | \eta_i \in H^1, \eta_i = 0 \text{ on } \phi(\partial \mathcal{B}_{u_i})\}. \quad (17)$$

The variational equation for static linear momentum balance treating the crack as an external boundary is given by

$$\int_{\mathcal{B} \setminus \mathcal{S}} \text{GRAD}(\boldsymbol{\eta}) : \mathbf{P} \mathrm{d}V - \int_{\mathcal{B} \setminus \mathcal{S}} \boldsymbol{\eta} \cdot \rho_0 \mathbf{G} \mathrm{d}V - \int_{\partial \mathcal{B}_t} \boldsymbol{\eta} \cdot \mathbf{t}_0 \mathrm{d}A + \mathcal{G}_c(\boldsymbol{\eta}, \mathbf{u}) = 0, \quad (18)$$

where GRAD is the gradient with respect to $\mathbf{X}$, and

$$\mathcal{G}_c(\boldsymbol{\eta}, \mathbf{u}) = - \int_{\tilde{\mathcal{S}}_-} \boldsymbol{\eta} \cdot \mathrm{d}\mathbf{f}_{\mathcal{S}_-} - \int_{\tilde{\mathcal{S}}_+} \boldsymbol{\eta} \cdot \mathrm{d}\mathbf{f}_{\mathcal{S}_+} \quad (19)$$

is a contact surface integral representing the virtual work produced by the differential contact forces $\mathrm{d}\mathbf{f}_{\mathcal{S}_-}$ and $\mathrm{d}\mathbf{f}_{\mathcal{S}_+}$ acting on the faces of the crack. The contact forces are continuous and

so are the Cauchy traction vectors; thus, we can write the contact surface integrals in alternative forms as

$$\begin{aligned}\mathcal{G}_c(\boldsymbol{\eta}, \mathbf{u}) &= - \int_{\phi(\tilde{\mathcal{S}}_-)} \boldsymbol{\eta} \cdot \mathbf{t}_{\mathcal{S}_-} \, da - \int_{\phi(\tilde{\mathcal{S}}_+)} \boldsymbol{\eta} \cdot \mathbf{t}_{\mathcal{S}_+} \, da \\ &= - \int_{\phi(\tilde{\mathcal{S}}_-)} (\boldsymbol{\eta}_- - \boldsymbol{\eta}_+) \cdot \mathbf{t}_{\mathcal{S}_-} \, da \\ &= - \int_{\tilde{\mathcal{S}}_-} (\boldsymbol{\eta}_- - \boldsymbol{\eta}_+) \cdot \mathbf{t}_{\mathcal{S}0_-} \, dA_-, \end{aligned} \quad (20)$$

where $\boldsymbol{\eta}_-$ and $\boldsymbol{\eta}_+$ are the weighting functions evaluated on the crack faces $\phi(\tilde{\mathcal{S}}_-)$ and $\phi(\tilde{\mathcal{S}}_+)$, and $\mathbf{t}_{\mathcal{S}0_-} = \mathbf{P}_- \cdot \mathbf{N}_-$ is the nominal traction vector evaluated on the negative side of the crack face. The variations are defined for the current configuration; thus, the difference in the variations, $\boldsymbol{\eta}_- - \boldsymbol{\eta}_+$, reflects the evolving geometry of the problem domain.

Consider the discontinuous displacement field given in (1) and note that the discontinuity does not belong to the domain of interest, $\mathcal{B} \setminus \mathcal{S}$. Therefore, we can consider a class of variations having a similar form

$$\boldsymbol{\eta}(X) = \bar{\boldsymbol{\eta}}(X) + M_{\mathcal{S}}(X) \tilde{\boldsymbol{\eta}}(X). \quad (21)$$

Substituting into (18) and noting that $\bar{\boldsymbol{\eta}}(X)$ and $\tilde{\boldsymbol{\eta}}(X)$ are two independent variations yield two independent variational equations. The first variational equation is given by

$$\int_{\mathcal{B} \setminus \mathcal{S}} \text{GRAD}(\bar{\boldsymbol{\eta}}) : \mathbf{P} \, dV - \int_{\mathcal{B} \setminus \mathcal{S}} \bar{\boldsymbol{\eta}} \cdot \rho_0 \mathbf{G} \, dV - \int_{\partial \mathcal{B}_t} \bar{\boldsymbol{\eta}} \cdot \mathbf{t}_0 \, dA + \bar{\mathcal{G}}_c(\bar{\boldsymbol{\eta}}, \mathbf{u}) = 0, \quad (22)$$

where

$$\bar{\mathcal{G}}_c(\bar{\boldsymbol{\eta}}, \mathbf{u}) = - \int_{\tilde{\mathcal{S}}_-} (\bar{\boldsymbol{\eta}}_- - \bar{\boldsymbol{\eta}}_+) \cdot \mathbf{t}_{\mathcal{S}0_-} \, dA_- \quad (23)$$

is the contact surface integral emanating from the *continuous* part of the weighting function. Observe here that the term ‘continuous weighting function’ pertains to a materially continuous domain. As finite deformation generates a material discontinuity on the crack, the continuous part of the weighting function is *discontinuous* on the crack, i.e. $\bar{\boldsymbol{\eta}}_- \neq \bar{\boldsymbol{\eta}}_+$. In contrast, the infinitesimal formulation, which utilizes only one configuration, remains materially continuous even in the presence of a crack, and thus $\bar{\mathcal{G}}_c(\bar{\boldsymbol{\eta}}, \mathbf{u}) = 0$ in the infinitesimal case. In Section 4 we shall carry out this important distinction as we formulate the finite-slip frictional contact law on the surface of discontinuity.

Noting that the function $M_{\mathcal{S}}(X)$ is zero outside $\mathcal{B}^h$, the second variational equation may be evaluated over a reduced domain

$$\begin{aligned} & \int_{\mathcal{B}^h \setminus \mathcal{S}} \text{GRAD}(M_{\mathcal{S}}(X) \tilde{\boldsymbol{\eta}}(X)) : \mathbf{P} \, dV - \int_{\mathcal{B}^h \setminus \mathcal{S}} M_{\mathcal{S}}(X) \tilde{\boldsymbol{\eta}}(X) \cdot \rho_0 \mathbf{G} \, dV \\ & - \int_{\partial \mathcal{B}_t} M_{\mathcal{S}}(X) \tilde{\boldsymbol{\eta}}(X) \cdot \mathbf{t}_0 \, dA + \tilde{\mathcal{G}}_c(\tilde{\boldsymbol{\eta}}, \mathbf{u}) = 0, \end{aligned} \quad (24)$$

where

$$\tilde{\mathcal{G}}_c(\tilde{\boldsymbol{\eta}}, \mathbf{u}) = \int_{\tilde{\mathcal{S}}_-} [f_-^h \tilde{\boldsymbol{\eta}}_- + (1 - f_+^h) \tilde{\boldsymbol{\eta}}_+] \cdot \mathbf{t}_{\mathcal{S}0_-} dA_- \quad (25)$$

is the contact surface integral emanating from the *discontinuous* part of the weighting function. Again, finite deformation results in material discontinuity on the crack; hence, both f^h and $\tilde{\boldsymbol{\eta}}$ are discontinuous on the crack. In contrast, f^h and $\tilde{\boldsymbol{\eta}}$ are both continuous in the infinitesimal case, and therefore the contact surface integral simplifies to the form

$$\tilde{\mathcal{G}}_c(\tilde{\boldsymbol{\eta}}, \mathbf{u}) = \int_{\tilde{\mathcal{S}}_-} \tilde{\boldsymbol{\eta}} \cdot \mathbf{t}_{\mathcal{S}0_-} dA. \quad (26)$$

The above expression is the same as that developed by Liu and Borja [20] and Borja [19] in the previous work.

