MONOTONE COMPARATIVE STATICS UNDER UNCERTAINTY

Susan Athey[*]
MIT and NBER

First Draft: June, 1994
Last Revision: August, 2000

ABSTRACT: This paper analyzes monotone comparative statics predictions in several classes of stochastic optimization problems. The main results characterize necessary and sufficient conditions for comparative statics predictions to hold based on properties of primitive functions, that is, utility functions and probability distributions. The results apply when the primitives satisfy one of the following two properties: (i) a single crossing property, which arises in applications such as portfolio investment problems and auctions, or (ii) log-supermodularity, which arises in the analysis of demand functions, affiliated random variables, stochastic orders, and orders over risk aversion.

KEYWORDS: Monotone comparative statics, single crossing, affiliation, monotone likelihood ratio, monotone probability ratio, log-supermodularity, investment under uncertainty, risk aversion.

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[*] I am indebted to Paul Milgrom and John Roberts for their advice and encouragement. For helpful discussions, I would like to thank Kyle Bagwell, Peter Diamond, Joshua Gans, Ed Glaeser, Christian Gollier, Bengt Holmstrom, Ian Jewitt, Miles Kimball, Jonathan Levin, Eric Maskin, Preston McAfee, Ed Schlee, Chris Shannon, Scott Stern, Nancy Stokey, several anonymous referees, and seminar participants at the Australian National University, Harvard/MIT Theory Workshop, Pennsylvania State University, University of Montreal, Yale University, the 1997 summer meetings of the econometric society in Pasadena, and the 1997 summer meeting of the NBER Asset Pricing group. Generous financial support was provided by the National Science Foundation (graduate fellowship and Grant SBR9631760) and the State Farm Companies Foundation. Correspondence: MIT Department of Economics, E52-252C, Cambridge, MA 02142; email: athey@mit.edu; url: http://mit.edu/athey/www.

1. Introduction

Since Samuelson, economists have studied and applied systematic tools for deriving comparative statics predictions. Recently, the theory of comparative statics has received renewed attention (Topkis (1978), Milgrom and Roberts (1990a, 1990b, 1994), Milgrom and Shannon (1994), Vives (1990)). The new literature provides general and widely applicable theorems about comparative statics, and further, it emphasizes the robustness of conclusions to changes in the specification of models. The literature shows that many of the robust comparative statics results that arise in economic theory rely on three main properties: supermodularity, log-supermodularity, and single crossing properties.[1]

This paper characterizes log-supermodularity and single crossing properties in stochastic problems, establishing necessary and sufficient conditions for comparative statics predictions.[2] More precisely, let u be an agent's payoff function, let f be a probability density function, and let the agent's objective function be given by $U(\mathbf{x},\theta) = \int u(\mathbf{x},\mathbf{s})f(\mathbf{s},\theta)d\mathbf{s}$, where $\mathbf{x}$ represents a choice vector and θ represents an exogenous parameter. The comparative statics question concerns conditions on primitives—the payoff function and probability density—under which the agent's optimal choice of $\mathbf{x}$ is nondecreasing in θ.

We begin with families of problems where one of the primitive functions is log-supermodular (abbreviated log-spm). For example, an agent's marginal utility $u'(w+s)$ is log-spm in (w,s) if the utility function satisfies decreasing absolute risk aversion; a parameterized demand function $D(p;\varepsilon)$ is log-spm if demand becomes less elastic as ε increases; a set of random variables is affiliated[3] if the joint density is log-spm; and a parameterized density of a single random variable, $f(s;\theta)$, is log-spm if the parameter shifts the distribution according to the monotone likelihood ratio property.

Our first result establishes that the agent's choice of $\mathbf{x}$ is nondecreasing in θ for all utility functions u that are log-spm, if and only if f is log-spm. One application considers a pricing game between firms with private information about their marginal costs; we provide conditions under which each firm's price increases in its marginal cost (and thus, a pure strategy Nash equilibrium exists). More generally, we show that U is log-spm for all u that are log-spm, if and only if f is log-spm. The characterizations of log-spm are used to establish relationships between several

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[1] A function on a product set is supermodular if increasing any one variable increases the returns to increasing each of the other variables; for differentiable functions this corresponds to non-negative cross-partial derivatives between each pair of variables. A positive function is log-supermodular if the log of that function is supermodular.

[2] Vives (1990) and Athey (1998) study supermodularity in stochastic problems. Vives (1990) shows that supermodularity is preserved by integration. Athey (1998) allows for payoff functions that satisfy properties which are preserved by convex combinations (including supermodularity, monotonicity and concavity). Independently, Gollier and Kimball (1995a, b) used convex analysis to analyze comparative statics of investment problems. The properties studied in this paper are not preserved by convex combinations, and so different techniques are used.

[3] A vector of random variables is affiliated if every nondecreasing function of the vector is positively correlated.

commonly used orders over distributions in investment theory, as well as to show that decreasing absolute risk aversion is preserved in the presence of independent or affiliated background risks.

Our second set of comparative statics theorems provide necessary and sufficient conditions for comparative statics predictions in problems with a single random variable, where one of the primitive functions satisfies a single crossing property and the other is log-spm.[4] The results are applied to portfolio and firm investment problems.[5] We further show how the results change when we impose additional structure (such as restrictions on risk preferences).

Finally, we extend the analysis to problems of the form $\int v(x,y,s)f(s,\theta)ds$, where we characterize single crossing of the x-y indifference curves; the results are applied to signaling games and consumption-savings problems.

2. Comparative Statics with Log-spm Primitives

This section considers problems where one or both of the primitives, u and f, are assumed to be non-negative and log-spm.

2.1. The Comparative Statics Problem

We begin with some notation. Let $\Theta \subseteq \mathbb{R}$, let $X = X_1 \times \cdots \times X_n$ (each $X_i \subseteq \mathbb{R}$),[6] and let $S = S_1 \times \cdots \times S_m$ (each $S_i \subseteq \mathbb{R}$). Let μ be a non-negative σ-finite product measure on S, and let $u : X \times S \to \mathbb{R}$ and $f : S \times \Theta \to \mathbb{R}$ be bounded, measurable functions.[7] Define $U : X \times \Theta \to \mathbb{R}$ by $U(\mathbf{x},\theta) \equiv \int u(\mathbf{x},\mathbf{s}) f(\mathbf{s};\theta) d\mu(\mathbf{s})$.

In this context, we seek conditions on u and f under which the following monotone comparative statics prediction holds:

$$\mathbf{x}^*(\theta,B) \equiv \arg\max_{\mathbf{x}\in B} U(\mathbf{x},\theta) \text{ is nondecreasing in } \theta \text{ and } B. \qquad \text{(MCS)}$$

In order to make (MCS) precise, we need to specify in what sense $\mathbf{x}^*$ should be nondecreasing, as well as what it means for the constraint set B to increase. To do so, we introduce some notation from lattice theory. Given a set X and a partial order $\geq$, the operations "meet" ($\vee$) and

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[4] Karlin and Rubin (1956) studied the preservation of single crossing properties under integration with respect to log-spm densities. A number of papers in the statistics literature have exploited this relationship, and Karlin's (1968) monograph presents the general theory of the preservation of an arbitrary number of sign changes under integration. Jewitt (1987) exploits the work on the preservation of single crossing properties and bivariate log-spm in his analysis of orderings over risk aversion and associate comparative statics; he makes use of the fact that orderings over risk aversion can be recast as statements about log-spm of a marginal utility.

[5] Many authors have studied the comparative statics properties of portfolio and investment problems, including Diamond and Stiglitz (1974), Eeckhoudt and Gollier (1995), Gollier (1995), Hadar and Russell (1978), Jewitt (1986, 1987, 1988b, 1989), Kimball (1990, 1993), Landsberger and Meilijson (1990), Meyer and Ormiston (1983, 1985, 1989), Ormiston (1992), and Ormiston and Schlee (1992, 1993); see Scarsini (1994) for a survey of the main results involving risk and risk aversion.

[6] Nothing would change in the paper if we let X be an arbitrary lattice; we use $X \subseteq \mathbb{R}^n$ for simplicity.

[7] More generally, whenever integrals appear, we assume that requirements of integrability and measurability are met.

"join" (∧) are defined as follows: $\mathbf{x} \vee \mathbf{y} = \inf\{\mathbf{z} | \mathbf{z} \geq \mathbf{x}, \mathbf{z} \geq \mathbf{y}\}$ and $\mathbf{x} \wedge \mathbf{y} = \sup\{\mathbf{z} | \mathbf{z} \leq \mathbf{x}, \mathbf{z} \leq \mathbf{y}\}$. For $\mathbb{R}^n$ with the usual order, these represent the component-wise maximum and component-wise minimum, respectively. A lattice is a set *X* together with a partial order, such that the set is closed under meet and join.

These definitions can be used to define the order over sets used in (MCS).

Definition 1 *A set A is greater than a set B in the* ***strong set order*** *(SSO), written A≥B, if, for any a in A and any b in B,* $a \vee b \in A$ *and* $a \wedge b \in B$*. A set-valued function* $A : \mathbb{R} \to 2^{\mathbb{R}^n}$ *is nondecreasing in the strong set order (SSO) if for any* $\tau_H > \tau_L$*,* $A(\tau_H) \geq A(\tau_L)$*. A set A is a* ***sublattice*** *if A≥A.*

If a set-valued function is nondecreasing in the strong set order, then the lowest and highest elements of this set are nondecreasing. For example, a set of the form $[a_1, b_1] \times [a_2, b_2]$ is nondecreasing (SSO) in a_i and b_i, *i*=1,2.

We assume throughout that the constraint set *B* is a sublattice, so that it is closed under the meet and join operations. Thus, if one component of **x** increases, the constraint set does not force other components of **x** to decrease. Then, Milgrom and Shannon (1994) show that (MCS) holds if and only if *U* is *quasi-supermodular* (abbreviated quasi-spm) in **x** and satisfies a single crossing property, referred to as *SC2*, in $(\mathbf{x};\theta)$.[8] These properties are defined as follows:

Definition 2 *Let* $g : \mathbb{R} \to \mathbb{R}$*. (i) g satisfies single crossing (****SC1****) in t if there exist* $t_0' \leq t_0''$ *such that* $g(t) < (\leq) 0$ *for all* $t < (\leq) t_0'$*,* $g(t) = 0$ *for all* $t_0' < t < t_0''$*, and* $g(t) > (\geq) 0$ *for all* $t > (\geq) t_0''$*.*
(ii) $h : X \times \mathbb{R} \to \mathbb{R}$ *satisfies single crossing in two variables (****SC2****) in* $(\mathbf{x};t)$ *if, for all* $\mathbf{x}_H > \mathbf{x}_L$*,* $g(t) \equiv h(\mathbf{x}_H; t) - h(\mathbf{x}_L; t)$ *satisfies SC1.*
(iii) $h : X \to \mathbb{R}$ *is* ***quasi-supermodular*** *if it satisfies SC2 in* $(x_i; x_j)$ *for all i≠j.*[9]

The definition of SC1 simply says that as *t* increases, *g*(*t*) crosses zero at most once and from below. Single crossing in two variables, SC2, in $(\mathbf{x};t)$ requires that the *incremental* returns to **x** satisfy SC1. When *U* satisfies SC2 in $(\mathbf{x};\theta)$, as θ increases, an agent choosing between $\mathbf{x}_L$ and $\mathbf{x}_H$ will first prefer $\mathbf{x}_L$, then become indifferent, and finally prefer $\mathbf{x}_H$. For establishing (MCS), the additional requirement that *U* is quasi-spm ensures that increases in the components of **x** reinforce one another.

2.2. Log-Spm Primitives

To understand when primitive functions might be log-spm, consider first the formal definition.[10]

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[8] Since an empty set is always larger and smaller than any other set in the strong set order, we do not state an assumption about the existence of an optimum (following Milgrom and Shannon (1994)).

[9] This definition applies because we assumed that *X* is a product set. When *X* is an arbitrary lattice, *h* is quasi-supermodular if for all $\mathbf{x}, \mathbf{y} \in X$, $h(\mathbf{x}) - h(\mathbf{x} \wedge \mathbf{y}) \geq (>) 0$ implies $h(\mathbf{x} \vee \mathbf{y}) - h(\mathbf{y}) \geq (>) 0$.

[10] Karlin and Rinott (1980) referred to log-spm as multivariate total positivity of order 2.

Definition 3** Let* $(X,\geq)$ *be a lattice. A function* $h{:}X{\to}\Re$ *is* ***supermodular *if, for all* $\mathbf{x},\mathbf{y}\in X$, $h(\mathbf{x}\vee\mathbf{y})+h(\mathbf{x}\wedge\mathbf{y})\geq h(\mathbf{x})+h(\mathbf{y})$. *h is* ***log-supermodular (log-spm)*** *if it is non-negative, and, for all* $\mathbf{x},\mathbf{y}\in X$, $h(\mathbf{x}\vee\mathbf{y})\cdot h(\mathbf{x}\wedge\mathbf{y})\geq h(\mathbf{x})\cdot h(\mathbf{y})$.

