

# Coherence Analysis of Iterative Thresholding Algorithms

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## Abstract

There is a recent surge of interest in developing algorithms for finding sparse solutions of underdetermined systems of linear equations  $y = \Phi x$ . In many applications, extremely large problem sizes are envisioned, with at least tens of thousands of equations and hundreds of thousands of unknowns. For such problem sizes, low computational complexity is paramount. The best studied  $\ell_1$  minimization algorithm is not fast enough to fulfill this need. Iterative thresholding algorithms have been proposed to address this problem. In this paper we want to analyze three of these algorithms theoretically, and give sufficient conditions under which they recover the sparsest solution.

## I. INTRODUCTION

Finding the sparsest solution of an underdetermined system of linear equations  $y = \Phi x$ , is a problem of interest in signal processing, data transmission, biology and statistics-just to name a few. Unfortunately, this problem is NP-hard and in general can not be solved by a polynomial time algorithm. Chen et al. [1] proposed the following convex optimization for recovering the sparsest solution;

$$(\mathcal{Q}_1) \quad \min \|x\|_1 \quad \text{s.t. } \Phi x = y,$$

where  $\ell_p$ -norm is defined as  $\|x\|_p = \sqrt[p]{\sum_i |x_i|^p}$ .

Greedy methods have also been proposed as another alternative for solving such a problem. One of the best known algorithms of this class is orthogonal matching pursuit (OMP) [2]. Intuitively speaking at each iteration, OMP finds a column of  $\Phi$  which has the maximum correlation with the error of the approximation up to this step, and adds it to the active set and projects  $y$  onto the range of the active set to get a new estimate. The third class of algorithms that has drawn a lot of attention recently is the class of iterative thresholding algorithms. This class has the least computational complexity and is the most suitable class for very large scale problems [3]. There are many theoretical results that prove the optimality of the first two classes of algorithms under certain conditions, but there are much less rigorous results for thresholding algorithms. Before mentioning some of the results, we first set up the notation we are going to use in the paper. Suppose that  $x_o \in \mathbb{R}^N$  is a  $k$  sparse vector (i.e. it has at most  $k$  non-zero elements). We observe the measurement vector  $y = \Phi x_o$  which is in  $\mathbb{R}^n$  ( $n < N$ ) and the goal is to reconstruct the original vector  $x_o$ . Without loss of generality, we assume that the columns of  $\Phi$  have unit  $\ell_2$  norm. Another notation that is used in the paper is the notion of restricted submatrices. For a subset of columns of  $\Phi$  called  $J$ ,  $\Phi_J$  includes all the columns of  $\Phi$  whose indices are in  $J$ , and  $x_J$  all the elements of  $x$  whose indices are in  $J$ . The coherence of  $\Phi$  is also defined as,

$$\mu = \max_{\{i,j:1 \leq i,j \leq N, i \neq j\}} |\langle \phi_i, \phi_j \rangle|. \quad (1)$$

where  $\phi_i$  is the  $i^{\text{th}}$  column of the matrix  $\Phi$ . The first question that shall be asked in sparse signal recovery is the uniqueness of the sparsest solution. The following theorem which is due to [4] characterizes the uniqueness of the solution.

**Theorem I.1.** *If  $k \leq (1 + \mu^{-1})$ , then the sparsest solution is unique.*

As mentioned before although under these conditions we know that the sparsest solution exists and is unique, but finding that solution is NP-complete and can not be found by polynomial time algorithm. Therefore  $\ell_1$  minimization and greedy methods are proposed for this task. In the following, a summary of the results proved for  $\ell_1$  minimization and OMP algorithms in [4] and [5] respectively, are presented.

**Theorem I.2.** *If  $k \leq \frac{1}{2}(1 + \mu^{-1})$ , then both the  $\ell_1$  minimization and the OMP recover the sparsest solution.*

When the matrix  $\Phi$  is drawn from a random ensemble [6], [7], we can bound the coherence [8], and find conditions for the exact sparse signal recovery. In this random setting, however, the results can be improved [9]. Although the theoretical results are basically focused on  $\ell_1$  relaxation and greedy methods, many large scale applications have already moved toward the thresholding algorithms [10], [11]. In a recent paper, we considered a few thresholding policies and showed that the results of these algorithms are very impressive in practical situations as compressed sensing [3]. In this paper we focus on the

theoretical aspects of these algorithms and we prove that similar guarantees can be provided for these algorithms as well. The organization of the paper is as follows. In Section II, we discuss the thresholding algorithms and the thresholding policy considered in the paper; The main results of the paper will also be reviewed. Section IV presents the convergence proof of the thresholding algorithms. In Section VI, we will briefly review the existing literature on iterative thresholding algorithms and compare those results to ours. Finally Section VII concludes the paper.