For the most part we will use the symmetric Kirchhoff stress tensor $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ for constitutive modeling of the solid domain response. The symmetric Kirchhoff stress tensor $\boldsymbol{\tau}$ is related to the first Piola–Kirchhoff stress tensor $\mathbf{P}$ through the relation

$$\boldsymbol{\tau} = \mathbf{P} \cdot \mathbf{F}^T. \quad (27)$$

With the above relation, the pair of variational equations can be written in the equivalent form

$$\int_{\mathcal{B} \setminus \mathcal{S}} \text{grad}(\tilde{\boldsymbol{\eta}}) : \boldsymbol{\tau} dV + \tilde{\mathcal{G}}_c(\tilde{\boldsymbol{\eta}}, \mathbf{u}) = \int_{\mathcal{B} \setminus \mathcal{S}} \tilde{\boldsymbol{\eta}} \cdot \rho_0 \mathbf{G} dV + \int_{\partial \mathcal{B}_t} \tilde{\boldsymbol{\eta}} \cdot \mathbf{t}_0 dA. \quad (28)$$

For the regular part, and

$$\int_{\mathcal{B}^h \setminus \mathcal{S}} \text{grad}(M_{\mathcal{S}} \tilde{\boldsymbol{\eta}}) : \boldsymbol{\tau} dV + \tilde{\mathcal{G}}_c(\tilde{\boldsymbol{\eta}}, \mathbf{u}) = \int_{\mathcal{B}^h \setminus \mathcal{S}} (M_{\mathcal{S}} \tilde{\boldsymbol{\eta}}) \cdot \rho_0 \mathbf{G} dV + \int_{\partial \mathcal{B}_t} (M_{\mathcal{S}} \tilde{\boldsymbol{\eta}}) \cdot \mathbf{t}_0 dA, \quad (29)$$

for the enhancement part. In the above equations, ‘grad’ is a spatial gradient operator denoting a gradient with respect to $\mathbf{x}$. Note that the first integrals are evaluated over the reference configuration $\mathcal{B} \setminus \mathcal{S}$ (or $\mathcal{B}^h \setminus \mathcal{S}$) although the inner product $\text{grad}(\circ) : \boldsymbol{\tau}$ is a spatial function.

4. FRICTIONAL CONTACT LAW

We follow the developments of nonlinear contact mechanics to address finite deformation frictional contact, see [29, 46]. For completeness, we outline the developments below in both rate and incremental forms.

4.1. General constitutive framework

Assume that $\mathbf{x}_-$ is a point on $\phi(\mathcal{S}_-)$ and $\mathbf{x}_+$ is its closest point on $\phi(\mathcal{S}_+)$, see Figure 2. Let $\mathbf{n}$ be the unit normal vector at $\mathbf{x}_-$ and pointing outward from $\phi(\mathcal{S}_-)$. The gap function at point $\mathbf{x}_-$ is given by

$$g_n = (\mathbf{x}_+ - \mathbf{x}_-) \cdot \mathbf{n}. \quad (30)$$

The condition that precludes penetration is $g_n \geq 0$, and the limit $g_n = 0$ defines the contact condition. At contact, the normal component of the Cauchy traction vector at $\mathbf{x}_-$ is $t_n = \mathbf{t}_{\mathcal{S}} \cdot \mathbf{n} \geq 0$, where the

Figure 2. Definition of gap function g_n .

inequality denotes a compressive normal stress when $\mathbf{x}_+$ presses against $\mathbf{x}_-$. Conditions for contact are thus given by the standard Kuhn–Tucker relations

$$g_n \geq 0, \quad t_n \leq 0, \quad g_n t_n = 0. \quad (31)$$

Let

$$\mathbf{t}_t = \mathbf{t}_{\mathcal{S}} - t_n \mathbf{n} \quad (32)$$

denote the tangential component of the Cauchy traction vector when the surfaces are in contact. Furthermore, denote the norm of $\mathbf{t}_t$ by

$$t_t = \|\mathbf{t}_t\|. \quad (33)$$

In the tangential direction, we distinguish between the stick and the slip conditions according to the Mohr–Coulomb frictional law

$$f = t_t + \mu t_n \leq 0, \quad (34)$$

where μ is the coefficient of friction on the contact surface. Stick condition is given by $f < 0$, whereas the slip condition is $f = 0$. The direction of plastic slip is determined by the flow rule

$$\dot{\mathbf{g}}_t = \dot{\lambda} \frac{\partial f}{\partial \mathbf{t}_t} = \dot{\lambda} \frac{\mathbf{t}_t}{\|\mathbf{t}_t\|}, \quad (35)$$

where $\dot{\lambda} \geq 0$ is a nonnegative plastic multiplier. In this case, the stick/slip conditions are once again defined by the Kuhn–Tucker relations

$$\dot{\lambda} \geq 0, \quad f \leq 0, \quad \dot{\lambda} f = 0. \quad (36)$$

The specific case of contact–slip condition is thus defined by two nested Kuhn–Tucker relations.

Consider a mapping in which dA_- in the reference configuration is mapped to da in the current configuration. By definition,

$$d\mathbf{f} = \mathbf{t}_{\mathcal{S}0} dA = \mathbf{t}_{\mathcal{S}} da, \quad (37)$$

where $\mathbf{t}_{\mathcal{S}0} \equiv \mathbf{t}_{\mathcal{S}0-}$ and $\mathbf{t}_{\mathcal{S}} = \mathbf{t}_{\mathcal{S}-}$ are, respectively, the nominal and Cauchy traction vectors on the negative face of the crack, and $dA \equiv dA_-$. It follows that the nominal traction vector can also be

decomposed into the normal and the tangential directions,

$$\mathbf{t}_{\mathcal{S}0} = t_N \mathbf{n} + t_T \mathbf{m}, \quad (38)$$

where

$$t_N = t_n \frac{da}{dA} \quad \text{and} \quad t_T = t_t \frac{da}{dA} \quad (39)$$

are, respectively, the normal and the shear forces in the current configuration per unit area in the reference configuration. We can view t_N and t_T as analogous to the elements of the (two-point) first Piola–Kirchhoff stress tensor $\mathbf{P}$.

In terms of the normal and tangential nominal traction components t_N and t_T , we can write a similar Mohr–Coulomb frictional law

$$\mathcal{F} = t_T + \mu t_N \leq 0. \quad (40)$$

Note that (34) and (40) are identical criteria provided that the mapping of the areas is unique. Letting $\mathbf{t}_T = t_T \mathbf{m}$, the plastic flow rule for slip may be written in the similar form as

$$\dot{\mathbf{g}}_T = \dot{\lambda} \frac{\partial \mathcal{F}}{\partial \mathbf{t}_T} = \dot{\lambda} \frac{\mathbf{t}_T}{\|\mathbf{t}_T\|}. \quad (41)$$

Similar Kuhn–Tucker conditions may also be written as follows:

$$\dot{\lambda} \geq 0, \quad \mathcal{F} \leq 0, \quad \dot{\lambda} \mathcal{F} = 0. \quad (42)$$

4.2. Rate formulation

Consider the same point $\mathbf{x}_- = \phi(\mathbf{X}_-)$ now on $\phi(\tilde{\mathcal{S}}_-)$ in contact with $\mathbf{x}_+ = \phi(\mathbf{X}_+)$ on $\phi(\tilde{\mathcal{S}}_+)$, where $\mathbf{x}_- = \mathbf{x}_+ \equiv \mathbf{x}$ by definition of contact condition. The instantaneous velocities of these two points are given by

$$\mathbf{v}_- = \bar{\mathbf{v}}_- + M_{\mathcal{S}}(\mathbf{X}_-) \tilde{\mathbf{v}}_-, \quad \mathbf{v}_+ = \bar{\mathbf{v}}_+ + M_{\mathcal{S}}(\mathbf{X}_+) \tilde{\mathbf{v}}_+, \quad (43)$$

where $\mathbf{v}_{\pm} = \mathbf{v}(\mathbf{X}_{\pm})$, etc. The instantaneous relative velocity is given by

$$\llbracket \mathbf{v}(\mathbf{x}) \rrbracket = \mathbf{v}_+ - \mathbf{v}_- = (\bar{\mathbf{v}}_+ - \bar{\mathbf{v}}_-) + \tilde{\mathbf{v}}_+ + f^h(\mathbf{X}_-) \tilde{\mathbf{v}}_- - f^h(\mathbf{X}_+) \tilde{\mathbf{v}}_+. \quad (44)$$

This expression for the jump in velocities (i.e. trial functions) is consistent with the expression for the jump in the variations considered in the previous section, and differs from the earlier expression (5) which is limited only to a materially continuous domain (i.e. slip less than element dimensions). Note that the relative velocity is a function of $\mathbf{x}$, the coordinates of the contact point, and not of any material point coordinates. However, it is always possible to map the contact point with a material point on either side of the crack, and here we select the mapping of $\mathbf{x}$ to $\mathbf{X}_-$. For brevity, we shall drop the negative subscript and take $\mathbf{X}$ to mean $\mathbf{X}_-$ unless otherwise noted. We recall that in the absence of material discontinuity (infinitesimal formulation), $\bar{\mathbf{v}}_+ = \bar{\mathbf{v}}_- \equiv \bar{\mathbf{v}}$, $f^h(\mathbf{X}_+) = f^h(\mathbf{X}_-) \equiv f^h(\mathbf{X})$, and $\tilde{\mathbf{v}}_+ = \tilde{\mathbf{v}}_- \equiv \tilde{\mathbf{v}}$, and thus $\llbracket \mathbf{v} \rrbracket = \tilde{\mathbf{v}}$. We also emphasize that the formulation is objective in the sense that one can choose either face of the crack as the negative side and obtain the same result.