To interpret this, observe that for a function $h:\mathbb{R}^2\to\mathbb{R}$, supermodularity requires that the incremental returns to increasing *x*, defined by $g(t)\equiv h(x_H;t)-h(x_L;t)$, must be nondecreasing in *t*; log-supermodularity of a positive function requires that the relative returns, $h(x_H;t)/h(x_L;t)$, are nondecreasing in *t*. The multivariate version of supermodularity simply requires that the relationship just described holds for each pair of variables.[11] Topkis (1978) proves that if *h* is twice differentiable, *h* is supermodular if and only if $\frac{\partial^2}{\partial x_i\partial x_j}h(\mathbf{x})\geq 0$ for all $i\neq j$. If *h* is positive, then *h* is log-spm if and only if $\log(h(\mathbf{x}))$ is supermodular. Both properties are stronger than quasi-supermodularity; thus, as properties of objective functions, both are sufficient for comparative statics predictions to hold. Observe that sums of supermodular functions are supermodular, and products of log-spm functions are log-spm. However, sums of log-spm functions are not necessarily log-spm.

Consider some examples where economic primitives are log-spm. A parameterized demand function $D(P;t)$ is log-spm if and only if the price elasticity, $\varepsilon(P;t)\equiv P\cdot D_P(P;t)/D(P;t)$, is nondecreasing in *t*. A marginal utility function $U'(w+s)$ is log-spm in (w,s) (where *w* often represents initial wealth and *s* represents the return to a risky asset) if and only if the utility satisfies decreasing absolute risk aversion (DARA). A parameterized distribution $F(\cdot;\theta)$ has a hazard rate $f(s)/(1-F(s;\theta))$ which is nonincreasing in θ if $1-F(s;\theta)$ is log-spm. Milgrom and Weber (1982) show that a vector of random variables is *affiliated* if and only if their joint density is log-spm (almost everywhere). When the support of $F(\cdot;\theta)$, denoted supp[*F*], is constant in θ, and *F* has a density *f*,[12] then the Monotone Likelihood Ratio Order (MLR) requires that *f* is log-spm, that is, the likelihood ratio $f(s;\theta_H)/f(s;\theta_L)$ is nondecreasing in *s* for all $\theta_H>\theta_L$.[13]

2.3. Necessary and Sufficient Conditions for Comparative Statics

Our first step in analyzing (MCS) for problems with log-spm primitives follows.

Lemma 1 *Suppose f is non-negative. Then (i) (MCS) holds for all* $u:X\times S\to\mathbb{R}_+$ *log-spm, if and only if (ii) U is log-spm in* $(\mathbf{x},\theta)$ *for all* $u:X\times S\to\mathbb{R}_+$ *log-spm.*

Lemma 1 states that if we want (MCS) to hold for all log-spm payoffs, the objective function

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[11] Although supermodularity can be checked pairwise (see Topkis, 1978), Lorentz (1953) and Perlman and Olkin (1980) establish that the pairwise characterization of log-spm requires additional assumptions, such as strict positivity (at least throughout "order intervals").

[12] The MLR can also be defined for distributions with varying supports, or distributions that do not have densities with respect to Lebesgue measure; see the Appendix for details.

[13] See Lehmann (1955) and Milgrom (1981). Related, in the statistics literature (Karlin, 1968), a strictly positive bivariate function satisfies total positivity of order 2 (TP-2) if and only if it is log-spm.

U must be log-spm (a stronger condition than quasi-spm). The proof shows that if *U* fails to be log-spm in $(\mathbf{x},\theta)$ for some *u* log-spm, then we can find another log-spm payoff, *v*, for which (MCS) fails (in particular, we show that $\int v(\mathbf{x},\mathbf{s})f(\mathbf{s};\theta)d\mu(\mathbf{s})$ fails to be quasi-spm in **x** or to be SC2 in $(\mathbf{x};\theta)$). Lemma 1 then motivates the main technical question for this section: under what conditions on *f* does log-supermodularity of *u* imply that *U* is log-spm? Consider a result from the statistics literature, which will be one of the main tools used in this paper.

Lemma 2 *(Ahlswede and Daykin, 1979) Let* h_i *(i=1,...,4) represent four non-negative functions,* $h_i : S \to \mathbb{R}$. *Then condition (L2.1) implies (L2.2):*

$$h_1(\mathbf{s})\cdot h_2(\mathbf{s}') \le h_3(\mathbf{s}\vee\mathbf{s}')\cdot h_4(\mathbf{s}\wedge\mathbf{s}') \text{ for } \mu\text{-almost all } \mathbf{s},\mathbf{s}'\in S. \tag{L2.1}$$

$$\int h_1(\mathbf{s})d\mu(\mathbf{s})\cdot\int h_2(\mathbf{s})d\mu(\mathbf{s}) \le \int h_3(\mathbf{s})d\mu(\mathbf{s})\cdot\int h_4(\mathbf{s})d\mu(\mathbf{s}). \tag{L2.2}$$

Karlin and Rinott (1980) provide a simple proof of this lemma.[14] They further explore a variety of interesting applications in statistics, though they do not consider the problem of comparative statics. While we will use this result in a variety of ways throughout the paper, the most important (and immediate) consequence of Lemma 2 for comparative statics is that log-supermodularity is preserved by integration. To see this, set

$$h_1(\mathbf{s}) = g(\mathbf{y},\mathbf{s}),\ h_2(\mathbf{s}) = g(\mathbf{y}',\mathbf{s}),\ h_3(\mathbf{s}) = g(\mathbf{y}\vee\mathbf{y}',\mathbf{s}), \text{ and } h_4(\mathbf{s}) = g(\mathbf{y}\wedge\mathbf{y}',\mathbf{s}).$$

Then (L2.1) states exactly that $g(\mathbf{y},\mathbf{s})$ is log-spm in $(\mathbf{y},\mathbf{s})$, while (L2.2) reduces to the conclusion is that $\int g(\mathbf{y},\mathbf{s})d\mu(\mathbf{s})$ is log-spm in **y**. Recall that arbitrary sums of log-spm functions are *not* log-spm, which makes this result somewhat surprising. But notice that Lemma 2 does not apply to arbitrary sums, only to sums of the form $g(\mathbf{y},\mathbf{s}^1) + g(\mathbf{y},\mathbf{s}^2)$, when *g* is log-spm in *all* arguments.

The preservation of log-spm under integration is especially useful for analyzing expected values of payoff functions. Since arbitrary products of log-spm functions are log-spm, a sufficient condition for $\int u(\mathbf{x},\mathbf{s})f(\mathbf{s};\theta)d\mu(\mathbf{s})$ to be log-spm is that *u* and *f* are log-spm. To understand the intuition, consider first the case where $X, S \subseteq \mathbb{R}$. Then, *u* is log-spm implies that the relative returns to *x* are nondecreasing in *s*. If *f* is log-spm, then θ increases the likelihood of high values of *s* relative to low values of *s*. Then, Lemma 2 implies that θ increases the *expected* relative returns to *x*. In the multivariate case, log-spm of *u* and *f* ensure that the interactions among components of **x** and **s** reinforce these effects. Karlin and Rinott give many examples of densities that are log-spm, and thus preserve log-spm of a payoff function.[15] Below, in Section 2.4.1, we

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[14] This result has a long history in statistics. Lehmann (1955) proved that bivariate log-supermodularity is preserved by integration. Ahlswede and Daykin (1979) extended the theory to multivariate functions. Karlin and Rinott (1980) present the theory of multivariate total positivity of order 2 (MTP-2) functions together with a variety of useful applications, though comparative statics results are not among them. Milgrom and Weber (1982) independently derived a variety of properties of affiliated random variables in auctions. See also Whitt (1982).

[15] For example, exchangeable, positively correlated normal vectors and absolute values of normal random vectors have log-supermodular densities (but arbitrary positively correlated normal random vectors do not; Karlin and Rinott (1980) give restrictions on the covariance matrix which suffice); and multivariate logistic, *F*, and gamma distributions

show how the result can be used in analyzing the preservation of ratio orderings, such as decreasing absolute risk aversion.

An especially important type of log-spm function for our purposes is an indicator function for a sublattice or a nondecreasing set-valued function.

Lemma 3 *Given* $A:\mathbb{R}\to 2^S$*, the indicator function* $\mathbf{1}_{A(\tau)}(\mathbf{s})$ *is log-spm in* $(\mathbf{s},\tau)$ *if and only if* $A(\tau)$ *is a sublattice for each* τ*, and is nondecreasing (strong set order).*

Lemma 3 implies, for example, that $\mathbf{1}_{[a,b]}(s)$ is log-spm in (a,b,s). To see how Lemma 3 can be used, observe that Lemmas 2 and 3 imply log-spm of a distribution is weaker than log-spm of a density. Formally, if $f(s;\theta)$ is log-spm, (that is, the distribution satisfies the MLR order), then the corresponding cumulative distribution function $F(s;\theta)=\int\mathbf{1}_{(-\infty,s]}(t)\,f(t;\theta)dt$ will be log-spm (that is, it satisfies the Monotone Probability Ratio Order),[16] which can be shown to be stronger than First Order Stochastic Dominance. This in turn implies that $\int_{-\infty}^{a}F(s;\theta)ds$ is log-spm, which is stronger than Second Order Stochastic Dominance if θ does not change the mean of s.

In a similar way, Lemma 3 can also be used to show that log-spm of f is a necessary condition for U to be log-spm whenever u is log-spm. Lemma 3 helps us identify a set of "test functions" that are a subset of the set of log-spm functions; these are the functions used to create counter-examples. Consider the special case where X=S. Define $B_\varepsilon(\mathbf{x})=\{\mathbf{y}\in X:\forall i,\ y_i\in[x_i-\varepsilon,x_i+\varepsilon]\}$. Because for each $\mathbf{x}$ and ε, $B_\varepsilon(\mathbf{x})$ is a sublattice and is nondecreasing in $\mathbf{x}$ in the strong set order, Lemma 3 implies that $\mathbf{1}_{B_\varepsilon(\mathbf{x})}(\mathbf{s})$ is log-spm in $(\mathbf{x},\mathbf{s})$. Define a set of such functions as follows:

$$\mathcal{T}(\beta)\equiv\{u:\ \exists\ \mathbf{x}\ \textit{and}\ 0\le\varepsilon<\beta\ \textit{such that}\ u(\mathbf{x},\mathbf{s})\equiv\mathbf{1}_{B_\varepsilon(\mathbf{x})}(\mathbf{s})\ \}.$$

Then, for any $\beta>0$, U is log-spm in $(\mathbf{x},\theta)$ whenever $u\in\mathcal{T}(\beta)$, if and only if f is log-spm in $(\mathbf{s},\theta)$ a.e.-μ.[17] To see this, let μ be Lebesgue and suppose f is continuous. Then $\lim_{\varepsilon\to 0}\int\mathbf{1}_{B_\varepsilon(\mathbf{x})}(\mathbf{s})\,f(\mathbf{s};\theta)d\mu(\mathbf{s})=f(\mathbf{x};\theta)$. This formalizes what we mean by a set of test functions: $\mathcal{T}(\beta)$ is a subset of log-spm functions, but the set is large enough to make log-spm of f a *necessary* condition for U to be log-spm.[18] The following result generalizes this discussion.

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have log-supermodular densities.

[16] See Eeckhoudt and Gollier (1995), who show that an MPR shift is sufficient for a risk-averse investor to increase his portfolio allocation.

[17] We say that f is log-spm a.e.-μ if the inequality in Definition 3 holds for almost all (w.r.t. the product measure on $S\times S$ induced by μ) $(\mathbf{s},\mathbf{s}')$ pairs in $S\times S$.

[18] Our approach to necessity can be understood with reference to the stochastic dominance literature, and more generally Athey (1998). Athey (1998) shows that in stochastic problems, if one wishes to establishes that a property P holds for $U(\mathbf{x},\theta)$ for all u in a given class $\mathcal{U}$, it is often necessary and sufficient to check that P holds for all u in the set of extreme points of $\mathcal{U}$. The extreme points can be thought of as test functions; for example, when $\mathcal{U}$ is the set of nondecreasing functions, the set of test functions is the set of indicator functions for sets $\mathbf{1}_A(\mathbf{s})$ that are nondecreasing in $\mathbf{s}$. Athey (1998) shows that this approach works well when P is a property preserved by convex combinations, *unlike* log-supermodularity.

Lemma 4 *Suppose f is nonnegative, and that n* ≥2 *if m*≥ 2 (*where m is the dimension of S and n is the dimension of X). The following two conditions are equivalent: (i) U is log-spm in* $(\mathbf{x},\theta)$ *for all* $u : X \times S \to \mathbb{R}_+$ *that are log-spm a.e.-*μ*; (ii) f is log-spm in* $(\mathbf{s};\theta)$ *a.e.-*μ*.*

Remark *Lemma 4 requires that* $\mathbf{x}$ *has at least two components (n≥2) if* $\mathbf{s}$ *has two or more components (m≥2).*[19] *However, even in cases where m≥2 and n=1, (ii) is sufficient for (i), and further, log-supermodularity of f in* (s_i,θ) *is necessary for (i) to hold.*

Lemma 4 is of interest in its own right, as we will see below. For the moment, however, we use Lemma 4 (together with Lemma 1) to prove our first comparative statics theorem.

Theorem 1 (Comparative Statics with Log-Supermodular Primitives) *Suppose f is nonnegative, and suppose n* ≥2 *if m*≥ 2 (*where m is the dimension of S and n is the dimension of X). The following two conditions are equivalent: (i) (MCS) holds for all* $u : X \times S \to \mathbb{R}_+$ *that are log-spm a.e.-*μ*; (ii) f is log-spm in* $(\mathbf{s},\theta)$ *a.e.-*μ*.*

Theorem 1 gives necessary and sufficient conditions for comparative statics when the payoff function is log-spm. Further, the approach based on test functions allows the modeler to immediately check whether any additional restrictions on the class of admissible payoff functions (where such restrictions might be motivated by an economic model) affect the necessity conclusion. For example, the elements of $\mathcal{T}(\beta)$ are clearly *not* monotonic, and thus the "only if" part of Theorem 1 does not hold under the additional assumption that *u* is nondecreasing. In contrast, we can approximate the elements of $\mathcal{T}(\beta)$ with smooth functions, so smoothness restrictions will not alter the conclusion of Theorem 1.