## II. ITERATIVE THRESHOLDING ALGORITHMS

### A. Abstracted thresholding algorithm

Consider two threshold functions  $\eta_t(x)$  to be applied elementwise to vectors: hard thresholding  $\eta_\mu^H(x) = x\mathbf{1}_{\{|x|>\mu\}}$  and soft thresholding  $\eta_\mu^S(x) = \text{sgn}(x)(|x| - \mu)_+$ , where  $\mathbf{1}$  is the indicator function and  $(a)_+$  is equal to  $a$  if  $a > 0$ , and zero otherwise. Iterative hard thresholding (IHT) and iterative soft thresholding (IST) algorithms are defined with the following iteration,

$$x^{t+1} = \eta_{\lambda_t}^*(x^t + \Phi^T(y - \Phi x^t)), \quad (2)$$

where  $\lambda_t$  is the threshold value at time  $t$  and as it is clear from the notation, may depend on time  $t$ .  $*$   $\in \{H, S\}$  represents hard or soft thresholding,  $\Phi^T$  is the transpose of the matrix  $\Phi$  and  $x^t$  is our estimate at time  $t$ . Note that the threshold value may depend on the iteration. The basic intuition is that since the solution satisfies the equation  $y = \Phi x$ , algorithm makes progress by moving in the direction of the gradient of  $\|y - \Phi x\|^2$  and then by thresholding the result, it tries to get a sparse vector closer to the hyperplane  $y = \Phi x$ . Another intuition for this algorithm comes from [12] and is as follows. Suppose that we want to solve the following optimization problem,

$$(\mathcal{P}_q) \quad \min_x \|y - \Phi x\|_2^2 + 2\lambda\|x\|_q.$$

It has been proved that the following IST algorithm converges to the solution of  $\mathcal{P}_1$  in case  $\|\Phi^T\Phi - I\|_{2,2} < 1$ ,

$$x^{t+1} = \eta_\lambda^S(x^t + \Phi^T(y - \Phi x^t)), \quad (3)$$

where  $\|A\|_{2,2}$  is the spectral norm of the matrix  $A$ . It may be noted that  $\lambda$  is fixed here and does not depend on the iteration. It is also well-known that as  $\lambda \rightarrow 0$  the solution of  $\mathcal{P}_1$  converges to the solution of  $\mathcal{Q}_1$ . But it is easy to see if  $\Phi$  is a fat matrix, setting  $\lambda$  to a very small value in (3) will not work and the iteration becomes unstable. Intuitively speaking, a proper thresholding policy is to set the threshold to a large value and gradually decrease it as the algorithm proceeds. The following theorem justifies this intuition.

Consider the iterative soft or hard thresholding algorithms introduced in equation (2). Suppose  $\lambda_t \rightarrow 0$  as  $t \rightarrow \infty$ , and  $\lambda_t$  is a decreasing sequence (this condition may not hold, but for the simplicity of the proof we assume it is true). Let  $J_t$  denote the union of the support of  $x^t$  and  $x_o$  and define  $L_t := J_{t+1} \cup J_t$ . Assume that  $L_t$  satisfies,  $\sup_t \|I - \Phi_{L_t}^T \Phi_{L_t}\|_{2,2} = \gamma < 1$ . Under these conditions:

**Theorem II.1.** *The iterative thresholding algorithm will converge to the sparsest solution.*

*Proof:*

$$\begin{aligned} \|x^{t+1} - x_o\|_2 &= \|x_{L_{t+1}}^{t+1} - x_{o_{L_{t+1}}}\|_2 \\ &\leq \|\eta_{\lambda_{t+1}}^*(x_{L_t}^t + \Phi_{L_t}^T(\Phi_{L_t} x_{o_{L_t}} - \Phi_{L_t} x_{L_t}^t)) - x_{o_{L_t}}\|_2, \\ &\leq \|(x_{L_t}^t + \Phi_{L_t}^T(\Phi_{L_t} x_{o_{L_t}} - \Phi_{L_t} x_{L_t}^t)) + \epsilon_{t+1} - x_{o_{L_t}}\|_2, \\ &\stackrel{1}{\leq} \|(I - \Phi_{L_t}^T \Phi_{L_t})(x_{L_t}^t - x_{o_{L_t}})\|_2 + \sqrt{n}\lambda_{t+1}, \\ &\leq \|(I - \Phi_{L_t}^T \Phi_{L_t})\|_{2,2} \|x_{L_t}^t - x_{o_{L_t}}\|_2 + \sqrt{n}\lambda_{t+1}, \end{aligned}$$

where  $\epsilon_{t+1}$  is an extra error introduced by the thresholding process and therefore each element of this vector is less than  $\lambda_{t+1}$ . Also all the elements that are not in  $L_t$  are zero. Inequality (1) is just the triangle inequality for  $\ell_2$  norm. For any  $\epsilon > 0$ , choose  $T_0$  such that  $\sqrt{n}\lambda_{T_0+1} < \frac{\epsilon(1-\gamma)}{2}$ , and let  $\|x^{T_0+1} - x_o\|_2 = e$ . Then, find  $T_1$  such that  $\gamma^{T_1} e < \epsilon/2$ . Now it is easy to prove that at  $t = T_0 + T_1$ , the error is less than  $\epsilon$  and therefore the total error goes to zero. ■

This theorem is not useful for practical purposes since we should have information on the size of  $L_t$ . In the section II-C we mention a practical thresholding policy that may be used in practice and under certain conditions will satisfy the properties mentioned above for the thresholds  $\lambda_t$ .

### B. Abstracted iterative thresholding with inversion

Another algorithm that we consider in this paper, is the Iterative Thresholding algorithm with Inversion (ITI). The algorithm is as follows,

$$\begin{aligned} u^t &= \eta_{\lambda_t}^* (x^{t-1} + \Phi^T (y - \Phi x^{t-1})) \\ I_t &= \text{supp}(u^t) \\ x_{I_t}^t &= (\Phi_{I_t}^T \Phi_{I_t})^{-1} \Phi_{I_t}^T y; \quad x_{I_t^c}^t = 0 \end{aligned} \quad (4)$$

where  $\text{supp}(u^t)$  includes the indices of the locations at which  $u^t$  is non-zero. This algorithm is similar to StOMP [19], CoSaMP [13] and subspace pursuit [14] and tries to get multiple elements into the active set in each step. This usually makes the algorithm faster than OMP in applications. The differences among these algorithms will be emphasized later in the discussion section. There is an interesting link between ITI and IHT which will be discussed later and will be helpful in understanding the behavior of IHT.

### C. Thresholding Policy

From theorem II.1 it is clear that one of the main questions in such algorithms is the way we choose the sequence of  $\lambda_k$ . Lots of methods have been proposed for setting the threshold. Two of the most successful heuristics are the following two. The first heuristic is the heuristic of multiple access interference noise which was proposed by Donoho et al. in [19]. It has been observed that  $x^{t-1} + \Phi^T (y - \Phi x^{t-1})$  can be modeled as original signal plus additive gaussian noise and they try to estimate the standard deviation of the noise and set the threshold according to the noise level. Since this heuristic is basically based on the heuristics of the central limit theorem for the noise, it works very well for compressed sensing problems where we deal with random measurement matrices. But it will not work as well when the measurement matrix has more structure. The other heuristic that can overcome this problem is as follows. Suppose that an oracle tells us the true underlying  $k$ . Then since the final solution is  $k$  sparse, the threshold can be set to the magnitude of the  $(k+1)^{\text{th}}$  largest coefficient. This type of thresholding policy has also been used in [13],[14], [15]. The only problem is how to get the oracle information. In a recent paper, we showed how one can de-oracle such algorithms for compressed sensing problems [3]. For other types of problems,  $k$  may be estimated using cross validation. If neither of these two methods is applicable, the bounds derived in this paper for the sparsity may be used for setting  $k$ . From now on, whenever we refer to IHT, IST or ITI, the thresholding policy is the  $k$  largest element thresholding policy unless otherwise stated.