The gap rate function is given by

$$\dot{g}_N(\mathbf{X}) = \llbracket \mathbf{v}(\mathbf{X}) \rrbracket \cdot \mathbf{n}(\mathbf{X}), \quad (45)$$

where $\mathbf{n}$ is the unit outward normal vector on $\phi(\tilde{\mathcal{S}}_-)$, which evolves according to Nanson's formula

$$\mathbf{n}(\mathbf{X}) = \frac{\mathbf{F}(\mathbf{X})^{-\top} \cdot \mathbf{N}(\mathbf{X})}{\|\mathbf{F}(\mathbf{X})^{-\top} \cdot \mathbf{N}(\mathbf{X})\|}. \quad (46)$$

The tangential component of velocity jump is

$$[\![\mathbf{v}(\mathbf{X})]\!]_{\text{T}} = [\![\mathbf{v}(\mathbf{X})]\!] - \dot{g}_{\text{N}}(\mathbf{X})\mathbf{n}(\mathbf{X}) = \dot{g}_{\text{T}}(\mathbf{X})\tilde{\mathbf{m}}(\mathbf{X}), \quad (47)$$

where $\dot{g}_{\text{T}}(\mathbf{X})$ is the instantaneous slip rate function in the tangent direction $\tilde{\mathbf{m}}(\mathbf{X})$, i.e.

$$\dot{g}_{\text{T}}(\mathbf{X}) = \|[\![\mathbf{v}(\mathbf{X})]\!]_{\text{T}}\|, \quad \tilde{\mathbf{m}}(\mathbf{X}) = [\![\mathbf{v}(\mathbf{X})]\!]_{\text{T}} / \dot{g}_{\text{T}}(\mathbf{X}). \quad (48)$$

We can thus write the velocity jump function on the crack in the complete form

$$[\![\mathbf{v}(\mathbf{X})]\!] = \dot{g}_{\text{N}}(\mathbf{X})\mathbf{n}(\mathbf{X}) + \dot{g}_{\text{T}}(\mathbf{X})\tilde{\mathbf{m}}(\mathbf{X}). \quad (49)$$

4.3. Incremental formulation

To evaluate the contact surface integral, it is necessary to develop a discrete evolution expression for the nominal traction vector $\mathbf{t}_{\mathcal{S}0}$. To this end, we consider some initially converged configuration $\phi_n(\mathcal{B})$ at which a material point X_- on the negative side of the crack is in contact with a point X^A on the positive side. We assume that at a current incremental load the crack faces slide from this converged configuration so that X_- is now in contact with another point X^B on the positive side of the crack, see Figure 3. Typically, we iterate to find a statically admissible configuration for the present incremental load, and here we assume that we are now working on the k th iteration toward the next statically admissible solution. The current positions (iterates) of the above material points at the k th iteration are

$$\mathbf{x}_- = \mathbf{X}_- + \mathbf{u}_-, \quad \mathbf{x}^A = \mathbf{X}^A + \mathbf{u}^A, \quad \mathbf{x}^B = \mathbf{X}^B + \mathbf{u}^B. \quad (50)$$

The contact condition is $\mathbf{x}_- = \mathbf{x}^B$, but in general this is not satisfied by the solution since the values of the penalty parameters are finite. This means that the negative and positive crack faces

Figure 3. Incremental evolution of frictional contact: material point X^A was initially in contact with X_- at $\mathbf{x}_- = \mathbf{x}^A$. After some incremental slip, material point X_- now finds itself in contact with a different material point X^B at $\mathbf{x}_- = \mathbf{x}^B$.

will not exactly coincide at the contact point, and there will be a small overlap. Dropping the negative subscript once again, this offset is calculated from the incremental version of (44) as

$$\Delta \mathbf{x} = \mathbf{x}^B - \mathbf{x} = \mathbf{X}^B + \bar{\mathbf{u}}^B + (1 - f^h(\mathbf{X}^B))\tilde{\mathbf{u}}^B - [\mathbf{X} + \bar{\mathbf{u}} - f^h(\mathbf{X})\tilde{\mathbf{u}}]. \quad (51)$$

The deformation gradient at contact point on the negative side is given by

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{1} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathbf{F}_n + \frac{\partial \Delta \mathbf{u}}{\partial \mathbf{X}} = \mathbf{f} \cdot \mathbf{F}_n, \quad (52)$$

where

$$\mathbf{F}_n = \frac{\partial \mathbf{x}_n}{\partial \mathbf{X}}, \quad \mathbf{f} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_n} = \mathbf{1} + \frac{\partial \Delta \mathbf{u}}{\partial \mathbf{x}_n}, \quad \Delta \mathbf{u} = \mathbf{u} - \mathbf{u}_n. \quad (53)$$

The total gap function is then given by

$$g_N(\mathbf{X}) = \Delta \mathbf{x} \cdot \mathbf{n}(\mathbf{X}), \quad (54)$$

while the normal component of nominal traction is obtained as

$$t_N(\mathbf{X}) = \varepsilon_N g_N(\mathbf{X}) \leq 0. \quad (55)$$

The tangential direction of the nominal traction vector is obtained from the relative slip of $\mathbf{X}_A$. The magnitude of incremental slip is given by

$$g_T = \|\Delta \mathbf{x}^{AB} - (\Delta \mathbf{x}^{AB} \cdot \mathbf{n})\mathbf{n}\|, \quad \Delta \mathbf{x}^{AB} = \mathbf{x}^A - \mathbf{x}^B, \quad (56)$$

while the instantaneous incremental slip direction is given by the unit tangent vector

$$\mathbf{m} = \frac{1}{g_T} [\Delta \mathbf{x}^{AB} - (\Delta \mathbf{x}^{AB} \cdot \mathbf{n})\mathbf{n}]. \quad (57)$$

From the hypothesis of slip occurring at $\mathbf{x}$, the frictional slip criterion (40) demands that

$$t_T(\mathbf{X}) = -\mu \varepsilon_N g_N \geq 0. \quad (58)$$

Hence, the final expression for the integrated nominal traction vector during incremental slip is given by

$$\mathbf{t}_{\mathcal{S}0} = \varepsilon_N g_N (\mathbf{n} - \mu \mathbf{m}). \quad (59)$$

Note that during incremental slip mode the nominal traction vector $\mathbf{t}_{\mathcal{S}0}$ is completely determined by the motions of $\mathbf{X}$, $\mathbf{X}^A$, and $\mathbf{X}^B$.

If the contact point has not slipped (stick mode), then $\mathbf{X}^A = \mathbf{X}^B$ and the discrete evolution of the nominal traction vector is given by

$$\mathbf{t}_{\mathcal{S}0} = \varepsilon_N g_N \mathbf{n} + \varepsilon_T \bar{g}_T \bar{\mathbf{m}}, \quad (60)$$

where

$$\bar{g}_T = \|\Delta \mathbf{x} - (\Delta \mathbf{x} \cdot \mathbf{n})\mathbf{n}\|, \quad \bar{\mathbf{m}} = \frac{\Delta \mathbf{x} - (\Delta \mathbf{x} \cdot \mathbf{n})\mathbf{n}}{\bar{g}_T} \quad (61)$$

define the tangential component of $\Delta \mathbf{x}$. As $\Delta \mathbf{x}$ generally is not parallel to $\Delta \mathbf{x}^{AB}$ in the presence of incremental slip, then $\mathbf{m}$ and $\bar{\mathbf{m}}$ generally are two different unit normal vectors.