Finally, we observe that the results of Lemmas 2 and 3 can be combined to provide sufficient conditions for comparative statics in problems outside the particular structure considered in (MCS). For example, suppose that for *j*=1,..,*J*, $A^j : \mathbb{R} \to 2^S$, $A^j(\tau_j)$ is a sublattice and is nondecreasing in τ_j, and let $\mathbf{A}(\boldsymbol{\tau}) = \cap_{j=1,..,J} A^j(\tau_j)$. Then if *g* and *h* are log-spm, $\mathbf{x}^*(\boldsymbol{\theta},\boldsymbol{\tau},B)$ $= \arg\max_{\mathbf{x}\in B} \int_{\mathbf{s}\in \mathbf{A}(\boldsymbol{\tau})} g(\mathbf{x},\boldsymbol{\theta},\mathbf{s})h(\mathbf{s};\mathbf{x},\boldsymbol{\theta})\,d\mu(\mathbf{s})$ is nondecreasing in all of its arguments. However, note that even if *h* is a probability density, the objective function in this problem cannot be interpreted as a conditional expectation of *g*, since we have not divided by the probability that $\mathbf{s} \in A(\boldsymbol{\tau})$. The working paper (Athey, 1997) uses Lemmas 2 and 3 to provide necessary and sufficient conditions for comparative statics predictions to hold when the objective function is a conditional expectation.

2.4. Applications

2.4.1. Risk Aversion and Background Risks

Lemma 4 can be applied to problems that involve orderings of ratios. For example, consider

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[19] If we extend the analysis so that *X* is an arbitrary lattice rather than a product set, we require that *X* has at least two elements that are not ordered.

$g, h\colon \mathbb{R} \to \mathbb{R}_+$. Then, define $u:\{0,1\}\times\mathbb{R}\to\mathbb{R}_+$ by $u(x,s) = g(s)$ if x=1 and $u(x,s) = h(s)$ if x=0. Now observe that if h>0, u is log-spm if and only if $g(s)/h(s)$ is nondecreasing in s. Using this construction, Lemma 4 implies that $\int g(s)f(s;\theta)ds \big/ \int h(s)f(s;\theta)ds$ is nondecreasing in θ for all $g, h\colon \mathbb{R}\to\mathbb{R}_+$ such that $g(s)/h(s)$ is nondecreasing in s, if and only if f is log-spm in $(s;\theta)$.

The Arrow-Pratt coefficient of absolute risk aversion is defined in terms of a ratio. Let the coefficient of risk aversion of the utility function u evaluated at w be $R(w;u(\cdot))\equiv -u''(w)/u'(w)$, and let $U(w;\gamma,\theta)\equiv\int u(w+s;\gamma)f(s;\theta)ds$. Then, Lemma 4 has the following implications:[20]

1. *MLR Shifts and Decreasing Absolute Risk Aversion* If $R(w;u(\cdot;\gamma))$ is nonincreasing in w and f is log-spm in (s,θ), then $R(w;U(\cdot;\gamma,\theta))$ is nonincreasing in θ.
2. *Preservation of Decreasing Absolute Risk Aversion Under Background Risks* If $R(w;u(\cdot;\gamma))$ is nonincreasing in w and γ, then $R(w;U(\cdot;\gamma,\theta))$ is nonincreasing in w and γ.
3. *Prudence* Results (1) and (2) hold for a risk-averse agent when $R(w;u)$ is replaced by prudence (Kimball (1990)), $P(w;u(\cdot))\equiv -u'''(w)/u''(w)$, if u'''>0 (or $-P$, if u'''<0).

More generally, Lemma 4 establishes that risk aversion orderings are preserved in the presence of multiple, affiliated background risks:

4. *Affiliated Background Risks* Let z be the "primary" risk, and suppose that the agent has a positive position (with portfolio weights $\boldsymbol{\alpha}$) in a vector of assets with conditional density function on S given by $g(\cdot|z)$. Let $U(z;w,\gamma) = \int u(w+\boldsymbol{\alpha}\cdot\mathbf{s};\gamma)g(\mathbf{s}|z)d\mathbf{s}$. If $g(\mathbf{s}|z)$ is log-spm (the risks are affiliated) and $R(w;u(\cdot;\gamma))$ is nonincreasing in w and γ, then $R(z;U(\cdot;w,\gamma))$ is nonincreasing in z, w and γ.[21]

2.4.2. Log-Spm Games of Incomplete Information

Consider a game of incomplete information where each player has private information about her own type, $t_i \in T_i \subseteq \mathbb{R}$, and chooses a strategy $\chi_i : T_i \to X_i$, where $X_i \subseteq \mathbb{R}$. Athey (forthcoming) shows that a pure strategy Nash equilibrium (PSNE) in nondecreasing strategies exists in games of incomplete information where an individual player's strategy $\chi_i(\cdot)$ is nondecreasing in her type, whenever all of her opponents use nondecreasing strategies.[22]

Let $T = T_1 \times\cdots\times T_I$, let $h: T\to\mathbb{R}_+$ be the joint density over types (with respect to Lebesgue measure), assumed strictly positive on T, and let $h_{-i}(\cdot\,|\,t_i)$ be the conditional density of player i's

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[20] Pratt (1988) first established that risk aversion orderings are preserved by expectations, while Jewitt (1987) first showed that the "more risk averse" ordering is preserved by a MLR shift. The results about prudence are new.

[21] An alternative sufficient condition requires that the conditional density of $y=\boldsymbol{\alpha}\cdot\mathbf{s}$ given z, $g(y|z)$, is log-spm.

[22] This result requires that the type distribution is atomless (Lebesgue), and that X is either finite or compact and convex. This result is distinct from results based on the theory of supermodular games. For example, Vives (1990) showed that if each player's payoff $v_i(\mathbf{x})$ is supermodular in the realizations of actions $\mathbf{x}$, the game has strategic complementarities in strategies (that is, a pointwise increase in an opponent's strategy leads to a pointwise increase in a player's best response). This in turn implies that a PSNE exists. In contrast, if each player's payoff $v_i(\mathbf{x})$ is log-spm, the game does not necessarily have strategic complementarities, and a different approach is required.

opponents' types. Let player *i*'s utility be given by $v_i : X \times T \rightarrow \mathbb{R}$. Expected payoffs are $V_i(x_i,t_i) \equiv \int_{\mathbf{t}_{-i}} v_i(x_i, \boldsymbol{\chi}_{-i}(\mathbf{t}_{-i}), \mathbf{t}) h_{-i}(\mathbf{t}_{-i} \mid t_i) d\mathbf{t}_{-i}$. The following result gives necessary and sufficient conditions for each player's best response to nondecreasing strategies to be nondecreasing.

Proposition 1 *The following two conditions are equivalent: (i) for all i,* $\chi_i(t_i) \equiv \arg\max_{x_i} V_i(x_i,t_i)$ *is nondecreasing in* t_i *for all* $\chi_j(\cdot)$ *nondecreasing for j≠i, and all* v_i *log-spm; (ii) the types are affiliated.*

Spulber (1995) recently analyzed how asymmetric information about a firms' cost parameters alters the results of a Bertrand pricing model, showing that there exists an equilibrium where prices are increasing in costs, and further firms price above marginal cost and have positive expected profits. Spulber's model assumes that costs are independently and identically distributed; Proposition 1 generalizes his result to asymmetric, affiliated signals, and to imperfect substitutes. To see this, let $v_i(\mathbf{x},\mathbf{t}) = (x_i - t_i)D_i(\mathbf{x})$, where **x** is the vector of prices, **t** is the vector of marginal costs, and $D_i(\mathbf{x})$ gives demand to firm *i* when prices are **x**. First observe that $x_i - t_i$ is log-spm. Then, by Lemma 4, the expected payoff function is log-spm if the signals are affiliated, each opponent uses a nondecreasing strategy, and $D_i(\mathbf{x})$ is log-spm. The interpretation of the latter condition is that the elasticity of demand is a non-increasing function of the other firms' prices.[23] When the goods are perfect substitutes, expected demand is given by

$$D_1(x_1)\int_{\mathbf{t}_{-1}} \mathbf{1}_{\chi_2(t_2)>x_1}(t_2) \cdot\cdot \mathbf{1}_{\chi_n(t_n)>x_1}(t_2)\, h(\mathbf{t}_{-1} | t_1) d\mathbf{t}_{-1}$$

and the set $\{t_j : \chi_j(t_j) > x_1\}$ is nondecreasing (strong set order) in x_1 when χ_j is nondecreasing. Then, by Lemma 3, expected payoffs must be log-spm when the density is log-spm.

3. Comparative Statics with Single-Crossing Primitives

This section studies single crossing properties in problems where there is a single real-valued random variable. Multivariate generalizations of the results are sufficiently restrictive that they are not considered here. However, many problems in the theory of investment under uncertainty concern a single random variable, and a number of other problems (such as auctions) can be reformulated so that the techniques from this section apply.

Formally, in this section we assume that $S \subseteq \mathbb{R}$, and for simplicity we also assume that $X \subseteq \mathbb{R}$. Then, we define $U(x,\theta) \equiv \int u(x,s) f(s;\theta) d\mu(s)$. Although the results of Section 2 apply to this problem, in this section we seek to relax the assumption that both primitives are log-spm. In particular, we consider the weaker assumption that *u*(*x*,*s*) satisfies SC2 in (*x*;*s*). Our comparative statics question becomes to find conditions on *u* and *f* under which

$$x^*(\theta, B) = \arg\max_{x \in B} U(x,\theta) \text{ is nondecreasing in } \theta \text{ and } B. \qquad \text{(MCS}'\text{)}$$

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[23] Demand functions which satisfy these criteria include logit, CES, transcendental logarithmic, and a set of linear demand functions (see Milgrom and Roberts (1990b) and Topkis (1979)).

Theorem 1 gives some initial insight into this problem: since the set of SC2 functions includes the set of log-spm functions, Theorem 1 implies that a *necessary* condition for (MCS′) to hold for all u which are SC2 is that f is log-spm. a.e.-μ. However, we have *not* established that u SC2 and f log-spm are sufficient for (MCS′), nor have we addressed the question of what conditions on u are necessary for the conclusion that (MCS′) holds for all f log-spm.

Before proceeding, we introduce a definition that will allow us to state concisely the theorems in this section. The definition helps to state theorems about *pairs* of hypotheses about u and f which guarantee a monotone comparative statics conclusion.

Definition 4 *Two hypotheses H-A and H-B are a* ***minimal pair of sufficient conditions (MPSC)*** *for the conclusion C if: (i) C holds whenever H-B does, if and only if H-A holds. (ii) C holds whenever H-A does, if and only if H-B holds.*

This definition captures the idea that we are looking for a pair of sufficient conditions that cannot be weakened without placing further structure on the problem. In some contexts, *H-A* will be given (such as an assumption on u), and we will search for the weakest hypothesis *H-B* (such as an assumption on f) that preserves the conclusion; in other problems, the roles of *H-A* and *H-B* will be reversed. The definition can be used to state this section's main comparative statics result:

Theorem 2 (Comparative Statics with Single Crossing Payoffs) *(A) u satisfies SC2 in (x;s) a.e.-μ. and (B) f is log-spm a.e.-μ; are a MPSC for (C) (MSC′) holds.*[24]

Theorem 2 has an interesting interpretation. Recall that SC2 of $u(x,s)$ (condition T2-A) is the necessary and sufficient condition for the choice of x which maximizes $u(x,s)$ (under certainty) to be nondecreasing in s. Thus, Theorem 2 gives necessary and sufficient conditions for the preservation of comparative statics results under uncertainty.[25] Any result that holds when s is known, will hold when s is unknown but the distribution of s experiences an MLR shift. Further, MLR shifts are the weakest distributional shifts that guarantee that conclusion. In Section 4, we illustrate in applications how additional commonly encountered restrictions on u or f can be used to relax (T2-A) and (T2-B).

We outline the proof of Theorem 2 in the text. Our first step is to transform the problem by taking the first difference with respect to x. In particular, $U(x,\theta)$ and $u(x,s)$ satisfy SC2 (in $(x;\theta)$ and $(x;s)$) if and only if, for all $x_H > x_L$, $U(x_H,\theta)-U(x_L,\theta)$ and $u(x_H,s)-u(x_L,s)$ satisfy SC1 (in θ and s). The following lemma characterizes SC1 for problems with a single random

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[24] The quantification over constraint sets B in (MCS′) can be dropped for the result that (B) is necessary for (C) to hold whenever (A) does.

[25] Jewitt (1987) and Ormiston and Schlee (1993) also give this interpretation in their analyses. Under additional regularity conditions, Ormiston and Schlee (1993) explicitly analyze the preservation of comparative statics with respect to MLR shifts, and further show, under additional regularity assumptions, that single crossing of u is a necessary condition for the result to hold for all MLR shifts.

variable.[26] It is stated in more general notation because not only will we apply it in cases where *u* is a payoff function and *f* is a probability density, but we will also apply it to transformed objective functions (where for example, *f* is a probability distribution) as well as in problems where the agent's choice variable affects the probability distribution directly.