### D. Main Results

We will prove three main theorems for the three thresholding algorithms that have been mentioned in the last section. In all these theorems active set of a vector is the set of indices at which that vector is non-zero. The correct active set is the active set of the original vector that we want to recover.

**Theorem II.2.** *Suppose that  $k < \frac{1}{3}\mu^{-1}$ . Then the ITI algorithm will find the correct active set in at most  $k$  steps and therefore converges to the correct answer in at most  $k$  iterations.*

**Theorem II.3.** *Suppose that  $k < \frac{1}{3.1}\mu^{-1}$  and  $\frac{|x_o(i)|}{|x_o(i+1)|} < 3^{\ell_i - 4}, \forall i, 1 \leq i < k$ . Then IHT finds the correct active set in at most  $\sum_{i=1}^k \ell_i + k$  steps. After this step all of these elements will remain in the active set and the error will go to zero exponentially fast.*

**Theorem II.4.** *Suppose that  $k < \frac{1}{4.1}\mu^{-1}$  and  $\forall i, 1 \leq i < k$ , we have  $\frac{|x_o(i)|}{|x_o(i+1)|} < 2^{\ell_i - 5}$ . Then IST recovers the correct active set in at most  $\sum_{i=1}^k \ell_i + k$  steps. After that all these coefficients will remain in the active set and the error will go to zero exponentially fast.*

The sufficient conditions provided here are slightly weaker than the conditions mentioned for  $\ell_1$  or OMP. Simulation results also confirm that all these algorithms are weaker than  $\ell_1$  in practice [3]. There are a few interesting facts that shall be emphasized here. First, the guarantees given here for ITI and IHT are very close. This is basically true because at each iteration IHT tries to solve the same problem as ITI but since it is not sure about the active set instead of fixing the active set it has the flexibility to change the active set at each iteration. As we will see later this will not affect the performance of the algorithm since the "important" elements remain the same. Another interesting fact is that in both IHT and IST the number of iterations needed, depends on the ratio of the coefficients but this dependency is roughly logarithmic and therefore it will work well in practice. Also, the algorithms find the correct active set in a finite number of iterations and once they find the correct active set, they will converge to the exact solution immediately (in case of ITI) or exponentially fast (in case of IHT and IST).

### III. PROOF OF CONVERGENCE FOR ITI

The goal of this section is to give an outline of the proof of Theorem II.2. The dynamics of the algorithm will be as follows. At the first iteration the algorithm will detect the largest element (the element with the maximum absolute value) and this element will get into the active set and will remain in the active set forever. At the second iteration the second largest element will go into the active set and will remain there forever. The same phenomena happens i.e. the  $i^{\text{th}}$  largest element will get into the active set at iteration  $i$  and will remain there after that forever, until all of the elements are in the active set. At the  $k^{\text{th}}$  iteration, the projection step will return the exact answer. In this section we prove all the above statements rigorously.

We define the following two variables,

$$z^{i+1} = x^i + \Phi^T(\Phi x_o - \Phi x^i), \quad (5)$$

$$w^i = x_o - x^i, \quad (6)$$

where  $x_o$  is the optimal value and  $x^i$  is our estimate at the  $i^{\text{th}}$  step. The  $j^{\text{th}}$  element of these two vectors will be denoted by  $z^i(j)$  and  $w^i(j)$ . The active set of  $x^i$  is called  $I^i$ . Since we always consider the active set of  $x^i$ , we may also call  $I^i$  the active set at time or step  $i$ . Finally,  $x_o(i)$  denotes the  $i^{\text{th}}$  element of  $x_o$ . Without loss of generality we assume that  $x_o(i)$ 's are sorted in descending order of their absolute values and therefore the only non-zero elements of  $x_o$  are the first  $k$  elements. It is also not difficult to see that  $z^{i+1} = x^i + \Phi^T(\Phi x_o - \Phi x^i) = x_o + (\Phi^T \Phi - I)(x_o - x^i)$ . I will call  $(\Phi^T \Phi - I)(x_o - x^i)$  the error term since it is something that has been added to the original vector that we want to recover.