5. FINITE ELEMENT EQUATIONS

This section develops the FE equations. An important aspect of the development is the enforcement of the contact condition in the current configuration.

5.1. Galerkin approximation

We construct finite-dimensional approximations of $\mathcal{U}$ and $\mathcal{V}$, denoted herein by the collection of functions $\mathcal{U}^h$ and $\mathcal{V}^h$. More specifically, we define

$$\mathcal{U}^h = \{u_i^h : \phi(\mathcal{B} \setminus \mathcal{S})|u_i^h \in H^1, u_i^h = u_{0i} \text{ on } \phi(\partial\mathcal{B}_{u_i})\} \quad (62)$$

and

$$\mathcal{V}^h = \{\eta_i^h : \phi(\mathcal{B} \setminus \mathcal{S})|\eta_i^h \in H^1, \eta_i^h = 0 \text{ on } \phi(\partial\mathcal{B}_{u_i})\}, \quad (63)$$

where $\phi(\mathcal{B} \setminus \mathcal{S})$ represents the current configuration. The Galerkin approximation considers the collection of functions $u_i^h \in \mathcal{U}^h$ given by

$$u_i^h = \eta_i^h + g_{0i}^h, \quad (64)$$

where g_{0i}^h is a given function satisfying the essential boundary condition $g_{0i}^h = u_{0i}$ on $\phi(\partial\mathcal{B}_{u_i})$. This construction means that $\mathcal{V}^h$ and $\mathcal{U}^h$ effectively contain identical collection of functions. Note that in the Galerkin approximation for the finite deformation frictional contact problem, the evolving state of contact impacts not only the approximation of $\mathbf{u}^h$ but also that of $\boldsymbol{\eta}^h$. Hence, the trial functions and variations must be iterated to determine the evolving state of contact.

Consider the approximations $u_i^h \in \mathcal{U}^h$ and $\eta_i^h \in \mathcal{V}^h$. The approximations to the variational equations are

$$\int_{\mathcal{B} \setminus \mathcal{S}} \text{grad}(\bar{\boldsymbol{\eta}}^h) : \boldsymbol{\tau} \, dV - \int_{\mathcal{B} \setminus \mathcal{S}} \bar{\boldsymbol{\eta}}^h \cdot \rho_0 \mathbf{G} \, dV - \int_{\partial\mathcal{B}_t} \bar{\boldsymbol{\eta}}^h \cdot \mathbf{t}_0 \, dA + \bar{\mathcal{G}}_c(\bar{\boldsymbol{\eta}}^h, \mathbf{u}^h) = 0 \quad (65)$$

and

$$\int_{\mathcal{B}^h \setminus \mathcal{S}} \text{grad}(M_{\mathcal{S}} \tilde{\boldsymbol{\eta}}^h) : \boldsymbol{\tau} \, dV - \int_{\mathcal{B}^h \setminus \mathcal{S}} M_{\mathcal{S}} \tilde{\boldsymbol{\eta}}^h \cdot \rho_0 \mathbf{G} \, dV - \int_{\partial\mathcal{B}_t} M_{\mathcal{S}} \tilde{\boldsymbol{\eta}}^h \cdot \mathbf{t}_0 \, dA + \tilde{\mathcal{G}}_c(\tilde{\boldsymbol{\eta}}^h, \mathbf{u}^h) = 0, \quad (66)$$

where

$$\bar{\mathcal{G}}_c(\bar{\boldsymbol{\eta}}^h, \mathbf{u}^h) = - \int_{\tilde{\mathcal{S}}_-} (\bar{\boldsymbol{\eta}}_-^h - \bar{\boldsymbol{\eta}}_+^h) \cdot \mathbf{t}_{\mathcal{S}0} \, dA \quad (67)$$

and

$$\tilde{\mathcal{G}}_c(\tilde{\boldsymbol{\eta}}^h, \mathbf{u}^h) = - \int_{\tilde{\mathcal{S}}_-} (M_{\mathcal{S}} \tilde{\boldsymbol{\eta}}_-^h - M_{\mathcal{S}} \tilde{\boldsymbol{\eta}}_+^h) \cdot \mathbf{t}_{\mathcal{S}0} \, dA. \quad (68)$$

It is implied that $\mathbf{t}_{\mathcal{S}0} = \mathbf{t}_{\mathcal{S}0-}$ and $dA = dA_-$.

A linearized form of the Galerkin approximation is useful to identify the consistent tangent operators. We recall the first variation

$$\delta \int_{\mathcal{B} \setminus \mathcal{S}} \text{grad}(\bar{\boldsymbol{\eta}}^h) : \boldsymbol{\tau} \, dV = \int_{\mathcal{B} \setminus \mathcal{S}} \text{grad}(\bar{\boldsymbol{\eta}}^h) : \delta \boldsymbol{\tau} \, dV + \int_{\mathcal{B} \setminus \mathcal{S}} \delta(\text{grad}(\bar{\boldsymbol{\eta}}^h)) : \boldsymbol{\tau} \, dV, \quad (69)$$

where

$$\delta\boldsymbol{\tau} = \boldsymbol{\alpha} : \text{grad}(\delta\mathbf{u}^h) = \boldsymbol{\alpha} : [\text{grad}(\delta\tilde{\mathbf{u}}^h) + \text{grad}(M_{\mathcal{S}}\delta\tilde{\mathbf{u}}^h)] \quad (70)$$

and

$$\delta(\text{grad}(\tilde{\boldsymbol{\eta}}^h)) = -\text{grad}(\tilde{\boldsymbol{\eta}}^h) \cdot \text{grad}(\delta\mathbf{u}^h), \quad (71)$$

and where $\boldsymbol{\alpha}$ is a non-symmetric rank-four tensor, see [37] for the properties of this tensor. The following tangent operator is useful for the undamaged part of the solid

$$\mathbf{a} = \boldsymbol{\alpha} - (\boldsymbol{\tau} \ominus \mathbf{1}), \quad (72)$$

where $(\boldsymbol{\tau} \ominus \mathbf{1})$ is a rank-four tensor with components $(\boldsymbol{\tau} \ominus \mathbf{1})_{ijkl} = \tau_{il}\delta_{jk}$.

For the nominal traction vector we recall that $\mathbf{t}_{\mathcal{S}0}$ is driven completely by the motion of material points X^A and X^B on the positive face, and material point X_- on the negative face of the crack (see Figure 3), hence we can write

$$\mathbf{t}_{\mathcal{S}0} = \mathbf{t}_{\mathcal{S}0}(\mathbf{u}^A, \mathbf{u}^B, \mathbf{u}_-). \quad (73)$$

The relevant tangent operators are defined from the linearized equation

$$\delta\mathbf{t}_{\mathcal{S}0} = \mathbf{E} \cdot (\delta\mathbf{u}^B - \delta\mathbf{u}_-) + \mathbf{S} \cdot (\delta\mathbf{u}^A - \delta\mathbf{u}^B) + \mathbf{W} : \text{grad}(\delta\mathbf{u}_-), \quad (74)$$

where $\mathbf{E}$ and $\mathbf{S}$ are the material tangent operators on the crack, and $\mathbf{W}$ accounts for the incremental spin of $\mathbf{n}$. Note that in the infinitesimal formulation, the variation of the traction vector depends only on the difference in the displacement variations, $(\delta\mathbf{u}^B - \delta\mathbf{u}_-)$, and hence, we only need to construct the tangent operator $\mathbf{E}$.

