Mathematical Programming and Combinatorial Optimization
MS&E 212, Spring 2005-2006, Stanford University
Instructor: Ashish Goel
Handout 8: Practice problems for the midterm.

Problems 1-4 formed last year's midterm. Problems 8 and 9 (specially 8) are much harder and problem 7 is much longer than any of the midterm problems are likely to be, but it would help clarify your understanding if you try to solve them. We have not yet discussed "polynomial time" algorithms – interpret that to mean "efficient". Brad will discuss problems 1, 2, 4, 5, 7, 9, and 8 on Thu in his discussion section (but I wouldn't be surprised if he doesn't get to problem 8 at all).

1. **[10 points]** Find a min-cut between r and s in the graph presented in figure 1. The numbers on the edges are the capacities. Prove that your cut is the minimum possible cut.

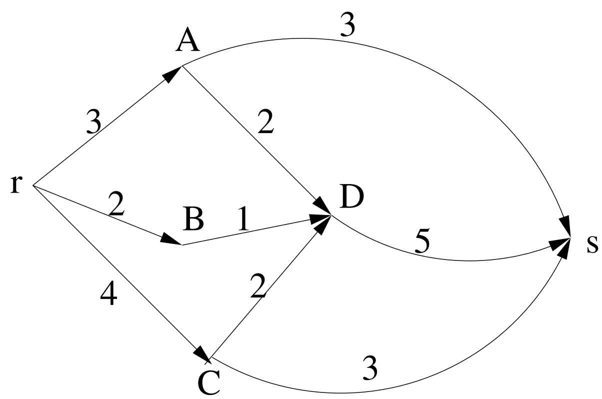

Figure 1:

2. **[10 points]** Give a polynomial time algorithm to find the smallest weight cycle in a weighted, directed graph. Assume there are no negative-weight cycles. Analyze the running time of your algorithm.

3. Given a graph $G = (V, E)$, consider the LP

$$\begin{aligned} &\text{maximize} \quad \sum_{v \in V} x_v \\ &\text{s.t.} \quad \forall (v,w) \in E: \quad x_v + x_w \leq 1 \\ &\qquad\qquad\qquad\qquad\qquad x \geq 0. \end{aligned}$$

(a) [**5 points**] Write down the dual of the LP.

(b) [**10 points**] Assume that the graph is bipartite, i.e., the set of vertices can be partitioned into two parts such that all the edges go from the first part to the second. Reduce the dual to a network flow problem.

4. You are given n goods. The i-th good weighs w_i pounds, and has value v_i dollars. Your goal is to find the minimum weight bundle of goods which has a total value of at least V. Assume that only one unit of each good is available, and that the goods can be chosen fractionally in the bundle.

   (a) [**5 points**] Write down a linear programming formulation of this problem.

   (b) [**10 points**] Model this as a min-cost flow problem. If all the v_i, w_i values are integers, does it follow that the solution will be integral? Explain your reasoning.

   (c) [**Extra credit**] Prove that there exists an optimum bundle where at most one good is chosen fractionally.

5. Given a graph with a capacity on each edge, present an algorithm to find the largest capacity path from a given source vertex s to a given destination vertex t. Analyze the running time of your algorithm.

6. Given a graph with a failure probability π_e defined on each edge, present an algorithm to find the most reliable path from s to t. Assume that the success probability of a path is the product of the success probabilities (i.e. $1 - \pi_e$) of all the edges in the path.

7. The function HEAPIFY does the following: It takes a binary tree that satisfies the heap structure property everywhere and satisfies the heap order property at every location except the root, and transforms this binary tree into a heap.

   (a) Describe how you would implement HEAPIFY. What would the running time of HEAPIFY be in terms of the height of the binary tree?

   (b) In the above description, there is no reason why you can not run HEAPIFY on a sub-tree of the binary tree. Consider the array representation of the heap. Let HEAPIFY(i) denote the operation of running HEAPIFY on the sub-tree rooted at the i-th position in the array. Prove that the running time of HEAPIFY(i)

is $O(H - l)$ where H is the height of the original heap, and l is $\lfloor \log_2 i \rfloor$, i.e. l is the level of element i in the heap.

(c) Consider an array of n elements. Describe what happens if you run HEAPIFY on each of the n positions successively, in reverse order.

(d) Analyze the total running time of the n HEAPIFY functions. *Hint: Make pictures.*

8. **Prove Hall's theorem:** You are given a set of n men and n women. You are also given a set of compatible (man,woman) pairs. A perfect matching is a one to one mapping of men to women such that every pair in the mapping is compatible. Hall's theorem states that a perfect matching exists if and only if for all possible sets S of men, the number of women compatible with at least one man in S is at least $|S|$. Use the max-flow min-cut theorem to prove Hall's theorem.

9. Prove that in a matrix the maximum number of nonzero entries such that no two are in the same row or column is equal to the minimum number of lines that include all the nonzero entries. How can you find this number?