Lemma 5 *Let* $g: S \to \mathbb{R}$ *and* $k: S \times \Theta \to \mathbb{R}$. *(A) g satisfies SC1 a.e.-*μ*;*[27] *and (B) k is log-spm a.e.-*μ*; are a MPSC for (C)* $G(\theta) \equiv \int g(s)k(s;\theta)d\mu(s)$ *satisfies SC1.*

Several extensions to Lemma 5 will be useful in our applications. To state them, we need another definition: we say that $g: \mathbb{R} \to \mathbb{R}$ satisfies *weak SC1* if there exists a t_0 such that $g(t) \leq 0$ for all $t < t_0$ and $g(t) \geq 0$ for all $t > t_0$, while $h: X \times \mathbb{R} \to \mathbb{R}$ satisfies *weak SC2* in $(\mathbf{x};t)$ if, for all $\mathbf{x}_H > \mathbf{x}_L$, $g(t) \equiv h(\mathbf{x}_H;t) - h(\mathbf{x}_L;t)$ satisfies weak SC1.

Extensions to Lemma 5*:*[28] *Lemma 5 also holds if: (i) g depends on* θ *directly, under the additional restrictions that g is piecewise continuous in* θ *and either (a) g is nondecreasing in* θ*, or (b) for all* θ*, g is non-zero except at a single (fixed) point* s_0*, and further, for all* $\theta_H > \theta_L$*,* $g(s,\theta_H)/g(s,\theta_L)$ *is nondecreasing in s.*

(ii) We allow that for each θ*, there exists a measure* μ^θ *such that* $K(s;\theta) \equiv \int_{-\infty}^{s} k(t,\theta)d\mu^\theta(t)$*, and we define* $G(\theta) \equiv \int g(s)dK(s;\theta)$*. Then, (L5-B) is equivalent to:* θ *orders* $K(\cdot;\theta)$ *by the MLR.*

(iii) $\mathrm{supp}[K(\cdot;\theta)]$ *is constant in* θ*, and (A) is replaced with (A′) g satisfies weak SC1.*

Consider first sufficiency. Intuitively, if *g* satisfies SC1 and crosses zero at s_0, then log-spm of *k* guarantees that as θ increases, the weight on *s* relative to s_0 increases (decreases) for values of *s* where *g* is nonnegative (nonpositive), i.e., where $s > (<) s_0$. More formally, suppose for simplicity that $k>0$. Define $l(s) \equiv k(s;\theta_H)/k(s;\theta_L)$. (L5-B) implies that $l(\cdot)$ is nondecreasing. In turn, this implies that for every possible crossing point s_0:

$$l(s) \leq l(s_0) \text{ for } s<s_0, \text{ while } l(s) \geq l(s_0) \text{ for } s>s_0. \tag{4.1}$$

Condition (4.1) is then used to establish sufficiency, as follows. Given $\theta_H > \theta_L$ and a point s_0 where *g* crosses zero,

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[26] The theory of the preservation of single crossing properties under uncertainty has been studied by a number of authors in the statistics literature. Karlin and Rubin (1956) establish sufficiency, and Karlin (1968) (pp. 233-237) analyzes necessary conditions, under some additional regularity conditions (including the assumption that Θ has at least three points; Karlin's proof is designed to solve a more complicated problem and thus requires an elaborate construction). In Extension (ii), we relax Karlin's maintained assumptions about absolute continuity. When *K* is a probability distribution, the maintained absolute continuity assumption may be undesirable; however, if *k* represents a utility function, it is the right assumption.

[27] We will say that *g*(*s*) satisfies SC1 almost everywhere-μ (a.e.-μ) if conditions in the definition of SC1 hold for almost all (w.r.t. the product measure on $S \times S$ induced by μ) (s_L, s_H) pairs in $S \times S$ such that $s_L < s_H$.

[28] A version of this theorem which gives minimal sufficient conditions for *strict* single crossing is provided in Athey (1996).

$$\begin{aligned}&\int g(s)k(s;\theta_H)d\mu(s)=\int g(s)l(s)k(s;\theta_L)d\mu(s)\\&\quad=-\int_{-\infty}^{s_0}|g(s)|l(s)k(s;\theta_L)d\mu(s)+\int_{s_0}^{\infty}g(s)l(s)k(s;\theta_L)d\mu(s)\qquad(4.2)\\&\quad\geq -l(s_0)\int_{-\infty}^{s_0}|g(s)|k(s;\theta_L)d\mu(s)+l(s_0)\int_{s_0}^{\infty}g(s)k(s;\theta_L)d\mu(s)=l(s_0)\int g(s)k(s;\theta_L)d\mu(s)\end{aligned}$$

The second equality holds because g satisfies SC1, while the inequality follows from (4.1). If, in addition, $l(s_0)\leq 1$, then $\int g(s)k(s;\theta_H)d\mu(s)\leq 0$ implies $\int g(s)k(s;\theta_L)d\mu(s)\leq\int g(s)k(s;\theta_H)d\mu(s)$.

The necessity parts of this theorem can be proved by constructing counterexamples, which are drawn from an appropriate set of test functions:

Lemma 6 (Test functions for single crossing problems) *Define the following sets:*

$\mathcal{G}(\beta)\equiv$ $\{g$: $\exists a,b>0$, $\beta>\varepsilon,\delta>0$, *and* $s_L<s_H$ *s.t.* $g(s)=-a$ *for* $s\in(s_L-\varepsilon,s_L+\varepsilon)$ *and* $g(s)=b$ *for* $s\in(s_H-\varepsilon,s_H+\varepsilon)$, $|g|<\delta$ *elsewhere, and g satisfies SC1*$\}$

$\mathcal{K}(\beta)\equiv$ $\{k$: $\exists\, a>0$ and $h:\Theta\to S$, *h increasing, and some* $\beta>\varepsilon>0$ *s.t.* $k(s,\theta)=a$ *for* $s\in(h(\theta)-\varepsilon,h(\theta)+\varepsilon)$, *and* $k(s,\theta)=0$ *elsewhere*$\}$.

Then Lemma 5 holds if, for any $\beta>0$, *(L5-A) is replaced with (L5-A') $g\in\mathcal{G}(\beta)$; Lemma 5 also holds if (L5-B) is replaced with (L5-B') $k\in\mathcal{K}(\beta)$.*

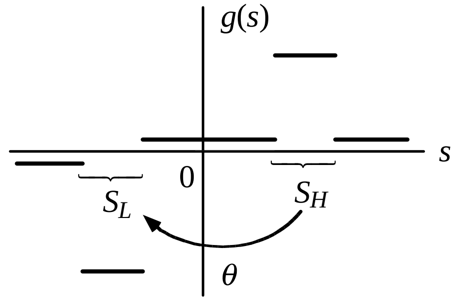

Figure 1: A test function for the set of single crossing functions.

As in Lemma 4, placing smoothness assumptions on g or k will not change the conclusions of Lemmas 5 and 6, but monotonicity or curvature assumptions will.

Now consider how the counter-examples are used. If g fails (L5-A), then there is a positive measure of pairs $s_L < s_H$ such that $g(s_L)>0$, but $g(s_H)<0$. But then, k can be defined so that $k(\cdot;\theta_H)$ places all of the weight on high points s_H, while $k(\cdot;\theta_L)$ places all of the weight on the low points s_L. This k is log-spm, but G fails SC1.

If k fails (L5-B), then there exist two sets of positive measure, S_H and S_L, such that increasing θ places more weight on S_L relative to S_H. Then we can construct a $g(s)$ that is negative on S_L, positive on S_H, and close to zero everywhere else.

3.1. Applications

3.1.1. The Choice of Distribution and Changes in Risk Preferences

This section considers investment problems where the agent chooses the probability distribution directly, from a parameterized family of distributions. For example, the agent chooses effort in a principal-agent problem, or makes an investment decision. The exogenous parameter (θ) describes risk preferences. We seek a MPSC for the conclusion that investment increases when risk preferences change. For this purpose, Lemma 5 provides succinct proofs of some existing results, and further suggests some new ones.

Let $v : S \times \Theta \to \mathbb{R}$ be an agent's utility function, suppose that $S = [\underline{s}, \overline{s}] \subset \mathbb{R}$, and let $H(\cdot; x)$ be a probability distribution on *S*, parameterized by *x*, with density $h(\cdot; x)$ with respect to *μ*. The agent solves $\max_{x \in B} \int_S v(s,\theta) h(s;x) d\mu(s)$.[29] For simplicity, consider the case where *v* is twice differentiable in *s*, and the derivatives (denoted v_s and v_{ss}) are absolutely continuous. Then:

$$\arg\max_{x \in B} \int_S v(s,\theta) h(s;x) d\mu(s) = \arg\max_{x \in B} - \int_S v_s(s,\theta) H(s;x) d\mu(s)$$

$$= \arg\max_{x \in B} v(\overline{s},\theta) \int_S s\, h(s;x) d\mu(s) + \int_S v_{ss}(s,\theta) \int_{t=\underline{s}}^{s} h(t;x) d\mu(t)$$

The following result follows directly from these equations and Lemma 5.[30]

Proposition 2 *Consider the following conclusion: (C)* $x^*(\theta, B) = \arg\max_{x \in B} \int_S v(s,\theta) h(s;x) d\mu(s)$ *is nondecreasing in θ and B. In each of the following, (A) and (B) are a MPSC for (C):*

	Additional Assumptions:	*(A)*	*(B)*
(i)	*v*≥0, *and* $\{s \mid v(s,\theta) \neq 0\}$ *is constant in θ.*	*v is log-spm. a.e.-μ.*	*h is WSC2 in (x;s) a.e.-μ.*
(ii)	$v_s \geq 0$, *and* $\{s \mid v_s(s,\theta) \neq 0\}$ *is constant in θ.*	v_s *is log-spm. in* $(s, -\theta)$ *a.e.-μ.*	*H is WSC2 in (x;s) a.e.-Lebesgue.*
(iii)	$\int_s s\, f(s;x) ds$ *is constant in x,* $v_{ss} \leq 0$, *and* $\{s \mid v_{ss}(s,\theta) \neq 0\}$ *is constant in θ.*	$-v_{ss}(s,\theta)$ *is log-spm. in* $(s, -\theta)$ *a.e.-μ.*	$\int_{-\infty}^{s} H(t;x) dt$ *is WSC2 in (x;s) a.e.-Lebesgue.*

This result provides necessary and sufficient conditions for comparative statics in a range of applications. Consider each of (i)-(iii) in turn.[31] Case (i) might apply if a principal offers a mechanism to an agent where the allocation that will be received is stochastic.[32] The payoff *u* is log-spm when higher types have a larger relative return to *s*. When the support of *s* is fixed, weak SC2 of a probability density is stronger than FOSD but weaker than MLR. In (ii), hypothesis (A) requires that the agent's Arrow-Pratt risk aversion is nondecreasing in *θ*. Further, (B) requires $H(s;x_H)$ crosses $H(s;x_L)$ at most once, from below, as a function of *s*. Under this assumption, it is possible that increasing *x* decreases the mean as well as the riskiness of the distribution; that is, it might incorporate a mean-risk tradeoff. In case (iii), the agents are restricted to be risk averse.

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[29] This is an example where it is useful to have results that do not rely on concavity of the objective: concavity of the objective in *x* requires additional assumptions (see Jewitt (1988a) and Athey (1998)), which may or may not be reasonable in a given application.

[30] Of these results, only (ii) has received attention in the literature. Diamond and Stiglitz (1974) established the sufficiency side of the relationship, and many authors have since exploited and further studied the result (such as Jewitt (1987, 1989)). Jewitt (1987) shows that (A) is necessary and sufficient for (C) to hold whenever (B) does.

[31] In each of (i)-(iii), the fact that (B) is necessary for the conclusion to hold whenever (A) holds relies crucially on the non-monotonicity of the relevant function in (A). Thus, while the sufficient conditions and the necessity of (A) in each case are quite general, one should be more careful in drawing conclusions about the necessity of (B).

[32] Such uncertainty might arise if the principal cannot observe the agent's choice perfectly, or if the principal must design an error-prone bureaucratic system to carry out the regulation.

We see that x always increases with an agent's prudence (as defined in Section 3.1) if and only if $\int_{-\infty}^{s} H(t;x)dt$ satisfies weak SC2 in (x;s).

3.1.2. How Much to Invest in a Risky Project

This subsection considers two classic problems, the portfolio investment problem and the decision problem of a risk averse firm. We generalize several comparative statics results previously established only for special functional forms.

Consider first the standard portfolio problem, where an agent with initial wealth w invests x in a risky asset with return s, and invests the remainder (w–x) in a risk-free asset with return r. Thus, the agent's payoff can be written $u((w-x)r+sx)$. The marginal returns to investment are given by $\int u'((w-x)r+sx)(s-r)f(s;\theta)d\mu(s)$. Notice that s–r satisfies single crossing, and so long as the utility function is nondecreasing, we can apply Lemma 5 to this problem. In this problem, the crossing point is fixed at s=r for all choices of x: thus, in applying (4.2), we could actually weaken the restriction that f is log-spm. In particular, we could assume that the likelihood ratio, $l(s)$, is less than $l(r)$ for $s<r$, and greater than $l(r)$ for $s>r$. That is, $l(s)$–$l(r)$ satisfies weak SC1(r). On the other hand, if we wish to obtain comparative statics results that hold for all risk-free rates, r, then it will be necessary that $l(s)$ is log-spm.