The linearized Galerkin approximation for dead loading ($\delta\rho_0 = 0$ and $\delta\mathbf{t}_0 = \mathbf{0}$) then takes the form

$$(\boldsymbol{\eta}^h, \delta\mathbf{u}^h)_{\mathbf{a}} - \langle (\boldsymbol{\eta}_-^h - \boldsymbol{\eta}_+^h), \delta\mathbf{t}_{\mathcal{S}0} \rangle_{\tilde{\mathcal{S}}_-} = 0, \quad (75)$$

where

$$(\circ, \bullet)_{\mathbf{a}} = \int_{\mathcal{B} \setminus \mathcal{S}} \text{grad}(\circ) : \mathbf{a} : \text{grad}(\bullet) \, dV, \quad (76)$$

$$\langle \circ, \bullet \rangle_{\tilde{\mathcal{S}}_-} = \int_{\tilde{\mathcal{S}}_-} (\circ) \cdot (\bullet) \, dA. \quad (77)$$

As the variations and trial functions are evaluated in the same (current) configuration, the Galerkin approximation preserves the symmetry of the variational structure.

5.2. FE equations

Before developing the FE equations, we first define the term ‘materially continuous’ in the context of the extended FE formulation. Consider a contact point $\mathbf{x} = \mathbf{x}_+ = \mathbf{x}_-$ with possibly distinct pullbacks X_+ and X_- in the reference configuration, respectively. If the associated material points X_- , $X_+ \in \Omega^e$, then we say that the crack is materially continuous at $\mathbf{x}$. In this case, the interpolations for $\bar{\mathbf{u}}^h$ and $\tilde{\mathbf{u}}^h$ are the same at $\mathbf{x}$. Note that this is a discrete (i.e. relaxed) definition of material continuity.

Figure 4. Illustration of material discontinuity in the discrete sense. After frictional sliding, elements 1 and 2 remain materially continuous at points denoted by the circular ticks, and materially discontinuous at points denoted by the square ticks. On the other hand, elements 3 and 4 are materially discontinuous everywhere. Sub-triangles (not shown) are employed for numerical integration.

Figure 4 illustrates the notion of discrete material continuity for constant strain triangular (CST) elements sliced by a crack.

The continuous part of the displacement field, $\bar{\mathbf{u}}(\mathbf{X})$, where $\mathbf{X} \in \mathcal{B}$, is approximated by

$$\bar{\mathbf{u}}^h(\mathbf{X}) = \sum_{I \in N_{\text{nodes}}} N_I(\xi) \mathbf{d}_I = \mathbf{N}(\xi) \mathbf{d}, \quad \xi = \xi(\mathbf{X}), \quad (78)$$

where $\mathbf{N}$ is the matrix of displacement shape functions, $\mathbf{d}$ is the vector of standard nodal displacements, and N_{nodes} is the number of nodes. We recall that the elements of the matrix $\mathbf{N}$ are typically given in natural coordinates ξ mapped either to coordinates $\mathbf{X}$ in the reference configuration or to coordinates $\mathbf{x} = \phi(\mathbf{X})$ in the current configuration. The displacement field $\tilde{\mathbf{u}}(\mathbf{X})$ is approximated by the same standard shape function, i.e.

$$\tilde{\mathbf{u}}^h(\mathbf{X}) = \sum_{I \in N_{\text{enr}}} N_I(\xi) \mathbf{a}_I, \quad \xi = \xi(\mathbf{X}), \quad (79)$$

where N_{enr} is the number of enriched nodes and $\mathbf{a}$ is the vector of nodal enrichments. The discontinuous part of the displacement field, $M_{\mathcal{G}}(\mathbf{X}) \tilde{\mathbf{u}}(\mathbf{X})$, is approximated similarly,

$$M_{\mathcal{G}}(\mathbf{X}) \tilde{\mathbf{u}}^h(\mathbf{X}) = \sum_{I \in N_{\text{enr}}} N_I(\mathbf{X}) [H_{\mathcal{G}}(\mathbf{X}) - H_{\mathcal{G}}(\mathbf{X}_I)] \mathbf{a}_I = \tilde{\mathbf{N}}(\mathbf{X}) \mathbf{a}. \quad (80)$$

We refer to [19, 20] for similar expressions developed for the infinitesimal case.

The FE matrix equation consistent with the Galerkin approximation (65) is (with a slight abuse in the notation by using $\boldsymbol{\tau}$ for both the vector and the tensor of Kirchhoff stresses)

$$\mathbf{F}_{\text{INT}}(\mathbf{d}, \mathbf{a}) = \mathbf{F}_{\text{EXT}}, \quad (81)$$

where

$$\mathbf{F}_{\text{INT}}(\mathbf{d}, \mathbf{a}) = \int_{\mathcal{B} \setminus \mathcal{G}} \mathbf{B}^T \boldsymbol{\tau}(\mathbf{d}, \mathbf{a}) dV - \int_{\tilde{\mathcal{G}}_-} (\mathbf{N}_-^T - \mathbf{N}_+^T) \mathbf{t}_{\mathcal{G}0}(\mathbf{d}, \mathbf{a}) dA, \quad (82)$$

$$\mathbf{F}_{\text{EXT}} = \int_{\mathcal{B} \setminus \mathcal{G}} \mathbf{N}^T \rho_0 \mathbf{G} dV + \int_{\partial \mathcal{B}_t} \mathbf{N}^T \mathbf{t}_0 dA. \quad (83)$$

In the above equation, $\mathbf{B} = \nabla_{\mathbf{x}} \mathbf{N}(\boldsymbol{\xi})$ is the gradient-displacement matrix, and $\mathbf{N}_-, \mathbf{N}_+$ are the shape function matrices on the opposite faces of the crack. Note that $\mathbf{B}$ takes the gradient of the standard displacement shape functions with respect to the spatial coordinates $\mathbf{x}$; however, the integral is evaluated over the reference configuration $\mathcal{B} \setminus \mathcal{S}$. Note also that $\mathbf{B}$ is not a symmetric operator in keeping with finite rotation in nonlinear continuum mechanics formulation.

The FE equation consistent with the Galerkin approximation (66) is

$$\mathcal{F}_{\text{INT}}(\mathbf{d}, \mathbf{a}) = \mathcal{F}_{\text{EXT}}, \quad (84)$$

where

$$\mathcal{F}_{\text{INT}}(\mathbf{d}, \mathbf{a}) = \int_{\mathcal{B}^h \setminus \mathcal{S}} \tilde{\mathbf{B}}^T \boldsymbol{\tau}(\mathbf{d}, \mathbf{a}) \, dV - \int_{\tilde{\mathcal{S}}_-} (\tilde{\mathbf{N}}_-^T - \tilde{\mathbf{N}}_+^T) \mathbf{t}_{\mathcal{S}0}(\mathbf{d}, \mathbf{a}) \, dA, \quad (85)$$

$$\mathcal{F}_{\text{EXT}} = \int_{\mathcal{B}^h \setminus \mathcal{S}} \tilde{\mathbf{N}}^T \rho_0 \mathbf{G} \, dV + \int_{\partial \mathcal{B}_t} \tilde{\mathbf{N}}^T \mathbf{t}_0 \, dA. \quad (86)$$

In the above equation, $\tilde{\mathbf{B}} = \nabla_{\boldsymbol{\xi}} \tilde{\mathbf{N}}(\boldsymbol{\xi})$, where the natural coordinates are now mapped to the current configuration according to the relation $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$.

5.3. Consistent tangent operators

It is necessary to construct consistent tangent operators to preserve the optimal rate of convergence of Newton–Raphson iterations. In this section we elucidate the general structure of such tangent operators for the finite deformation frictional contact problem via the extended FE method. We emphasize that the evaluation of the tangent matrices is done from the usual assembly of individual element contributions. The objective of this section is simply to show the general structure of such global tangent matrices and not the detail of the assembly operations.

Consider the following FE interpolations for the displacements at $\mathbf{x}_-$ and $\mathbf{x}_+$:

$$\mathbf{u}_-^h(\Omega^A) = \mathbf{N}^A \mathbf{d}^A + \tilde{\mathbf{N}}_-^A \mathbf{a}^A, \quad (87)$$

$$\mathbf{u}_+^h(\Omega^B) = \mathbf{N}^B \mathbf{d}^B + \tilde{\mathbf{N}}_+^B \mathbf{a}^B, \quad (88)$$

where $\mathbf{N}^I$, $\tilde{\mathbf{N}}_{\pm}^I$, $\mathbf{d}^I$, and $\mathbf{a}^I$ are the element shape functions and nodal displacement vectors for element I . Note that the element shape function matrix $\mathbf{N}^I$ is continuous on the crack when $A = B$ (i.e. the crack is materially continuous) but is discontinuous when $A \neq B$. However, the element shape function matrix $\tilde{\mathbf{N}}_{\pm}^I$ is always discontinuous on the crack.

