Mathematical Programming and Combinatorial Optimization
MS&E 212, Spring 2005-2006, Stanford University
Instructor: Ashish Goel
Handout 6: Homework 2. Given 5/2/06. Due 5/9/06 in class.

Collaboration policy: Limited collaboration is allowed – you can discuss general strategies with other students in this class but can not collaborate on the actual final answer. Please do not look at someone else's solution and do not share your solution with anyone else.

1. **[10 pts.]** Consider problem 2.35 from the text. Ignore the solution suggested in the text. Instead, model it as a shortest path problem.

2. **[10 pts.]** Prove that for any network without negative cost cycles, there exists a sequence of only $n-1$ relaxations which will result in the termination of Ford's algorithm. How does this imply the correctness of Bellman-Ford?

3. Suppose you are given k tasks and n agents. There is an edge from task i to agent j if this task can be performed by this agent. Each task takes one day to complete. Your goal is to assign tasks to agents such that the completion time of the tasks (i.e. the time when the last task is completed) is minimized. Assume that a task can not be split between two days or two agents.

   (a) **[5 pts.]** Reduce this problem to a single instance of the min-cost flow problem.

   (b) **[15 pts.]** Reduce this to solving a sequence of $O(\log k)$ max-flow problems. Write down the AMPL model for one general problem (i.e. not just the first or the last) in this sequence and run the AMPL model on a data set. Email the model and the data set to the TA but also submit a hard copy.

   (c) **[Extra credit]** Which of the two approaches is superior? Why?

4. **[10 pts.]** Give an algorithm to find a negative cost cycle in a graph if such a cycle exists. Analyze its running time, and mention one possible application.

5. **[10 pts.]** The government of a country gives subsidies to a farmers co-operative on the first 500 tonnes of grain transported on a government operated train route[1]. Hence, the cost of transporting x tonnes of grain over a government train route is $20x$ if $x \leq 500$ and $20x + 50(x - 500)$ if $x > 500$. All other transportation methods are privately run and charge proportional to the amount of grain transmitted. The co-operative has many farms in diverse locations and many cities where there is demand. How can the co-operative determine the cheapest way of transporting its grain? Assume that the total demand is equal to the total supply. Your solution should require solving a single LP or a single flow problem. Explain why your solution is correct. *Something to think about: Would the same approach work if the government charged an extra tax on the first 500 tonnes as opposed to giving a subsidy? Why not?*

[1] This is a per-route subsidy, not an aggregate subsidy. If the co-operative uses multiple routes, it gets the subsidy separately for each route.

Problem 2.35: Suppose that we are given tasks $t_1, t_2, \ldots, t_k$. Each task t_i has a processing time p_i. For certain pairs (i, j), t_i *must precede* t_j, that is, the processing of t_j can not begin till the processing of t_i is completed. We wish to schedule the processing of the tasks so that all of the tasks are completed as soon as possible. Solve this problem as a maximum feasible potential problem on an acyclic digraph.

Ignore the last sentence in the problem. Instead, show how this problem can be solved using a shortest path algorithm. Clarification: multiple tasks can be processed at the same time.