Mathematical Programming and Combinatorial Optimization
MS&E 212, Spring 2005-2006, Stanford University
Instructor: Ashish Goel
Handout 10: Homework 3. Given 5/16/06. Due 5/25/06 in class.

Collaboration policy: Limited collaboration is allowed – you can discuss general strategies with other students in this class but can not collaborate on the actual final answer. Please do not look at someone else's solution and do not share your solution with anyone else.

1. **[10 pts]** You are given an undirected graph $G = (V, E)$ with weights w on edges. Consider the following integer program:

$$\begin{aligned} \textbf{minimize} \ & \sum_e w_e x_e \\ \text{subject to:} & \\ \forall e \in E: \quad & x_e \in \{0,1\} \\ \forall R \subset V, |R| \geq 1: \quad & \sum_{e \in \delta(R)} x_e \geq 1 \end{aligned}$$

Here $\delta(R)$ is the set of edges that go between R and $V - R$. Prove that the above integer program captures the minimum spanning tree problem (i.e., prove that any spanning tree is feasible for this program and any feasible solution corresponds to a spanning tree, possibly with some extra edges).

2. **[10 pts]** Take the linear relaxation of the above integer program (i.e. replace the constraints $x_e \in \{0,1\}$ by $x_e \geq 0$). Solve the linear relaxation when the graph G is a 3-cycle with all edges having the same weight. Are all vertex-optimal solutions of this linear program always integral?

3. **[10 pts]** Give an algorithm for the **maximum** weight spanning tree problem which just invokes the minimum spanning tree algorithm once (on a modified problem). Why would a similar approach not work for converting Dijkstra's algorithm into a **longest** path algorithm?

4. The President of a country with n families decides to offer a tax-break. Let x_i denote the tax-break offered to family i. The President, not being as all-powerful as she initially believed, must abide by the constraints $Ax \leq c$ where A is an $m \times n$ matrix. Assume that x_i's must be non-negative. Assume that all entries in the matrix A and all entries in the vector c are non-negative. The goal of the President, who wants to be re-elected, is to maximize the total tax-break for those $K = n/2$ (assume n is even) families which receive the smallest tax-break. You must have seen these kinds of numbers reported in the newspapers. For example, if $n = 10$, and the tax-breaks are $100, 20, 40, 60, 70, 10, 15, 25, 30, 80$ then the 5 families receiving the smallest tax-break are families 2, 6, 7, 8, and 9, and the total value of the objective function is $20 + 10 + 15 + 25 + 30 = 100$.

   (a) **[20 pts]** Write down an LP to solve the above problem. *Hint: Use an extra variable u to "guess" the value of the median, and then use extra variables y_i such that y_i somehow captures* $\min\{x_i, u\}$.

(b) **[Extra credit]** Suppose all feasible vertex solutions to $x \geq 0; Ax \leq c$ are integral. Is it true that the above LP would always return an integer solution? Either give a proof or a counter-example.