Consider the internal nodal force $\mathbf{F}_{\text{INT}}(\mathbf{d}, \mathbf{a})$. The consistent tangent operators have the form

$$\begin{aligned} \frac{\partial \mathbf{F}_{\text{INT}}}{\partial \mathbf{d}} &= \int_{\mathcal{B}^h \setminus \mathcal{S}} \mathbf{B}^T \mathbf{A} \mathbf{B} \, dV + \int_{\tilde{\mathcal{S}}_-} (\mathbf{N}_+^T - \mathbf{N}_-^T) \mathbf{E} (\mathbf{N}_+ - \mathbf{N}_-) \, dA \\ &\quad + \int_{\tilde{\mathcal{S}}_-} (\mathbf{N}_+^T - \mathbf{N}_-^T) \mathbf{S} \Delta \mathbf{N}_+ \, dA + \int_{\tilde{\mathcal{S}}_-} (\mathbf{N}_+^T - \mathbf{N}_-^T) \mathbf{W} \mathbf{B}_- \, dA \end{aligned} \quad (89)$$

and

$$\begin{aligned} \frac{\partial \mathbf{F}_{\text{INT}}}{\partial \mathbf{a}} = & \int_{\mathcal{B}^h \setminus \mathcal{S}} \mathbf{B}^T \mathbf{A} \tilde{\mathbf{B}} dV + \int_{\tilde{\mathcal{S}}_-} (\mathbf{N}_+^T - \mathbf{N}_-^T) \mathbf{E} (\tilde{\mathbf{N}}_+ - \tilde{\mathbf{N}}_-) dA \\ & + \int_{\tilde{\mathcal{S}}_-} (\mathbf{N}_+^T - \mathbf{N}_-^T) \mathbf{S} \Delta \tilde{\mathbf{N}}_+ dA + \int_{\tilde{\mathcal{S}}_-} (\mathbf{N}_+^T - \mathbf{N}_-^T) \mathbf{W} \tilde{\mathbf{B}}_- dA, \end{aligned} \quad (90)$$

where $\mathbf{A}$, $\mathbf{E}$, $\mathbf{S}$, and $\mathbf{W}$ are matrix forms of tensors $\mathbf{a}$, $\mathbf{E}$, $\mathbf{S}$, and $\mathbf{W}$, respectively, while $\Delta \mathbf{N}_+$ and $\Delta \tilde{\mathbf{N}}_+$ are the incremental shape function matrices interpolating slip of material point X^A relative to X^B on the positive face of the crack (see Figure 3). The surface integral involving $\mathbf{W}$ is a consistent linearization of the evolving contact surface unit normal $\mathbf{n}$ on the negative face of the crack.

Similar tangent matrices may be derived for the enhancement part of the FE equations, and for completeness we outline the two additional tangent operators as follows:

$$\begin{aligned} \frac{\partial \mathcal{F}_{\text{INT}}}{\partial \mathbf{d}} = & \int_{\mathcal{B}^h \setminus \mathcal{S}} \tilde{\mathbf{B}}^T \mathbf{A} \mathbf{B} dV + \int_{\tilde{\mathcal{S}}_-} (\tilde{\mathbf{N}}_+^T - \tilde{\mathbf{N}}_-^T) \mathbf{E} (\mathbf{N}_+ - \mathbf{N}_-) dA \\ & + \int_{\tilde{\mathcal{S}}_-} (\tilde{\mathbf{N}}_+^T - \tilde{\mathbf{N}}_-^T) \mathbf{S} \Delta \mathbf{N}_+ dA + \int_{\tilde{\mathcal{S}}_-} (\tilde{\mathbf{N}}_+^T - \tilde{\mathbf{N}}_-^T) \mathbf{W} \mathbf{B}_- dA \end{aligned} \quad (91)$$

and

$$\begin{aligned} \frac{\partial \mathcal{F}_{\text{INT}}}{\partial \mathbf{a}} = & \int_{\mathcal{B}^h \setminus \mathcal{S}} \tilde{\mathbf{B}}^T \mathbf{A} \tilde{\mathbf{B}} dV + \int_{\tilde{\mathcal{S}}_-} (\tilde{\mathbf{N}}_+^T - \tilde{\mathbf{N}}_-^T) \mathbf{E} (\tilde{\mathbf{N}}_+ - \tilde{\mathbf{N}}_-) dA \\ & + \int_{\tilde{\mathcal{S}}_-} (\tilde{\mathbf{N}}_+^T - \tilde{\mathbf{N}}_-^T) \mathbf{S} \Delta \tilde{\mathbf{N}}_+ dA + \int_{\tilde{\mathcal{S}}_-} (\tilde{\mathbf{N}}_+^T - \tilde{\mathbf{N}}_-^T) \mathbf{W} \tilde{\mathbf{B}}_- dA. \end{aligned} \quad (92)$$

6. NUMERICAL EXAMPLES

6.1. Mesh convergence study: hyperelastic solid

For the mesh convergence study we consider a rectangular block of dimensions $0.08\text{ m} \times 0.04\text{ m}$ clamped at both its top and bottom faces and containing a 45° edge crack. Placing the bottom left corner of the block at coordinates $(0, 0)$, the crack edge is located at $(0.0, 0.01833)$ and the tip is at coordinates $(0.0349, 0.05323)$. We assume that the block to be isotropic and hyperelastic with Young's modulus $E = 10000\text{ MPa}$ and Poisson's ratio $\nu = 0.25$ (see [40] on how we used these hyperelastic material parameters); and the crack to be purely frictional with a constant coefficient of friction $\mu = 0.30$. The normal and tangential penalty parameters are $\varepsilon_N = \varepsilon_T = 10^6 h / h_0 \text{ MN/m}^3$, where h is a characteristic length scale for the mesh with reference value $h_0 = 0.005\text{ m}$. In principle, the functional dependence of the penalty parameters with the mesh length scale should not be significant and may be neglected for sufficiently large values of the penalty parameters. However, they have the physical meaning of spring constants, thus we consider their explicit functional dependence on h for the present mesh convergence study. In this example the crack tip is not allowed to advance.

Figure 5. Undeformed meshes for plane strain compression of rectangular block with a 45° edge crack. From left to right: Mesh 1 (153 nodes, 256 elements), Mesh 2 (561 nodes, 1024 elements), Mesh 3 (2145 nodes, 4096 elements), and Mesh 4 (8385 nodes, 16 384 elements).

Four FE meshes consisting of CST elements are considered in this study, shown in Figure 5. The driving force is a vertical downward movement of the top boundary by 0.01 m combined with a leftward movement of the same boundary by 0.01 m, while holding the bottom boundary fixed. This caused the body to slide on the crack and create traction-free surfaces from the sliding faces. Figure 6 shows the deformed meshes generated by the solution at the conclusion of the simulations. Note that as the mesh is refined the opening mode in the neighborhood of the crack tip progressively resolves into a smooth traction-free surface characteristic of a blunt crack, leaving only a shorter segment of the crack in direct frictional contact. We can thus expect some global softening in the structural response as a result of the reduced length of the crack in direct frictional contact.

Figure 7 shows the total horizontal and vertical support reactions calculated by the proposed method. As noted earlier, the reduced frictional surface area causes a global softening response, which is more apparent for the horizontal support reactions. The softening response in the vertical reaction is somewhat offset by the vertical compression of the right stem in front of the crack, which has a competing hardening effect.

Figure 8 shows the normal component of the Cauchy traction vector acting on the top and bottom faces of the crack halfway through and at the end of the simulations. As expected, the contact pressure is smaller and the contact area is larger during the early stage of deformation. However, as the deformation becomes larger and free surfaces form, the contact area becomes smaller and the contact pressure becomes more intense. In general, the pressure distribution oscillates with coarser meshes, but this oscillation tends to go away with finer meshes. Such oscillation is largely a result of a non-smooth approximation of the crack face geometry, which goes away as the mesh is refined.