**Lemma III.1.** *Suppose that  $k < \frac{1}{3}\mu^{-1}$  then at the first stage of the ITI,  $x_o(1)$  will be in the active set<sup>1</sup>.*

*Proof:*

$$\begin{aligned} |z^1(1)| &= |x_o(1) + \sum_{j=2}^k \langle \phi_1, \phi_j \rangle x_o(j)| \geq |x_o(1)| - \mu \sum_{j=2}^k |x_o(j)| \\ &\geq |x_o(1)| - k\mu |x_o(1)|. \end{aligned}$$

On the other hand,

$$\max_{\{i:k < i\}} |z^1(i)| = \max_{\{i:k < i\}} \left| \sum_{j=1}^k \langle \phi_i, \phi_j \rangle x_o(j) \right| \leq k\mu |x_o(1)|$$

Since  $k\mu < 1 - k\mu$ , the first element will get into the active set after the first step. ■

**Lemma III.2.** *Suppose that  $k\mu < \frac{1}{3}$  and that  $x_o(1), x_o(2), \dots, x_o(r)$  are in the active set  $I_m$  at the  $m^{\text{th}}$  step. Then,*

$$\max_{i \in I_m} |x^m(i) - x_o(i)| \leq \frac{k\mu |x_o(r+1)|}{1 - k\mu}. \quad (7)$$

*Proof:* Suppose  $I_m^c$  is the complement of  $I_m$ .

$$x_{I_m} = (\Phi_{I_m}^T \Phi_{I_m})^{-1} \Phi_{I_m}^T \Phi x_o = x_{o_{I_m}} + (\Phi_{I_m}^T \Phi_{I_m})^{-1} \Phi_{I_m}^T \Phi_{I_m^c} x_{o_{I_m^c}},$$

and therefore

$$\begin{aligned} \|x_{I_m} - x_{o_{I_m}}\|_{\infty} &= \|(\Phi_{I_m}^T \Phi_{I_m})^{-1} \Phi_{I_m}^T \Phi_{J_m} x_{o_{J_m}}\|_{\infty} \leq \\ &\|(\Phi_{I_m}^T \Phi_{I_m})^{-1}\|_{\infty, \infty} \|\Phi_{I_m}^T \Phi_{J_m} x_{o_{J_m}}\|_{\infty}, \end{aligned} \quad (8)$$

where  $J_m = I_m^c \cap \{1, 2, \dots, k\}$  and  $\|A\|_{\infty, \infty}$  is the operator norm of the matrix  $A$  when it is considered as a linear operator from  $\ell_{\infty}$  to  $\ell_{\infty}$ . We bound the above two terms separately.

$$\begin{aligned} \|(\Phi_{I_m}^T \Phi_{I_m})^{-1}\|_{\infty, \infty} &\leq \|I\|_{\infty, \infty} + \|I - \Phi_{I_m}^T \Phi_{I_m}\|_{\infty, \infty} + \dots \\ &\leq \frac{1}{1 - \|I - \Phi_{I_m}^T \Phi_{I_m}\|_{\infty, \infty}} \leq \frac{1}{1 - k\mu}. \end{aligned}$$

$$\|\Phi_{I_m}^T \Phi_{J_m} x_{o_{J_m}}\|_{\infty} \leq \sum_{i \in J_m} \mu |x_o(i)| \leq k\mu |x_o(r+1)|.$$

In these equation  $I$  represents the identity matrix. By combining the above two bounds with equation (8) we achieve the desired bound. ■

<sup>1</sup>It should be mentioned that this result holds even if  $k\mu < \frac{1}{2}$ . The only reason we are stating it in this way is for the consistency with the other parts of the proof

This lemma shows that if the thresholding step is successful in detecting the correct positions, the inversion step will be successful in "reducing" the error on those elements. The next lemma shows that the thresholding step is also successful in finding the correct positions.