While the portfolio problem has been widely studied, far fewer results have been obtained for more general investment problems, where potentially risk-averse firms invest in a risky projects $\pi(x,s)$, or make pricing or quantity decisions under uncertainty about demand. Suppose that an agent's objective is as follows: $\max_{x\in B}\int u(\pi(x,s))f(s;\theta)d\mu(s)$, and the solution set is denoted $x^*(\theta,B)$. Thus, π represents a general return function which depends on the investment amount, x, and the state of the world, s. Notice that in this problem, the crossing point of π_x is not the same for all x (as it will be in the portfolio problem).

When this objective function is differentiable, it suffices to check that the marginal returns to x, denoted $\int u'(\pi(x,s))\pi_x(x,s)f(s;\theta)d\mu(s)$, satisfy SC1. The following result treats this case, as well as cases where investment is a discrete choice, or the agent's risk preferences change with the exogenous parameter.[33]

Proposition 3 *Consider the problem* $\max_{x\in B}\int u(\pi(x,s))f(s;\theta)d\mu(s)$. *Assume that* $u(y,\theta)$ *is*

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[33] This result generalizes the existing investment literature in several ways. This literature typically considers the problem where the objective is differentiable and strictly quasi-concave. Landsberger and Meilijson (1990) show that the MLR is sufficient for comparative statics in the portfolio problem, and Ormiston and Schlee (1993) show that general comparative statics results are preserved by the MLR. A few papers consider comparative statics when $\pi(x,s) = h(x)\cdot s$, as in Sandmo's (1971) classic model of a firm facing demand uncertainty. Milgrom (1994) shows that comparative statics results derived for the portfolio problem also hold for Sandmo's model. Proposition 2 highlights the critical role played by the assumption that π satisfies SC2, but not multiplicative separability, extending the existing results to more general models of firm objectives.

increasing and differentiable[34] *in y, and* $\pi(x,s)$ *is nondecreasing in s. Then:*

(A) $\pi(x,s)$ *satisfies SC2 in* $(x;s)$ *a.e.-*μ*, and (B)* $u_1(y,\theta)\,f(s;\theta)$ *is log-spm. in* (s,y,θ) *a.e.-*μ*,*[35] *are a MPSC for the conclusion (C)* $x^*(\theta,B)$ *is nondecreasing in* θ *and B.*

Hypothesis (B) is satisfied if (i) θ decreases the investor's absolute risk aversion, and (ii) θ generates an MLR shift in F. Thus, this result provides a generalization of two basic results in the theory of investment under uncertainty, illustrating that log-supermodularity links the seemingly unrelated conditions on the distribution and the investor's risk aversion.

4. Incorporating Additional Structure on Primitives

Many economic primitives have more structure than a single crossing property. One particular kind of structure that arises in auction games and investment problems is that the incremental return function $g(s)$ is quasi-concave in s. For example, in a portfolio problem, the marginal returns to investment are quasi-concave if the agent is risk averse and the coefficient of relative risk aversion is greater than 1.

Notice that quasi-concavity of g restriction prevents us from constructing the counter-examples of Lemma 5: the test function illustrated in Figure 1 is clearly not quasi-concave. Theorem 2 can then be extended, as follows:

Theorem 3 *For each* $\theta \in \Theta$*, let* $K(\cdot;\theta)$ *be a probability distribution on S. Then (C) (MCS′) holds for all sets B whenever (A) u satisfies SC2 in (x;s) a.e.-*μ*.,*[36] *and for all* $x_H > x_L$*,* $u(x_H,s)-u(x_L,s)$ *is weakly quasi-concave in s a.e.-*μ*,* if and only if *(B) K is log-supermodular.*

By Lemma 2 (and recalling the discussion in Section 2), log-spm of K (corresponding to the Monotone Probability Ratio Order) is weaker than log-spm of k (corresponding to MLR). To understand the difference between the two orders, observe that the Monotone Probability Ratio Order (MPR) requires that θ increases the weight on s relative to the aggregate of all states $s'<s$, while the MLR requires that θ increases the weight on s relative to every individual state $s'<s$.[37] One consequence is that the MPR is more robust to perturbations.

Necessity of (T3-B) can be established using the test functions approach:[38]

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[34] Differentiability is not essential but it simplifies the proof.

[35] If u is not everywhere differentiable in its first argument, the corresponding hypothesis can be stated as follows: $[u(y_H,\theta)-u(y_L,\theta)]\cdot f(s;\theta)$ is log-supermodular in (y_L,y_H,θ,s) for all $y_L < y_H$.

[36] We will say that $g(s)$ satisfies SC1 almost everywhere-μ (a.e.-μ) if conditions (a) and (b) of the definition of SC1 hold for almost all (w.r.t. the product measure on $\mathbb{R}^2$ induced by μ) (s_L,s_H) pairs in $\mathbb{R}^2$ such that $s_L < s_H$. The definition of SC2 a.e.-μ is defined analogously.

[37] Another way to describe the difference is that the MLR requires that a distribution be ordered by First Order Stochastic Dominance conditional on every two-point set, while the MPR requires a distribution to be ordered by First Order Stochastic Dominance conditional on every interval $(-\infty,s]$ (the latter result is established in the working paper (Athey, 1997)).

[38] It turns out that quasi-concavity of g is not necessary for the comparative statics conclusion. This follows because,

Lemma 7 (Test functions for single crossing/quasi-concave problems) *Define the following sets:* $\mathcal{G}(\beta) \equiv \{g: \exists a,b>0, \beta>\varepsilon,\delta>0$, *and* s_0 *s.t.* $g(s)=-a$ *for* $s \in (-\infty, s_0]$, $g(s)=b$ *for* $s \in (s_0, s_0+\varepsilon]$ *and* $g(s)=\delta$ *for* $s \in (s_0, \infty]\}$

Then Theorem 3 holds if, for any $\beta>0$, *(T3-A) is replaced with (T3-A') for all* $x_H > x_L$, $u(x_H,\cdot)-u(x_L,\cdot)=g(\cdot)\in\mathcal{G}(\beta)$.

Figure 2 illustrates a test function from Lemma 7. The expected value of such a function is approximately $b\cdot k(s_0;\theta)-a\cdot K(s_0;\theta)$, highlighting the role of monotonicity of k/K in θ (i.e. log-spm of K).

Athey (1999) further explores how restrictions on risk preferences affect the comparative statics of portfolio and investment problems; here, we present an example from auction theory.

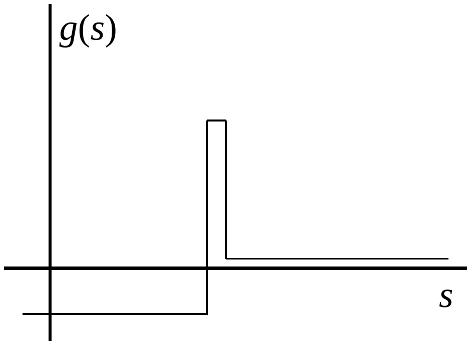

Figure 2: A test function for the set of single-crossing, quasi-concave functions.

4.1. *Application: Mineral Rights Auction with Asymmetries and Risk Aversion*

This section studies Milgrom and Weber's (1982) model of a mineral rights auction, generalized to allow for risk averse, asymmetric bidders whose utility functions are not necessarily differentiable.[39] We focus on the case of two bidders. Suppose that bidders one and two observe signals s_1 and s_2, respectively, where each agent's utility (written $v_i(b_i,s_1,s_2)$) satisfies:

$$v_i(b_i,s_1,s_2) \text{ is nondecreasing in } (-b_i,s_1,s_2) \text{ and supermodular in } (b_i,s_j),\ j=1,2. \quad (4.5)$$

The signals have a joint density h with respect to Lebesgue measure, and the conditional distribution of s_{-i} given s_i is written $H_{-i}(\cdot\,|\,s_i)$, with density $h_{-i}(\cdot\,|\,s_i)$. Condition (4.5) holds, for example, if $v_i(b_i,s_1,s_2)\equiv\int\hat{v}_i(y-b_i)g(y\,|\,s_1,s_2)dy$, where the "common value" y is affiliated with s_1 and s_2, $g(\cdot\,|\,s_1,s_2)$ is the conditional density of y, and $\hat{v}_i$ is nondecreasing and concave.[40]

When player two uses the bidding function $\beta_2(\cdot)$, then the set of best reply bids for player one given her signal (s_1) can be written (assuming ties are broken randomly):

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as a probability distribution, K is restricted to be monotone. This restriction prevents us from constructing the requisite counter-examples. It can, however, be established that g cannot be decreasing and then increasing at the upper end of the support; more generally, any violations of quasi-concavity must not be too severe relative to the magnitude of the function.

[39] As in the pricing game studied in Section 2, this is *not* a game with strategic complementarities between players bidding functions, so Vives (1990) may not be applied to establish existence of a PSNE.

[40] To see why (4.5) holds, note that s_i and s_j each induce a first order stochastic dominance shift on F, and $\hat{u}_i$ is supermodular in (b_i,v). Supermodularity of the expectation in (b_i,s_i) and (b_i,s_j) follows (see Athey (1998)).

$$b_1^*(s_1) = \arg\max_{b_1\in B} \int_{s_2} v_1(b_1,s_1,s_2)\mathbf{1}_{b_1>\beta_2(s_2)}(s_2)h_2(s_2\,|\,s_1)ds_2 + \tfrac{1}{2}\int_{s_2} v_1(b_1,s_1,s_2)\mathbf{1}_{b_1=\beta_2(s_2)}(s_2)h_2(s_2\,|\,s_1)ds_2$$

When bidder two plays a nondecreasing strategy, (4.5) implies that bidder one's payoff function given a realization of s_2 satisfies weak SC2 in $(b_1;s_2)$. Consider first the case where player 2's strategy is strictly increasing. The returns to increasing the bid from b_L to b_H are strictly negative for low values of s_2, when the opponent bids less than b_L; the returns are increasing in s_2 on the region where raising the bid causes the player to win, where she would have lost with b_L; and the effect is zero for s_2 so high that even b_H does not win. Thus, the incremental returns to the bid are quasi-concave as well as weak single crossing. When ties are permitted, the single crossing property remains, but quasi-concavity fails.

Observe further that the payoff function depends directly on s_1 as well. By assumption, the returns to b_1 are increasing in s_1. This yields (using Extension (i) to Lemma 5, and Theorem 4):

Proposition 4 *Consider the 2-bidder mineral rights model, where the utility function satisfies (4.5) above, and the support of the random variables is a product set. (i) Suppose that bidder 2 uses a strategy* $\beta_2(\cdot)$ *which is nondecreasing in* s_2*. Then if the types are affiliated,* $b_1^*(\cdot)$ *is nondecreasing in* s_1*. (ii) Suppose that bidder 2 uses a strategy* $\beta_2(\cdot)$ *which is strictly increasing in* s_2*. Then if* $H_2(s_2\,|\,s_1)$ *is log-spm in* (s_1,s_2)*,* $b_1^*(\cdot)$ *is nondecreasing in* s_1*.*

Athey (forthcoming) applies this result to show that in auction games such as the first price auction described above, a PSNE exists.[41]

5. Single Crossing of Indifference Curves

The Spence-Mirrlees single crossing property (SM) is central to the analysis of monotonicity in standard signaling and screening games, as well as many other mechanism design problems. For an arbitrary differentiable function $h:\mathbb{R}^3 \to \mathbb{R}$ that satisfies $\frac{\partial}{\partial y}h(x,y,t)\neq 0$, (SM) is defined as follows:

$$\frac{\partial}{\partial x}h(x,y,t)\Big/\left|\frac{\partial}{\partial y}h(x,y,t)\right| \text{ is nondecreasing in } t. \qquad \text{(SM)}$$

When the (*x*,*y*) indifference curves are well defined, (SM) is equivalent to the requirement that the indifference curves cross at most once as a function of *t*. We will make use the following assumption which guarantees that the (*x*,*y*)-indifference curves are well-behaved:[42]

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[41] What happens when we try to extend this model to $I\geq 2$ bidders? If the bidders face a symmetric distribution, and all opponents use the same symmetric bidding function, then only the maximum signal of all of the opponents will be relevant to bidder one. Define $s_M = \max(s_2,..,s_I)$. Milgrom and Weber (1982) show that (s_1,s_M) are affiliated when the distribution is exchangeable. Further, if the opponents are using the same strategies, whichever opponent has the highest signal will necessarily have the highest bid. Then we can apply Proposition 4 to this problem exactly as if there were only two bidders. Unfortunately, this approach does not extend directly to *n*-bidder, asymmetric auctions with common value elements. Under asymmetric distributions (or if players use asymmetric strategies), affiliation of the signals is not sufficient to guarantee that the signal of the highest bidder is affiliated with a given player's signal, nor is it sufficient to guarantee log-supermodularity of the conditional distribution.

[42] It is also possible to generalize SM to the case where *h* is not differentiable, but we maintain (WB) for simplicity.

h is differentiable in (x,y); $\frac{\partial}{\partial y} h(x,y,t) \neq 0$; the (x,y)-indifference curves are closed curves. (WB)

The following result characterizes (SM) for objective functions of the form $V(x,y,\theta) \equiv \int v(x,y,s) f(s;\theta) d\mu(s)$.