An important aspect of the finite deformation solution is its ability to detect the relative stretching of the top and bottom faces of the crack. This is illustrated in Figure 9, which shows the contact pressure distributions at 50% total imposed boundary displacement. The inset shows lower crack segment AB in direct frictional contact with upper crack segment $A'B'$ in the current configuration. A pull-back to the reference configuration locates point A on the right vertical boundary and point

Figure 6. Deformed meshes for plane strain compression of rectangular block with a 45° edge crack: Galerkin solution.

A' in the interior of the block. In Figure 9 the contact pressure distribution for the upper face of the crack is shifted to the left so that the point A' falls on the origin of the reference coordinate frame. We see that the two curves do not fall one on top of the other. Instead, the span of the top-face curve in the reference configuration is slightly shorter, which means that relative to the reference configuration the top face of the crack has stretched more than the bottom face.

Earlier we noted that the mathematical formulation is objective with respect to the choice of $\mathcal{S}_-$ and $\mathcal{S}_+$, i.e. one can choose either side of the crack as the negative face and still obtain the same result. Figure 10 affirms this statement as the spatial variations of normal component of Cauchy traction vectors are plotted and compared for two sets of numerical simulations: the first in which the upper face of the crack was chosen as the negative side ('master upper'), and the second in which the lower face was chosen as the negative side ('master lower'). Except for some minor difference due to spatial discretization of the crack faces, the curves are one on the top of the other, particularly with Mesh 4 where no noticeable difference can be observed.

Figure 7. Plane strain compression of rectangular block with a 45° edge crack: horizontal and vertical support reactions (Force X and Force Y, respectively).

We should note that the normal component of the traction vector is the state variable most affected by the unit normal vector to the crack face. If we had plotted the displacements of the structure, they would be identical to plotting the accuracy. Finally, Figure 11 shows the convergence profiles of Newton–Raphson iterations from the simulations utilizing Mesh 2. With the use of a consistent tangent operator, the iterations demonstrate optimal (asymptotically quadratic) convergence.

6.2. Crack propagation in hyperelastic solid

Advancing a crack requires a criterion for propagation that generally depends on the state of stress prevailing in the neighborhood of the crack tip. In this example we propagate a crack in a hyperelastic solid experiencing large deformation. We used the Rankine criterion to predict the crack propagation. According to this criterion failure initiates when the maximum principal Cauchy stress $\sigma_{\max}$ exceeds a material-dependent stress limit, and the normal vector $\mathbf{n}$ of the advancing failure surface may be determined by the direction of the maximum principal stress. To enable us to track the crack path in the reference configuration, we perform a pull-back of $\mathbf{n}$ in the reference configuration via Nanson's formula,

$$\mathbf{N} = \mathbf{n} \cdot \mathbf{F} / \|\mathbf{n} \cdot \mathbf{F}\|. \quad (93)$$

As the normal vector field near the crack tip is highly scattered, we evaluate the near-tip stress using the averaging procedure [47–49]

$$\bar{\sigma} = \frac{\sum_{R_i < R_c} \sigma_i A_i}{\sum_{R_i < R_c} A_i}, \quad (94)$$

where R_i is the distance of the center of element i to the crack tip in the reference configuration, A_i is the area of element A_i , and R_c is a user-specified parameter that depends on factors such as

Figure 8. Plane strain compression of rectangular block with a 45° edge crack: contact pressure at 50% and 100% of total imposed deformation. TOP is the top face of the crack and BOT is the bottom face.

the element size, which is chosen as small as possible but large enough to avoid the scatter of the normal vector field (R_c is often assigned values in the range of 2–4 times a characteristic element size to achieve reliable numerical results).

We now examine a $1\text{ m} \times 1\text{ m}$ square domain with an initial diagonal crack inside shown in Figure 12. The initial coordinates of the crack tips are $[0.298, 0.297]$ and $[0.702, 0.701]$. The solid is assumed hyperelastic with Young's modulus $E = 10\text{ GPa}$ and Poisson's ratio $\nu = 0.3$. The crack growth is $da = 0.065$ at each step and $R_c = 0.05$. We used a mesh with 676 nodes and 1250 CST elements. The bottom boundary is fixed, the top boundary is loaded by a uniform vertical displacement of 0.03 m , and the rest of the boundaries are assumed to be traction free. The tensile strength of the bulk material is taken as 50 MPa . The coefficient of friction for the interface is $\mu = 0.1$; the penalty parameters are $\varepsilon_N = \varepsilon_T = 10^6\text{ MN/m}^3$.

Figure 12 shows a four-step crack propagation through a fixed FE grid and suggests that the orientation of the advancing crack continually changes with the prevailing stress field at the moment of rupture. In this simulation the diagonal crack is initially stationary until the wing cracks

Figure 9. Plane strain compression of rectangular block with a 45° edge crack: relative stretching of the top and bottom faces of the crack at 50% total deformation using Mesh 4.

Figure 10. Objectivity of the formulation with respect to the choice of the negative crack face: save for minor errors due to spatial discretization of the crack faces, the normal components of the Cauchy traction vector are the same irrespective of the choice of $\mathcal{S}_-$ and $\mathcal{S}_+$. Upper figure: simulations with Mesh 2; lower figure: simulations with Mesh 4.

propagate in four steps, after which the tips are prevented from advancing further. As deformation progresses, the old crack changes orientation, which in turn influences the prevailing near-tip stress field. The wing crack experiences an opening mode, and frictional contact is restricted to the initial diagonal crack. This justifies the adopted crack propagation criterion. The total vertical reaction

Figure 11. Typical convergence profiles of Newton–Raphson iterations for plane strain compression of rectangular block with a 45° edge crack.

Figure 12. Displacement contours for a four-step wing crack propagation.

acting on the upper boundary is plotted against the vertical displacement in Figure 13 and shows the four-step crack propagation phase sandwiched between two stationary crack analyses. The overall stiffness of the plate degrades slightly during the crack propagation phase.

6.3. Inelastic deformation

The region around a crack trip is a zone of intense deformation where yielding occurs even with a modest amount of load. Theoretically, the strain is asymptotically infinite as the stress point

Figure 13. Reaction force versus displacement for a four-step wing crack propagation.

approaches the crack tip, and where the strain is large enough a ‘small-scale’ yield zone forms around the crack tip. As loading continues, the yield zone expands and propagates away from the crack tip to cover a much larger region, which may include points where the singular component of the strain field is no longer dominant. In this paper, we refer to such condition as ‘large-scale’ yielding. The following example shows how an expanding yield zone around a crack tip may be modeled using the proposed formulation.

We consider the same rectangular block with a 45° edge crack described in Section 6.1, but now assume that the bulk solid exhibits an elastic-perfectly plastic response. We assume the following elastic material properties: Young’s modulus $E = 10000 \text{ MPa}$ and Poisson’s ratio $\nu = 0.25$. Yielding is assumed to obey a non-associated Drucker–Prager yield criterion with yield surface F and plastic potential function G given by the expressions

$$F = \|s\| - (\alpha - \beta p) = 0, \quad Q = \|s\| + bp, \quad (95)$$

where $s = \sigma - p\mathbf{1}$ is the deviatoric component of the Cauchy stress tensor σ , $p = \text{tr}(\sigma)/3$ is the Cauchy mean normal stress, $\alpha = 1500 \text{ MPa}$ is the equivalent bulk cohesion parameter of the solid, $\beta = 0.8$ is the equivalent bulk friction parameter, and $b = 0.7$ the equivalent bulk dilatancy parameter. In the finite deformation regime, the symmetric Kirchhoff stress tensor $\tau = J\sigma$ replaces the Cauchy stress tensor σ in the above definitions of the yield and plastic potential functions.