**Lemma III.3.** *Suppose that  $k < \frac{1}{3}\mu^{-1}$  and that  $x_o(1), x_o(2), \dots, x_o(r)$  are in the active set at the  $m^{\text{th}}$  step. Then at the next step all of them will remain in the active set and at least  $x_o(r+1)$  will get into the active set.*

*Proof:* We proved that  $\max_{i \in I^m} |x^m(i) - x_o(i)| \leq \frac{k\mu|x_o(r+1)|}{1-k\mu}$ . We also have,

$$\begin{aligned} & |x_o(i) - z^{m+1}(i)| \\ & \stackrel{1}{=} \left| \sum_{j \in I_m \setminus \{i\}} \langle \phi_i, \phi_j \rangle (x_o(j) - x^m(j)) + \sum_{j \in \{1, 2, \dots, k\} \setminus (I_m \cup \{i\})} \langle \phi_i, \phi_j \rangle x_o(j) \right| \\ & \stackrel{2}{\leq} k\mu \frac{k\mu|x_o(r+1)|}{1-k\mu} + k\mu|x_o(r+1)| < \frac{|x_o(r+1)|}{2}. \end{aligned}$$

Equality 1 is based on equation (5) and the fact that  $x^i + \Phi^T(\Phi x_o - \Phi x^i) = x_o + (\Phi^T \Phi - I)(x_o - x^i)$ . In order to derive inequality 2 we are using a few facts. First, since  $i \neq j$ ,  $|\langle \phi_i, \phi_j \rangle| < \mu$ . Second, for  $j \in I_m$ ,  $|x_o(j) - x^m(j)| \leq \frac{k\mu|x_o(r+1)|}{1-k\mu}$  according to the previous lemma. Third,  $\{1, 2, \dots, r\} \in I_m$  and therefore for  $j \in \{1, 2, \dots, k\} \setminus (I_m \cup \{i\}) \subset \{r+1, \dots, k\}$  and therefore the maximum absolute value of  $x_o(j)$  on this set is  $|x_o(r+1)|$ .

Now, by using this bound we have,

$$\max_{\{i: i > k\}} |z^{m+1}(i)| \leq k\mu x_o(r+1) + k\mu \frac{k\mu x_o(r+1)}{1-k\mu x_o(r+1)} < \frac{x_o(r+1)}{2},$$

and

$$\min_{\{i: 1 \leq i \leq r+1\}} |z^{m+1}(i)| \geq |x_o(r+1)| - |x_o(i) - z^{m+1}(i)| \geq |x_o(r+1)| - k\mu x_o(r+1) - k\mu \frac{k\mu x_o(r+1)}{1-k\mu x_o(r+1)} > \frac{x_o(r+1)}{2}.$$

By comparing these two we see that  $(r+1)^{\text{th}}$  element will also get into the active set while  $x_o(1), \dots, x_o(r)$  remain in the active set. ■

*Outline of the proof of Theorem II.2:* The proof is an induction that combines the above lemmas. Suppose that  $x_o(1), x_o(2), \dots, x_o(r)$  are in the active set according to the above lemma at the next iteration, all of them will remain in the active set and  $x_o(r+1)$  will also get into the active set. The base of this induction is also lemma III.1. Therefore after  $k$  steps all the correct elements are in the active set and the projection step will give us the exact solution. ■

#### IV. PROOF OF CONVERGENCE FOR THE IHT ALGORITHM

The goal of this section is to give an outline of the proof of Theorem II.3. Let me first summarize the behavior of the algorithm intuitively. It will help the reader understand the steps of the proof more easily. All these things will be proved rigorously later in this section. When we run the algorithm, at the first iteration the largest element of  $x_o$  will get into the active set. The nice fact is that once this element gets into the active set it will always remain in the active set. What happens in the next few iterations of the algorithm is that the first element remains in the active set and the error term decreases and after a few steps it will be so small that the second largest element will be detected (This statement is not exactly right. The error term may go up for a finite number of iterations, but eventually it will decrease. You will see the rigorous bound of this error and its performance in the next lemma). Here we can see the similarity between this algorithm and ITI. Since the first element remains in the active set and in each iteration the algorithm try to decrease the error on each element, we can view it as an iterative method to estimate the inversion that we had in the last section. Once the second largest term gets into the active set, the first and second elements will remain in the active set and the same process will happen again, i.e. the error term will decrease and eventually the third largest element will get into the active set. The goal of this section is to make all the above statements precise.

The next lemma will be useful later when we try to bound the error at each iteration.