Lemma 8 *Let* $v : \mathbb{R}^3 \to \mathbb{R}$ *and* $f : \mathbb{R}^2 \to \mathbb{R}_+$, *and suppose that v and V satisfy (WB). Then (A) v(x,y,s) satisfies (SM) a.e.-μ, and (B) f is log-spm in (s;θ) a.e.-μ, are a MPSC for (C)* $V(x,y,\theta)$ *satisfies (SM).*

Theorem 4 (Comparative Statics and the Spence-Mirrlees SCP) *Lemma 8 also holds if (L8-C) is replaced with (C)* $x^*(\theta,B) = \arg\max_{x\in B} V(x,b(x),\theta)$ *is nondecreasing in θ and B for all* $b : \mathbb{R} \to \mathbb{R}$.

Sufficiency in Lemma 8 can be shown using Lemma 2. Let: $h_1(s)=\left|v_y(x,y,s)\right| f(s;\theta_L)$, $h_2(s)=v_x(x,y,s) f(s;\theta_H)$, $h_3(s)=v_x(x,y,s) f(s;\theta_H)$, $h_4(s)=\left|v_y(x,y,s)\right| f(s;\theta_L)$, and note that (T5-A) and (T5-B) imply:

$$\frac{v_x(x,y,s_L)}{\left|v_y(x,y,s_L)\right|}\frac{f(s_H;\theta_L)}{f(s_L;\theta_L)} \leq \frac{v_x(x,y,s_H)}{\left|v_y(x,y,s_H)\right|}\frac{f(s_H;\theta_H)}{f(s_L;\theta_H)} \tag{4.5}$$

This in turn implies by Lemma 2 that $\frac{\partial}{\partial x} V(x,y,\theta_L) \Big/ \left|\frac{\partial}{\partial y} V(x,y,\theta_L)\right| \leq \frac{\partial}{\partial x} V(x,y,\theta_H) \Big/ \left|\frac{\partial}{\partial y} V(x,y,\theta_H)\right|$.

5.1. Applications to Signaling Games and Savings Problems

Theorem 4 can be applied to an education signaling model, where *x* represents a worker's choice of education, *y* is monetary income, and *θ* is a noisy signal of the worker's ability, *s* (for example, the workers' experience in previous schooling). If the worker's preferences *u*(*x*,*y*,*s*) satisfy (SM) and higher signals increase the likelihood of high ability, the worker's education choice will be nondecreasing in the signal *θ* for any wage function *w*(*x*).

In another example, consider a consumption-savings problem. Let *x* denote savings and *b*(·) the value function of savings. The agent has an endowment (*θ*) of a risky asset (*s*). The probability distribution over asset returns is given by *F*(·;*θ*). The agent's utility given a realization of *s* is $u(z+s-x, b(x))$, which is assumed to be nondecreasing. The agent solves $\max_{x\in[0,z]} \int u(z+s-x, b(x)) dF(s;\theta)$. The following two conditions are a minimal pair of sufficient conditions for the conclusion that savings increases in *θ*: (A) the marginal rate of substitution of current for future utility, u_1 / u_2, is nondecreasing in *s*, and (B) *F* satisfies MLR.

6. Conclusions

This paper studies conditions for comparative statics predictions in stochastic problems, and shows how they apply to economic problems. The main theorems are summarized in Table 1. The properties of primitives considered in this paper, the various single crossing properties and log-supermodularity, are each necessary and sufficient for comparative statics in an appropriately defined class of problems. Several variations of these results are analyzed, each of which exploits additional structure that can arise in economic problems. Because the properties studied in this paper, and the corresponding comparative statics predictions, do not rely on differentiability or concavity, the results from this paper can be applied in a wider variety of economic contexts than similar results from the existing literature.

This paper builds on results from the statistics literature to establish sufficient conditions for comparative statics. We further provides new results about necessity, while highlighting the limitations of these results. Table 1 summarizes the tradeoffs that must occur between weakening and strengthening assumptions about various components of economic models. Together with Athey (1998)'s analysis of stochastic supermodularity, concavity, and other differential properties, the results in this paper can be used to identify which assumptions are the appropriate ones to guarantee robust monotone comparative statics predictions in a wide variety of stochastic problems in economics.

Table 1: Summary of Results

Thm #	**A: Hypothesis on *u* (a.e.-μ)**	**B: Hypothesis on *f* (a.e.-μ)**	**C: Conclusion**	**Corresponding Comparative Statics Conclusion**
Lem 4; Thm 1	$u(\mathbf{x},\mathbf{s})\geq 0$ is log-spm.	$f(\mathbf{s};\theta)$ is log-spm.	$\int u(\mathbf{x},\mathbf{s})f(\mathbf{s};\theta)d\mu(\mathbf{s})$ is log-spm. in $(\mathbf{x},\theta)$.	$\arg\max_{x\in B}\int u(\mathbf{x},\mathbf{s})f(\mathbf{s};\theta)d\mu(\mathbf{s})$ $\uparrow$ in θ and B.
Lem 5; Thm 2	$u(x,s)$ satisfies SC2 in $(x;s)$.	$f(s;\theta)$ is log-spm.	$\int u(x,s)f(s;\theta)d\mu(s)$ satisfies SC2 in $(x;\theta)$.	$\arg\max_{x\in B}\int u(x,s)f(s;\theta)\, d\mu(s)$ $\uparrow$ in θ and B.
Lem 7; Thm 3	$u(x,s)$ satisfies SC2 and the returns to x are quasi-concave in s.	$F(s;\theta)\geq 0$ is log-spm.	$\int u(x,s)f(s;\theta)d\mu(s)$ satisfies SC2 in $(x;\theta)$.	$\arg\max_{x\in B}\int u(x,s)f(s;\theta)\, d\mu(s)$ $\uparrow$ in θ and B.
Lem 8, Thm 4	$u(x,y,s)$ satisfies SM.	$f(s;\theta)$ is log-spm.	$\int u(x,y,s)f(s;\theta)d\mu(s)$ satisfies SM.	$\arg\max_{x\in B}\int u(x,b(x),s)f(s;\theta)d\mu(s)$ $\uparrow$ in θ for all $b:\mathbb{R}\rightarrow\mathbb{R}$.

Notes: In rows 1, 2, and 4: (A) and (B) are a *minimal pair of sufficient conditions* (Definition 4) for the conclusion (C); further, (C) is equivalent to the comparative statics result in column 4. In row 3, the same relationships hold except that (A) is not necessary for (C) to hold whenever (B) does.

Notation and Definitions: Bold variables are real vectors; italicized variables are real numbers; *f* is non-negative; log-spm. indicates log-supermodular (Definition 3); sets are increasing in the strong set order (Definition 1); SC2 indicates single crossing of incremental returns to *x* (Definition 2); and SM indicates single crossing of *x*-*y* indifference curves (Section 5). Arrows indicate weak monotonicity.

7. Appendix

Definition A1 *For each θ, let* $F(\cdot;\theta):S\to\mathbb{R}_+$ *be nondecreasing and right-continuous. For all* $\theta_H>\theta_L$, *let* $C^{LH}(\cdot)=\frac{1}{2}[F(\cdot;\theta_H)+F(\cdot;\theta_L)]$, *and define* $f^{LH}:S\times\{\theta_L,\theta_H\}\to\mathbb{R}_+$ *so that for all* (s,θ), $F(s;\theta)=\int_{-\infty}^{s}f^{LH}(t;\theta)dC^{LH}(t)$. *The parameter θ indexes* $F(\cdot;\theta)$ *according to the* ***Monotone Likelihood Ratio Order*** *(MLR) if, for all* $\theta_H>\theta_L$, f^{LH} *is log-spm a.e.-μ.*[43]

Proof of Lemma 1: If U is log-spm in ($\mathbf{x}$,θ), then it must be quasi-spm, which Milgrom and Shannon (1994) show implies the comparative statics conclusion. Now suppose that U fails to be log-spm in ($\mathbf{x}$,θ) for some u. For simplicity, consider the case where n=2, and start with the case where U>0. Then, there exists an $x_{1H}>x_{1L}$, $x_{2H}\geq x_{2L}$ and $\theta_H\geq\theta_L$, with one of the weak inequalities holding strictly, such that $U(x_{1H},x_{2H},\theta_H)/U(x_{1L},x_{2H},\theta_H)$ $<U(x_{1H},x_{2L},\theta_L)/U(x_{1L},x_{2L},\theta_L)$. Let $\gamma=U(x_{1L},x_{2L},\theta_L)/U(x_{1H},x_{2L},\theta_L)$ >0. Then, define $b:X_1\to\mathbb{R}$ by $b(x_1)$ = 1 if $x_1\neq x_{1H}$, and $b(x_{1H})$ = γ. Since log-spm is preserved by multiplication, $v(\mathbf{x},\mathbf{s})\equiv u(\mathbf{x},\mathbf{s})\cdot b(x_1)$ is log-spm in ($\mathbf{x}$,$\mathbf{s}$). Define $V(\mathbf{x},\theta)\equiv\int v(\mathbf{x},\mathbf{s})f(\mathbf{s};\theta)d\mu(\mathbf{s})$. But then, $V(x_{1H},x_{2H},\theta_H)/V(x_{1L},x_{2H},\theta_H)=\gamma U(x_{1H},x_{2H},\theta_H)/U(x_{1L},x_{2H},\theta_H)<1=$ $=\gamma U(x_{1H},x_{2L},\theta_L)/U(x_{1L},x_{2L},\theta_L)$ $=V(x_{1H},x_{2L},\theta_L)/V(x_{1L},x_{2L},\theta_L)$. Thus, $V(x_{1H},x_{2L},\theta_L)=V(x_{1L},x_{2L},\theta_L)$ while $V(x_{1H},x_{2H},\theta_H)<V(x_{1L},x_{2H},\theta_H)$, violating the requirement that (a) V is quasi-spm in $\mathbf{x}$ *and* (b) V satisfies SC2 in ($\mathbf{x}$;θ).

Now, return to the case where we might have $U(\mathbf{x},\theta)$=0 for some ($\mathbf{x}$,θ). Note that since $U\geq 0$, log-spm of U can fail in the example above only if $U(x_{1H},x_{2L},\theta_L)>0$ and $U(x_{1L},x_{2H},\theta_H)>0$. If, in addition, $U(x_{1L},x_{2L},\theta_L)$ >0, the argument above is unchanged. Now suppose $U(x_{1L},x_{2L},\theta_L)$ =0. If $U(x_{1H},x_{2H},\theta_H)$ =0, then quasi-supermodularity fails as well; if $U(x_{1H},x_{2H},\theta_H)$ >0, then define v as above except with $\gamma=U(x_{1L},x_{2H},\theta_H)/U(x_{1H},x_{2H},\theta_H)$. Then, we have $V(x_{1H},x_{2L},\theta_L)>V(x_{1L},x_{2L},\theta_L)=0$ while $V(x_{1H},x_{2H},\theta_H)=V(x_{1L},x_{2H},\theta_H)$, another violation of Milgrom and Shannon's (1994) necessary and sufficient conditions for comparative statics.

Proof of Lemma 4: Sufficiency follows from Lemma 2. Necessity is treated in two main cases, with other cases following in a similar manner:

(1) n,m=1: Define $v(A|\theta)\equiv\int_A f(s;\theta)d\mu(s)$. Pick any two intervals of length ε, $S_H(\varepsilon)$ and $S_L(\varepsilon)$, such that $S_H(\varepsilon)\geq S_L(\varepsilon)$ and $S_H(\varepsilon)\cap S_L(\varepsilon)=\varnothing$. Define $u:X\times S\to\mathbb{R}$ by $u(x_L,s)\equiv\mathbf{1}_{S_L(\varepsilon)}(s)$, and let $u(x_H,s)\equiv\mathbf{1}_{S_H(\varepsilon)}(s)$. Then $\int u(x_L,s)f(s,\theta)d\mu(s)=v(S_L|\theta)$ and $\int u(x_H,s)f(s;\theta)d\mu(s)=v(S_H|\theta)$. Since u is log-spm, then $\int u(x,s)f(s;\theta)d\mu(s)$ must be log-spm by (C), i.e., $v(S_H,\theta_H)v(S_L,\theta_L)\geq v(S_H,\theta_L)v(S_L,\theta_H)$. Taking the limit as $\varepsilon\to 0$, standard limiting arguments (e.g. Martingale convergence theorem) imply that f must be log-spm in (s,θ) almost everywhere-μ.