We now impose vertical and horizontal top boundary displacements of -0.002 m . Although this movement is small enough to warrant a small-strain analysis, it still produces a singular strain field at the crack tip. In order to resolve this singularity, we introduce the following enrichment functions appropriate for near-tip displacements under combined Modes I and II loading as derived

from linear elastic fracture mechanics [50–53]

$$\begin{aligned}\gamma_i(r, \theta) &= \sqrt{r} f_i, \quad f_1 = \cos\left(\frac{\theta}{2}\right), \quad f_2 = \sin\left(\frac{\theta}{2}\right), \\ f_3 &= \sin\left(\frac{\theta}{2}\right) \sin \theta, \quad f_4 = \cos\left(\frac{\theta}{2}\right) \sin \theta,\end{aligned}\tag{96}$$

where θ ranges from $-\pi$ to $+\pi$, and with $\theta=0$ representing the crack line pointing away from the tip. Strictly speaking, the above near-tip enrichment functions are valid only for isotropic linearly elastic response. For elastoplastic response the near-tip enrichment function depends not only on the plasticity model but also on the imposed loading direction. Derivation of an exact enrichment function for the elastoplastic case is not trivial even for simple loading cases, and in this work we thus consider the above expressions only as an approximation in the elastoplastic regime. In this work we apply the near-tip enrichment to all FEs contained in a ball of radius R (taken as 0.003 m in this example) such that the yield zone is fully contained in this ball.

Figure 14 shows the resulting small-scale plastic yield zone surrounding the crack tip, along with the enrichment ball as represented by the red circle. Yielding is evident on both sides of the crack tip although plastic deformation appears to be more intense on the upper face of the crack. The contour of determinant function in Figure 14, used by Rudnicki and Rice [54] to predict the onset of a shear band, suggests a potential wing crack initiation upward from the crack tip. Note that the determinant function is all negative throughout the yield zone because of the intense deformation prevailing around the crack tip. With a singular near-tip strain field it would not be possible to identify when this determinant function vanished for the first time. However, it is reasonable to assume that the shear band would likely initiate in the region with the most negative value of the localization function.

Large-scale yielding occurs when the structure is loaded further and the yield zone propagates at distances away from the crack tip where the singular component of the strain field no longer dominates. To illustrate the effect of large deformation on the shape of the yield zone, we perform two elastoplastic analyses: one employing an infinitesimal formulation and the other employing a finite deformation formulation. Figure 15 shows the yield zone for the infinitesimal deformation solution when the vertical and horizontal displacements of the top boundary have reached the value -0.01 m. In this analysis, we have utilized Mesh 2 and a Heaviside enrichment, without a crack tip enrichment. The plastic zone is most intense near the crack tip and spreads to the neighboring area. However, the shape of the large-scale plastic yield zone is quite different from the small-scale yield zone shown in Figure 14. More importantly, if the determinant function of Rudnicki and Rice is to be used as a measure of the potential of a crack to advance, then Figure 15 suggests that the infinitesimal formulation without a crack tip enhancement would predict a crack tip advancing away in opposite directions perpendicular to the crack line.

For the same imposed boundary displacement and using the same Mesh 2, Figure 16 shows the yield zone and localization function obtained from the finite deformation simulation employing the standard multiplicative plasticity theory. In contrast to the small strain simulation, plasticity is now seen to be localized to the top face of the crack. Moreover, the finite deformation solution predicts a more prominent opening mode near the tip, as well as a greater amount of stretching of the upper face of the crack resulting in a more negative value of the localization function. Note that the infinitesimal formulation does not require a contact search and therefore leads to simpler

Figure 14. Plane strain compression of rectangular block with a 45° edge crack: small-scale plastic yield zone and localization function for a non-associated Drucker–Prager elastoplastic material formulated with infinitesimal deformation and simulated with Mesh 4. Left block is total mesh; right insets show magnification of the crack tip region. The enrichment ball (red circle) has a radius $R=0.003$ m. All finite elements inside this ball are enriched with the crack tip functions.

calculations. However, it imposes the contact condition in the undeformed configuration; hence, the contact faces appear to overlap in Figure 15.

6.4. S-shaped frictional crack

Stresses and deformation along wavy frictional faults are of interest in computational geosciences. The roughness on the fault produces a variation in the normal stress that complicates the stress distribution critical for seismic rupture propagation [55–58]. In this final example, we consider an S-shaped (sine-wave) frictional crack embedded in an elastoplastic rectangular domain 20 m wide and 10 m high. The FE mesh is shown in Figure 17 and consists of 890 nodes and 1732 CST elements many of which have been clustered around the crack to provide higher resolution spatial interpolation in that area. Placing the lower left corner of the block at the origin of the Cartesian coordinate system, the sine wave crack with an amplitude of 1 m begins at coordinates (5.0, 5.0) and terminates at coordinates (15.0, 5.0). Employing once again the standard multiplicative plasticity theory, the bulk material is assumed to obey the Drucker–Prager yield criterion with material parameters $E=10$ GPa (Young’s modulus), $\nu=0.3$ (Poisson’s ratio), $c=3$ GPa (cohesion), $\beta=1.2$ (bulk friction), and $b=1.0$ (bulk dilatancy); the crack has a coefficient of friction of $\mu=0.27$. The

Figure 15. Plane strain compression of rectangular block with a 45° edge crack: plastic yield zone (left) and localization function (right) for a non-associated Drucker–Prager elastoplastic material formulated with infinitesimal deformation and simulated with Mesh 2.

Figure 16. Plane strain compression of rectangular block with a 45° edge crack: plastic yield zone (left) and localization function (right) for a non-associated Drucker–Prager elastoplastic material formulated with finite deformation and simulated with Mesh 2.

rectangular domain is subjected to pure shearing by applying a horizontal displacement of 4 m in 200 equal increments at the top boundary while holding the bottom boundary fixed, for a total imposed nominal engineering shear strain of $\gamma = \frac{4}{10} = 40\%$. This very large movement severely deforms the mesh and subjects the crack to extreme geometrical distortion.

Figure 18 shows the deformations, yield zones, and localization functions generated by the extended FE solutions. Yield zones form near the crack tips and propagate in an antisymmetric fashion. The localization functions are all negative [59, 60], implying that the bifurcation condition of Rudnicki and Rice has been met near the crack tip right at the onset of loading owing to the

Figure 17. Finite element mesh with embedded S-shaped frictional crack. The rectangular domain is 20 m wide and 10 m high. The crack is denoted by the thick curve passing through the interior of CST elements.

Figure 18. Evolutions of plastic yield zone (left) and localization function (right) superimposed on deformed meshes for rectangular block with an embedded S-shaped frictional crack. Numbers in percent are imposed nominal engineering shear strains.

Figure 19. Evolutions of slip (left) and contact pressure (right) on S-shaped crack as functions of the imposed nominal engineering shear strain (in percent). Both slip and contact pressure distributions are relatively smooth (i.e. no oscillation is observed) where the crack faces remained in frictional contact.

Figure 20. Elastoplastic solid with embedded S-shaped crack: convergence profile of Newton-Raphson iterations. Ticks in percent denote values of nominal engineering shear strain.

nearly singular strain field present there. The spatial variation of slip all throughout the crack portion that remained in frictional contact remains smooth (Figure 19). Newton-Raphson iterations also exhibit stability as demonstrated by the global convergence profile of Figure 20.

7. SUMMARY AND CONCLUSIONS

We have reformulated a recently proposed extended FE framework for frictional crack problems to accommodate finite deformation and bulk inelasticity. A novel feature of the formulation is a variational equation that accommodates material discontinuity in the sense that slip magnitudes much larger than element dimensions are allowed. The formulation is objective in the sense that one can choose either face of a crack as a ‘master segment’ and obtain the same result. Conditions for contact and frictional sliding are imposed in the current configuration, and integration on all types of nonlinearity, including material, geometric, and contact search, is done implicitly. We have pursued a standard Galerkin approximation where the variations and trial functions are evaluated in the same (current) contact configuration.

The results of plane strain simulations demonstrated various aspects of the extended FE solutions for solving finite deformation frictional crack problems. The mechanisms considered included combined opening and frictional sliding in initially straight, curved, and S-shaped cracks, with and without bulk plasticity. Our mesh convergence studies focused on both the global and local responses of structures with embedded cracks, including the distribution of the normal component of traction on the contacting faces of the crack. In the example presented with uniform meshes, the spatial oscillation in the normal contact pressure appears to go away as the mesh is refined. This may not necessarily hold true with irregular meshes, and so work is currently underway toward a stabilized FE formulation that can be used for both regular and irregular meshes.

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