**Lemma IV.1.** *Consider the following sequence for  $s \geq 0$ ,*

$$f_s = \alpha^1 + \dots + \alpha^s + \beta\alpha^{s+1},$$

where  $0 < \alpha < 1$ . The following statements are true;

- 1) If  $\beta(1 - \alpha) < 1$ , then for every  $s$ ,  $f_s < \frac{\alpha}{1-\alpha}$ .
- 2) If  $\beta(1 - \alpha) > 1$ , then for every  $s$ ,  $f_s < \beta\alpha$ .
- 3) If  $\beta(1 - \alpha) = 1$ , then  $f_s$  is a constant sequence and is always equal to  $\frac{\alpha}{1-\alpha}$ .

It is easy to see that the sequence is either increasing or decreasing or constant depending on the values of  $\alpha$  and  $\beta$ . The proof is simple and is omitted for the sake of brevity.

**Lemma IV.2.** *Suppose that  $x_o(1), x_o(2), \dots, x_o(r-1)$ ,  $r-1 < k$ , are in the active set at the  $m^{\text{th}}$  step. Also assume that,*

$$|z^m(j) - x_o(j)| \leq 1.5k\mu|x_o(r-1)| \quad \forall j.$$

*If  $k\mu < \frac{1}{3.1}$ , then at stage  $m+s$  and for every  $j$  we will have the following upper bound for  $|z^{m+s}(j) - x_o(j)|$ ,*

$$|x_o(r)|(k\mu + \dots + (k\mu)^s) + 1.5(k\mu)^{s+1}|x_o(r-1)|. \quad (9)$$

*Moreover,  $x_o(1), x_o(2), \dots, x_o(r-1)$  will remain in the active.*

Before proving this theorem it should be mentioned that at this point the factor 1.5 may seem unnecessary in the proof. But as will be seen later in lemma IV.3, this factor is necessary and can not be omitted.

*Proof:* We prove this by induction; Assuming that the bound holds at stage  $m+s$  and  $x_o(1), x_o(2), \dots, x_o(r-1)$  are in the active set, we show that the upper bound holds at stage  $m+s+1$  and the first  $r-1$  elements will remain in the active set.

$$\begin{aligned} & |z^{m+s+1}(i) - x_o(i)| \\ & \leq \sum_{j \in I^{m+s} \setminus \{i\}} |\langle \phi_i, \phi_j \rangle w^{m+s}(j)| + \sum_{j \in \{1, 2, \dots, k\} \setminus I^{m+s} \cup \{i\}} |\langle \phi_i, \phi_j \rangle w^{m+s}(j)|, \\ & \stackrel{1}{=} \sum_{j \in I^{m+s} \setminus \{i\}} |\langle \phi_i, \phi_j \rangle w^{m+s}(j)| + \sum_{j \in \{r, \dots, k\} \setminus I^{m+s} \cup \{i\}} |\langle \phi_i, \phi_j \rangle w^{m+s}(j)|, \\ & \stackrel{2}{\leq} \sum_{j \in I^{m+s} \setminus \{i\}} |\langle \phi_i, \phi_j \rangle (z^{m+s}(j) - x_o(j))| + k\mu x_o(r), \\ & \leq k\mu|x_o(r)|(k\mu + \dots + (k\mu)^s) + 1.5(k\mu)^{s+2}|x_o(r-1)| \\ & \quad + k\mu|x_o(r)|, \\ & \leq |x_o(r)|(k\mu + \dots + (k\mu)^{s+1}) + 1.5(k\mu)^{s+2}|x_o(r-1)|. \end{aligned}$$

In these calculations equality (1) is due to the assumptions of the induction, i.e. the first  $r-1$  elements are in the active set at stage  $m+s$ . To get inequality (2) we have used two different facts. The first one is that when  $j \in I^{m+s}$ ,  $w^{m+s}(j) = x_o(j) - z^{m+s}(j)$  and the second one is that when  $j \in \{r, \dots, k\} \setminus I^{m+s}$  then  $w^{m+s}(j) = x_o(j)$  and therefore  $|x_o(j)| \leq |x_o(r)|$ . The last step is to prove that all the first  $r-1$  elements remain in the active set. For  $i \in \{1, 2, \dots, r-1\}$ ,