(2) n,m≥2: Define $v(A\,|\,\boldsymbol{\theta})=\int_A f(\mathbf{s};\boldsymbol{\theta})d\mu(\mathbf{s})$. We partition $\mathbb{R}^m$ into m-cubes with ith element

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[43] Observe that absolute continuity of $F(\cdot;\theta_H)$ with respect to $F(\cdot;\theta_L)$ on the intersection of their supports is a consequence of the definition, not prerequisite for comparability.

given by $[i_1 - \frac{1}{2^t}, (i_1+1)\frac{1}{2^t}] \times \cdots \times [i_m - \frac{1}{2^t}, (i_m+1)\frac{1}{2^t}]$, and let $Q^t(\mathbf{s})$ be the unique cube containing **s**. Consider a t, and take any $\mathbf{a},\mathbf{b} \in \mathbb{R}^m$. Further, let $C = \{\mathbf{y}, \mathbf{z}, \mathbf{y} \vee \mathbf{z}, \mathbf{y} \wedge \mathbf{z}\}$, where each element of C is distinct. Define $u(\mathbf{x},\mathbf{s})$ on C as follows: $u(\mathbf{y},\mathbf{s}) = \mathbf{1}_{Q^t(\mathbf{a})}(\mathbf{s})$, $u(\mathbf{z},\mathbf{s}) = \mathbf{1}_{Q^t(\mathbf{b})}(\mathbf{s})$, $u(\mathbf{y} \vee \mathbf{z},\mathbf{s}) = \mathbf{1}_{Q^t(\mathbf{a}\vee\mathbf{b})}(\mathbf{s})$, and $u(\mathbf{y} \wedge \mathbf{z},\mathbf{s}) = \mathbf{1}_{Q^t(\mathbf{a}\wedge\mathbf{b})}(\mathbf{s})$. Let $u(\mathbf{x},\mathbf{s}) = 0$ for $\mathbf{x} \notin C$. It is straightforward to verify that u is log-spm. If $\int u(\mathbf{x},\mathbf{s}) f(\mathbf{s};\boldsymbol{\theta}) d\mu(\mathbf{s})$ is log-spm, it follows that $\nu(Q^t(\mathbf{a}\vee\mathbf{b})|\boldsymbol{\theta}\vee\boldsymbol{\theta}') \cdot \nu(Q^t(\mathbf{a}\wedge\mathbf{b})|\boldsymbol{\theta}\vee\boldsymbol{\theta}') \geq \nu(Q^t(\mathbf{a})|\boldsymbol{\theta}) \cdot \nu(Q^t(\mathbf{b})|\boldsymbol{\theta}')$ for all $\boldsymbol{\theta},\boldsymbol{\theta}'$. Since this must hold for all t and for all **a**,**b**, we can use the Martingale convergence theorem to conclude that $f(\mathbf{a}\vee\mathbf{b};\boldsymbol{\theta}\vee\boldsymbol{\theta}') \cdot f(\mathbf{a}\wedge\mathbf{b};\boldsymbol{\theta}\wedge\boldsymbol{\theta}') \geq f(\mathbf{a};\boldsymbol{\theta}) \cdot f(\mathbf{b};\boldsymbol{\theta}')$ for μ-almost all **a**,**b** (recalling that μ is a product measure).

Proof of Proposition 1: Sufficiency follows from Lemma 2 and Milgrom and Shannon (1994). Following the proof of Theorem 1, it is possible to show that $U_i(x_i,t_i)$ is log-supermodular for all $u_i(x_i,\mathbf{t})$ log-supermodular, only if, for all $\mathbf{t}_{-i}^H > \mathbf{t}_{-i}^L$ and all $t_i^H > t_i^L$, $h_i(\mathbf{t}_{-i}^H|t_i^H)h_i(\mathbf{t}_{-i}^L|t_i^L) \geq h_i(\mathbf{t}_{-i}^L|t_i^H)h_i(\mathbf{t}_{-i}^H|t_i^L)$. But, since for a positive function, log-spm can be checked pairwise, this condition holds for all i if and only if h is log-spm. Apply Lemma 1.

Lemma A1 *Let* $K(s;\theta) \equiv \int_{-\infty}^{s} k(s;\theta) d\mu(s)$. $G(\theta) = \int g(s;\theta)k(s;\theta)d\mu(s)$ *satisfies SC1 in* θ *under the following sufficient conditions: (i)(a) For each* θ, *g satisfies WSC1 in s a.e.-*μ*; for* μ*-almost all s, g is nondecreasing in* θ*. (i)(b) k is log-spm in* (s,θ) *a.e-*μ*. (i)(c) Either g satisfies SC1 in s a.e-*μ*, or else* supp$[K(\cdot;\theta)]$ *is constant in* θ*.*

Proof of Lemma A1: Pick $\theta_H > \theta_L$. Suppose that $\int g(s;\theta_L)k(s;\theta_L)d\mu(s) \geq (>) 0$. Choose s_0 so that $g(s,\theta_L) \geq 0$ for μ- almost all $s > s_0$ and $g(s,\theta_L) \leq 0$ for μ- almost all $s \leq s_0$ (s_0 exists by the definition of WSC1). Let $\hat{s}_0 = \min\{s \geq s_0 \text{ and } s \in \text{supp}[K(\cdot;\theta_L)]\}$. First, a fact that we will use repeatedly: if k is log-spm, then $\text{supp}[K(\cdot;\theta)]$ is increasing in the strong set order. Now, observe that if $\hat{s}_0 \notin \text{supp}[K(\cdot;\theta_H)]$, then $\text{supp}[K(\cdot;\theta_H)] > \hat{s}_0$ since $\text{supp}[K(\cdot;\theta)]$ is nondecreasing in the strong set order. This in turn implies that, for all s in $\text{supp}[K(\cdot;\theta_H)]$, $g(s,\theta_H) \geq g(s,\theta_L) \geq 0$, which implies $G(\theta_H) \geq 0$. Further, if $G(\theta_L) > 0$, then we can conclude that $G(\theta_H) > 0$ by (i)(c). Second, observe that the case where $\text{supp}[K(\cdot;\theta_H)] \leq \hat{s}_0$ is degenerate, since this would imply by the strong set order that $\text{supp}[K(\cdot;\theta_L)] \leq \hat{s}_0$ as well. But then our hypothesis that $G(\theta_L)$ is nonnegative would imply that $g(s,\theta_L) = 0$ a.e.-μ on $\text{supp}[K(\cdot;\theta_L)]$. Since supp$[K(\cdot;\theta)]$ is nondecreasing in the strong set order, this in turn implies $G(\theta_H) = 0$ by (i)(c). So, we consider the third case where $\hat{s}_0 \in \text{supp}[K(\cdot;\theta_H)]$, but there exist $s', s'' \in \text{supp}[K(\cdot;\theta_H)]$ such that $s' < \hat{s}_0 < s''$. Notice that, by the strong set order, $\text{supp}[K(\cdot;\theta_H)] = \text{supp}[K(\cdot;\theta_L)]$ on an interval surrounding $\hat{s}_0$. It further implies that $g(s,\theta_L) \leq 0$ for all $s \in \text{supp}[K(\cdot;\theta_L)] \backslash \text{supp}[K(\cdot;\theta_H)]$ (because everything in the set lies below $\text{supp}[K(\cdot;\theta_H)]$, which contains $\hat{s}_0$). By the same reasoning, $g(s,\theta_L) \geq 0$ on $\text{supp}[K(\cdot;\theta_H)] \backslash \text{supp}[K(\cdot;\theta_L)]$.

Now, define a modified likelihood ratio $\hat{l}(s)$, as follows: $\hat{l}(s) = 0$ for $s \notin \text{supp}[K(\cdot;\theta_L)]$, $\hat{l}(s) = k(s;\theta_H)/k(s;\theta_L)$ for $s \in \text{supp}[K(\cdot;\theta_L)]$ and $k(s;\theta_L) > 0$, and then extend the function so that $\hat{l}(s) = \max\left(\lim_{s' \downarrow s} \hat{l}(s'), \lim_{s' \uparrow s} \hat{l}(s')\right)$ for $s \in \text{supp}[K(\cdot;\theta_L)]$ and $k(s;\theta_L) = 0$ (recalling that the likelihood ratio can be assumed to be nondecreasing in s on $\text{supp}[K(\cdot;\theta_L)]$ without loss of generality by (i)(b)). Thus, we know $\hat{l}(\hat{s}_0)>0$, since $\hat{s}_0 \in \text{supp}[K(\cdot;\theta_L)]$ and since $\text{supp}[K(\cdot;\theta_H)]=\text{supp}[K(\cdot;\theta_L)]$ on an open interval surrounding $\hat{s}_0$. We use this to establish:

$$\int g(s;\theta_H)k(s;\theta_H)d\mu(s) \geq \int g(s;\theta_L)k(s;\theta_H)d\mu(s) \geq \int g(s;\theta_L)\hat{l}(s)k(s;\theta_L)d\mu(s)$$

$$\geq -l(\hat{s}_0)\int_{-\infty}^{\hat{s}_0} |g(s;\theta_L)| k(s;\theta_L)d\mu(s) + l(\hat{s}_0)\int_{\hat{s}_0}^{\infty} g(s;\theta_L)k(s;\theta_L)d\mu(s) = l(\hat{s}_0)\int g(s;\theta_L)k(s;\theta_L)d\mu(s)$$

The first inequality follows by the fact that $g(s,\theta_H) \geq g(s,\theta_L)$. The second inequality follows by the definition of $\hat{l}(s)$ and since $g(s,\theta_L) \geq 0$ on $\text{supp}[K(\cdot;\theta_H)]\backslash\text{supp}[K(\cdot;\theta_L)]$. The equality follows by WSC1 of g. The third inequality is true because $\hat{l}(s)$ is nondecreasing a.e.-μ on $\text{supp}[K(\cdot;\theta_L)]$ since k is log-spm. The last equality is definitional. Thus, since $\hat{l}(s)>0$, $G(\theta_L) \geq (>) 0$ implies $G(\theta_H) \geq (>) 0$.

Lemma A2 *If* $\int g(s)dK(s;\theta)$ *satisfies SC1 whenever g satisfies SC1, then (i)* $\text{supp}[K(\cdot;\theta)]$ *is nondecreasing in the strong set order, and (ii) for all* $\theta_H > \theta_L$, $K(\cdot;\theta_H)$ *is absolutely continuous with respect to* $K(\cdot;\theta_L)$ *on* $(\inf_s\text{supp}[K(\cdot;\theta_H)], \sup_s\text{supp}[K(\cdot;\theta_L)])$.

Proof of Lemma A2: Pick $\theta_H > \theta_L$. Define measures ν_L and ν_H as follows: $\nu_L(A)=\int_A dK(s;\theta_L)$ and $\nu_H(A) = \int_A dK(s;\theta_H)$. Define $a = \inf\{s | s \in \text{supp}[K(\cdot;\theta_H)]\}$ and $b= \sup\{s | s \in \text{supp}[K(\cdot;\theta_L)]\}$. Let $C(\cdot;\theta_H,\theta_L) = \frac{1}{2}[K(\cdot;\theta_L) + K(\cdot;\theta_H)]$, and define $h(s;\theta) \equiv dK(s;\theta_H)/dC(s;\theta_H,\theta_L)$. Let $D \equiv \text{supp}[K(\cdot;\theta_L)] \cup \text{supp}[K(\cdot;\theta_H)]$. Since the behavior of $h(s;\theta)$ outside of D will not matter, we will restrict attention to D. The proof proceeds in several steps.

Part (a): If $a \geq b$, then the conclusions hold automatically. Throughout the rest of the proof, we treat the case where $a < b$.

Part (b): For any $S \equiv (s_L, s_H] \subset [a,b]$, $\nu_L(S) > 0$ implies that $\nu_H(S) > 0$. Proof: Suppose that $\nu_L(S) > 0$ and $\nu_H(S) = 0$. Note that $0 < K(s_L;\theta_H)$ since $a < s_L$. If $\text{supp}[K(s_L;\theta_H)] \leq s_L$, then define g as follows. $g(s)=-1$ for $s \in (-\infty, s_L)$, while $g(s)=K(s_L;\theta_L)/\nu_L([s_L,\infty))$ for $s \in [s_L,\infty)$. Otherwise, define g as follows: $g(s)=-1$ for $s \in (-\infty, s_L)$, $g(s)=K(s_L;\theta_L)/\nu_L(S)$ for $s \in S$, and $g(s)=.9 \cdot K(s_L;\theta_H)/\nu_H([s_H,\infty))$ for $s \in [s_H,\infty)$. Now, it is straightforward to verify that SC1 is violated for this g.

Part (c): For any $S \equiv (s_L, s_H] \subset [a,b]$, $\nu_H(S) > 0$ implies that $\nu_L(S) > 0$. Proof: Suppose that $\nu_H(S) > 0$ and $\nu_L(S) = 0$. If $K(s_L;\theta_L)=0$, then define g as follows: $g(s)=-1$ for $s \in (-\infty, s_H)$, while $g(s)=K(s_H;\theta_H)/(2\nu_H([s_H,\infty))$ for $s \in [s_H,\infty)$. Otherwise, define g as follows: $g(s)=-\nu_L([s_H,\infty))/K(s_L;\theta_L)$ for $s \in (-\infty, s_L)$, $g(s)=-\nu_H([s_H,\infty))/\nu_H(S)$ for $s \in S$, and $g(s)=1$ for $s \in [s_H,\infty)$. Now, it is straightforward to verify that SC1 is violated for this g.

Part (d): $\text{supp}[K(\cdot;\theta_H)] \geq \text{supp}[K(\cdot;\theta_L)]$ in the strong set order. Proof: Parts (c) and (d) imply that ν_H is absolutely continuous with respect to ν_L on $[a,b]$, and vice versa. It now suffices to show that if $s' \in \text{supp}[K(\cdot;\theta_H)]$ and $s'' \in \text{supp}[K(\cdot;\theta_L)]$, and $s''>s'$, then $s',s'' \in \text{supp}[K(\cdot;\theta_L)] \cap \text{supp}[K(\cdot;\theta_H)]$. Since $s \notin \text{supp}[K(\cdot;\theta_L)]$ for all $s>b$, we may restrict attention to $s'' \leq b$. Likewise we may restrict attention to $s'>a$. But, if (as we argued in part (b)) ν_H is absolutely continuous with respect to ν_L and vice versa on [*a*,*b*], then supp[$K(\cdot;\theta_H)$]=supp[$K(\cdot;\theta_L)$] on [*a*,*b*], and we are done.

