MS&E 319/CS 369X: Topics in Network Algorithms. Winter 2005-06
Course URL: `http://www.stanford.edu/~ashishg/network-algorithms`.
Instructor: Ashish Goel, Stanford University.

Handout 1: Chernoff Bounds and the Lovász Local Lemma

These two probabilistic inequalities lie at the heart of packet routing theory with small buffers. This is a recurring phenomenon in packet routing – often, simple randomized techinques are easier to analyze than their deterministic counterparts.

Let $X_1, X_2, \ldots, X_m$ be m independent (but not necessarily identical) Bernoulli variables. Let p_i denote $\mathbf{Pr}[X_i = 1]$. Further, define

$$\begin{aligned} S &= \sum_i X_i, \text{ and} \\ \mu &= \mathbf{E}[S]. \end{aligned}$$

Observe that $\mu = \sum_i \mathbf{E}[X_i] = \sum_i p_i$. Chernoff bounds are useful for placing a limit on the probability that S is much larger than μ. In particular, for any $\delta > 0$,

$$\mathbf{Pr}[S > \mu(1+\delta)] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}} \right)^\mu. \quad (1)$$

Lovász's local lemma is useful when the X_i's are not independent, but where the dependence can be bounded. Let $p = \max_i p_i$. Further, let us assume that for each random variable X_i, there is a set T_i containing at least $m - b - 1$ other random variables X_j such that X_i is independent of all variables in the set T_i. Thus, there are at most b "degrees of dependence". The local lemma states that

$$\text{If } 4pb < 1, \text{ then } \mathbf{Pr}[S = 0] > 0. \quad (2)$$

Thus, if we think of the X_i's as bad events, there is a non-zero probability that a bad event does not happen. Unlike Chernoff bounds, this is not a high probability result. Its power lies in the existential statement: "there must be one way of choosing the variables X_i such that the dependencies between them are respected and no bad event happens."

References

[1] R. Motwani and P. Raghavan. *Randomized Algorithms*. Cambridge University Press, 1995.