Proof of Lemmas 5 and 6: Sufficiency follows from Lemma A1. Necessity will follow by constructing counter-examples from the relevant sets. Necessity of (L5-A) follows by observing that for any $s_L < s_H$, if we let $S_L(\varepsilon)$= $(s_L-\varepsilon, s_L]$ and $S_H(\varepsilon)$ = $(s_H-\varepsilon, s_H]$, the function *k* defined by $k(s;\theta_L)\equiv \mathbf{1}_{S_L(\varepsilon)}(s)$, $k(s;\theta_H)\equiv \mathbf{1}_{S_H(\varepsilon)}(s)$ is log-spm. Thus, (L5-C) will imply that *g* is SC1 a.e.-μ. Necessity of (L5-B): Pick $\theta_H > \theta_L$, and define ν_L, ν_H, *a*, *b*, *h*, and *D* as in the proof of Lemma A2. It suffices to consider log-spm of *h*. Note that Lemma A2 implies that ν_H is absolutely continuous w.r.t. ν_L and that supp$[K(\cdot;\theta_H)]\geq$supp$[K(\cdot;\theta_L)]$ in the strong set order.

First, notice that if $a \geq b$, then *h* is log-spm in (s;θ) a.e.-μ. To see this, observe that $a \geq b$ implies that $h(s,\theta_H)$ = 0 and $h(s,\theta_L)$ = 2 for *s* on supp$[K(\cdot;\theta_L)]$, while $h(s,\theta_L)$ = 0 and $h(s,\theta_H)$ = 2 on supp$[K(\cdot;\theta_H)]$. This implies that $h(s;\theta)$ is log-spm.

Now, suppose $a<b$. Let B = supp$[K(\cdot;\theta_L)]\cap$supp$[K(\cdot;\theta_H)]$, which we have shown is equivalent to $D\cap[a,b]$. Pick any $s_L, s_H \in B$, and define $S_L(\varepsilon)$= $(s_L-\varepsilon, s_L]$ and $S_H(\varepsilon)$ = $(s_H-\varepsilon, s_H]$, such that $s_H-\varepsilon \geq s_L$. For the moment, we will suppress the ε in our notation. Suppose further that $s_L, s_H \in B$, but $\nu_H(S_H)\cdot\nu_L(S_L) < \nu_H(S_L)\cdot\nu_L(S_H)$. By definition, if $s_L > a$, then $\nu_H([a, s_L-\varepsilon])>0$; then, by absolute continuity, $\nu_L([a, s_L-\varepsilon])>0$ and thus $K(s_L-\varepsilon;\theta_L)>0$. Let $g(s;\delta)$ be defined as follows:

$$g(s;\delta)=\begin{cases} -\delta\cdot\dfrac{\nu_L([s_L,\infty))-\nu_L(S_H)}{K(s_L-\varepsilon;\theta_L)} & s\in(-\infty, s_L-\varepsilon) \\ -1 & s\in S_L \end{cases} \quad \begin{cases} \delta & s\in[s_L,\infty)\setminus S_H \\ \dfrac{\nu_L(S_L)}{\nu_L(S_H)} & s\in S_H \end{cases}$$

It is straightforward to verify that there exists a $\delta>0$ such that the single crossing property fails with this *g*. But this implies that for any ε positive and in the relevant range, $\nu_H(S_H(\varepsilon))\cdot\nu_L(S_L(\varepsilon)) \geq \nu_H(S_L(\varepsilon))\cdot\nu_L(S_H(\varepsilon))$. But this implies that *h* is log-spm in (s,θ) a.e.-*C* on *B*. Since supp$[K(\cdot;\theta_H)]\geq$supp$[K(\cdot;\theta_L)]$, this implies that *h* is log-spm in (s,θ) a.e.-*C* on *D*.

Extensions to Lemma 5: Part (i): Case (a) follows from Lemma A1. Case (b): Consider the case where *g* crosses 0 (otherwise, the expectation is always non-negative) at $s=s_0$, and $k>0$; the other cases can be handled in a manner similar to the proof of Theorem 2. Then, we extend (4.2) as follows:

$$\begin{aligned} &\int g(s;\theta_H)k(s;\theta_H)d\mu(s) \\ &\quad\geq \lim_{s\to s_0}\frac{g(s;\theta_H)k(s;\theta_H)}{g(s;\theta_L)k(s;\theta_L)}\left[-\int_{-\infty}^{s_0}|g(s;\theta_L)|k(s;\theta_L)d\mu(s)+\int_{s_0}^{\infty}g(s;\theta_L)k(s;\theta_L)d\mu(s)\right] \\ &\quad= \lim_{s\to s_0}\frac{g(s;\theta_H)k(s;\theta_H)}{g(s;\theta_L)k(s;\theta_L)}\int g(s;\theta_L)k(s;\theta_L)d\mu(s) \end{aligned}$$

The inequality follows, as in Theorem 2, because *g* crosses zero at s_0 and $\dfrac{g(s;\theta_H)k(s;\theta_H)}{g(s;\theta_L)k(s;\theta_L)}$ is non-negative and nondecreasing. Part (ii): Sufficiency follows from Lemma A2. Necessity of (A) follows from Lemma 5. Necessity of (B) follows by Lemma A2, since once the properties of Lemma A2 are established, the proof of Lemma 5 applies. Part (iii): follows from Lemma A1.

Proof of Proposition 2: (A) and (B) imply (C): If π is differentiable in *x* and *B* is a convex set, we can analyze whether $\int u'(\pi(x,s))\pi_x(x,s)\,f(s;\theta)\,d\mu(s)$ satisfies SC1. We can let $g=\pi_x$, and let $k=u_1 f$, and apply Theorem 2. Now consider the investor's choice between two values of *x*, $x_H > x_L$. Then, define $g(s,\theta)\equiv u(\pi(x_H,s),\theta)-u(\pi(x_L,s),\theta)$, and let $k(s;\theta)\equiv f(s;\theta)$. First, observe that SC2 is preserved under monotone transformations, so that if *u* is nondecreasing in its first argument, then by (A), $u(\pi(x,s),\theta)$ must satisfy SC2 in $(x;s)$, and $g(s,\theta)$ satisfies SC1 in *s*. Let s_0 be the crossing point of π_x. First restrict attention to $s \geq s_0$, where $\pi(x_H,s)\geq\pi(x_L,s)$. Define $h(a,b,\theta)\equiv\int_{y=a}^{b} u_1(y,\theta)dy$, and note that *h* is log-spm in (a,b,θ) for all $a<b$, by Lemmas 2 and 3 and (B). This in turn implies that $g(s,\theta)= h(\pi(x_L,s),\pi(x_H,s),\theta)$ is log-spm in (s,θ) on $s\geq s_0$ since π is nondecreasing in *s*, and thus $g(s;\theta_H)/g(s;\theta_L)$ is nondecreasing in *s* on $s\geq s_0$. On the other hand, if $s<s_0$, $\pi(x_H,s)\leq\pi(x_L,s)$, and $g(s,\theta)=-h(\pi(x_L,s),\pi(x_H,s),\theta)$. Then, $g(s;\theta_H)/g(s;\theta_L)$ is nondecreasing in *s* on $s<s_0$ since *h* is log-spm, and we apply extension (i) of Lemma 5. Necessity follows by Theorem 2 for the case where π is differentiable; the proof is omitted for the more general case.

Proof of Lemma 7: Consider sufficiency first. The argument is easiest to see when *g* is absolutely continuous, so that integration by parts may be used. We also work backwards in establishing the single crossing property: we show that $G(\theta_H)\leq(<)0$ implies $G(\theta_L)\leq(<)0$. To proceed, observe that $G(\theta_H)\leq 0$ if and only if

$$\lim_{s\to\infty} g(s)-\int g'(s)K(s;\theta_H)ds\leq 0,$$

which requires $-\int g'(s)K(s;\theta_H)ds\leq 0$ since *g* is single crossing. But, quasi-concavity of *g* implies that $-g'$ is SC1. Thus, equation (4.2) can be applied, letting $-g'$ play the role of *g*, *K* play the role of *k*, so that, if we define $L(s)\equiv K(s;\theta_H)/K(s;\theta_L)$, we have:

$$-\int g'(s)K(s;\theta_H)ds\geq -L(s_0)\int g'(s)K(s;\theta_L)ds$$

Now, since *K* is a probability distribution, if $L(\cdot)$ is nondecreasing, it must always be less than one. Thus, if $-\int g'(s)K(s;\theta_H)ds\leq 0$, then $-\int g'(s)K(s;\theta_H)ds\geq -\int g'(s)K(s;\theta_L)ds$. But then,

$$G(\theta_L)=\lim_{s\to\infty} g(s)-\int g'(s)K(s;\theta_L)ds\leq \lim_{s\to\infty} g(s)-\int g'(s)K(s;\theta_H)ds=G(\theta_H),$$

and we have established that $G(\theta_H)\leq(<)0$ implies $G(\theta_L)\leq(<)0$. Now consider necessity. Suppose that *K* is not log-supermodular. Then, there exists an open interval $(s_0,s_0+\varepsilon)$ such that $(K(s_0+\varepsilon;\theta)-K(s_0;\theta))/K(s_0;\theta)$ is decreasing in θ. Then, let $g(s;\delta)$ be defined as follows:

$$g(s;\delta)=\begin{cases} -1 & s\in(-\infty,s_0] \\ \dfrac{K(s_0;\theta_L)-\delta(1-K(s_0+\varepsilon;\theta_L))}{K(s_0+\varepsilon;\theta_L)-K(s_0;\theta_L)} & s\in(s_0,s_0+\varepsilon] \end{cases} \qquad \begin{cases} \delta & s\in(s_0+\varepsilon,\infty) \\ & \end{cases}$$

so that $G(\theta_L)=0$, while for δ sufficiently small, $G(\theta_H)<0$.

Proof of Lemma 8: (i): Under the assumptions of the theorem, $v(x,y,s)$ satisfies (SM) if and only if $u(x,s;b)\equiv v(x,b(x),s)$ has SC2 in $(x;s)$ for all functions *b*. Furthermore, $V(x,y,\theta)$ satisfies (SM) if and only if $U(x,\theta;b)\equiv V(x,b(x),\theta)$ has SC2 in $(x;\theta)$ for all functions *b*. So, if we know that $v(x,y,s)$ satisfies (SM), then $u(x,s;b)$ has SC2 in $(x;s)$ for all functions *b*. If *k* is log-spm a.e.-μ, then Theorem 2.1 implies that $U(x,\theta;b)$ has SC2 in $(x;s)$ for all functions *b*. But this in turn implies that $V(x,y,\theta)$ satisfies (SM).

(ii): Consider any $f: S\times\Theta\rightarrow\mathbb{R}_+$. Let $F(s;\theta)$ = $\int_{-\infty}^{s} f(t;\theta)d\mu(t)$. The working paper (Athey, 1997) shows that if $F(\cdot;\theta)$ is not ordered by (MLR), then there exists a continuous g which satisfies SC1 so that $\int g(s)dF(s;\theta)$ fails SC1 (this is a continuous approximation to the test functions from Lemma 5). Consider this function g. We know that, since g is continuous and crosses zero only once, it must be monotone nondecreasing in some neighborhood of the crossing point/region. Now, define the following points, which are the boundaries of the region where $g(s)$ = 0: $c=\inf_{s\in\mathbb{R}}\{s: g(s)=0\}$, $d=\sup_{s\in\mathbb{R}}\{s: g(s)=0\}$. Now, we can find a $\delta > 0$ and two corresponding points, $c_\delta\equiv\sup_{s<c}\{s|g(s)=-\delta\}$ and $d_\delta\equiv\inf_{s>d}\{s|g(s)=\delta\}$, such that $(c,d)\subset(c_\delta,d_\delta)$ and g is nondecreasing on (c_δ,d_δ). Let us define a new function, $\alpha(s)$, as follows: $\alpha(s) = \delta$ for $s\in(c_\delta,d_\delta)$, $\alpha(s) = |g(s)|$ elsewhere. Now, pick any $x_H > x_L$, and let $v(x,y,s)=x\cdot g(s)/(x_H-x_L)+\alpha(s)\cdot y$. Since $\alpha(s)$ and $g(s)$ are continuous and $\alpha(s) > 0$, v satisfies (WB). Finally, $\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y}=g(s)/((x_H-x_L)\cdot\alpha(s))$ is nondecreasing in s. Thus, v satisfies the assumptions of the theorem as well as (SM). Now, $\int v(x,y,s)dF(s;\theta)$ satisfies (SM) if and only if $\int v(x,b(x),s)dF(s;\theta)$ satisfies SC2 in $(x;\theta)$ for all b. Let $b(x) = 0$. But, $v(x_H,0,s)-v(x_L,0,s)=g(s)$, and by construction $\int g(s)dF(s;\theta)$ fails SC1, which in turn implies that $\int v(x,0,s)dF(s;\theta)$ fails SC2 in $(x;\theta)$. Thus, $\int v(x,y,s)dF(s;\theta)$ fails (SM).

(iii): If $v(x,y,s)$ fails the (SM), then there exists a $b(x)$ such that $v(x,b(x),s)$ fails SC2 in $(x;s)$. But then, Theorem 2 implies that there exists an $f(s;\theta)$ which is log-spm a.e.-μ such that $\int v(x,b(x),s)f(s;\theta)d\mu(s)$ fails SC2 in $(x;\theta)$. We conclude that $\int v(x,y,s)f(s;\theta)d\mu(s)$ fails (SM).

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