$$\begin{aligned} |z^{m+s+1}(i)| & \geq |x_o(i)| - |z^{m+s+1}(i) - x_o(i)|, \\ & \stackrel{1}{\geq} |x_o(i)| - (k\mu|x_o(r-1)| + \dots + (k\mu)^{s+1}|x_o(r-1)|) \\ & \quad - 1.5(k\mu)^{s+2}|x_o(r-1)| \stackrel{2}{\geq} |x_o(i)| - \frac{|x_o(r-1)|}{2.05} \\ & \geq |x_o(r-1)| - \frac{|x_o(r-1)|}{2.05}. \end{aligned}$$

In inequality (1) we have used the bound in (9) by replacing  $x_o(r)$  with  $x_o(r-1)$ . Inequality (2) is the result of Lemma IV.1. For  $i \notin \{1, 2, \dots, k\}$ , we have

$$|z^{m+s+1}(i)| \leq \frac{|x_o(r-1)|}{2.05},$$

and since  $\min_{\{i: i \leq r-1\}} |z^{m+s+1}(i)| > \max_{\{i: i > k\}} |z^{m+s+1}(i)|$ , the first  $r-1$  elements will remain in the active set. The base of the induction is the same as the assumptions of this lemma and the proof is complete. ■

**Lemma IV.3.** *Suppose that  $k < \frac{1}{3.1}\mu^{-1}$ , and  $x_o(1), x_o(2), \dots, x_o(r)$ ,  $r < k$ , are in the active set at the  $m^{\text{th}}$  step. Also assume that  $\frac{|x_o(r)|}{|x_o(r+1)|} \leq 3^{\ell_r - 4}$ . If*

$$|z^m(j) - x_o(j)| \leq 1.5k\mu|x_o(r)| \quad \forall j,$$

*after  $\ell_r$  more steps  $x_o(r+1)$  will get into the active set, and*

$$|z^{m+\ell_r+1}(j) - x_o(j)| \leq 1.5k\mu|x_o(r+1)| \quad \forall j.$$

*Proof:* By setting  $s = \ell_r$  in the upper bound of the last lemma we get,

$$|z^{m+\ell_r}(j) - x_o(j)| \leq \frac{1.5|x_o(r+1)|}{273} + \frac{|x_o(r+1)|}{2.1}.$$

Similar to the last lemma it is also not difficult to see that

$$\begin{aligned} |z^{m+\ell_r}(r+1)| &= |z^{m+\ell_r}(r+1) - x_o(r+1) + x_o(r+1)| \\ &\geq |x_o(r+1)| - |z^{m+\ell_r}(r+1) - x_o(r+1)| \\ &\geq |x_o(r+1)| - \frac{|1.5x_o(r+1)|}{273} - \frac{|x_o(r+1)|}{2.1}. \end{aligned}$$

But,

$$|z^{m+\ell_r}(r+1)| > \max_{\{i:i>k\}} |z^{m+\ell_r}(i)|,$$

and therefore  $x_o(r+1)$  will be detected at this step. It may also be noted that at this stage the error is less than  $|x_o(r+1)|/2$ . For the next stage we will have at most  $k$  active elements the error of each is less than  $|x_o(r+1)|/2$  and at most  $k-r$  non-zero elements of  $x_o$  that have not passed the threshold and whose magnitudes are smaller than  $|x_o(r+1)|$ . Therefore, the error of the next step is less than  $1.5k\mu|x_o(r+1)|$ . ■

Our goal is to prove the correctness of IHT by induction and we have to know the correctness of IHT at the first stage. The following lemma provides this missing step.

**Lemma IV.4.** *Suppose that  $k < \frac{1}{3.1}\mu^{-1}$ , then at the first stage of the IHT,  $x_o(1)$  will be in the active set<sup>2</sup> and  $|z^1(j) - x_o(j)| \leq k\mu|x_o(1)|$ .*

The proof is exactly similar to the proof of lemma III.1.

Finally the following lemma describes the performance of the algorithm after detecting all the non-zero elements.

**Lemma IV.5.** *Suppose that  $x_o(1), x_o(2), \dots, x_o(k)$ , are in the active set at the  $m^{\text{th}}$  step. Also assume that,*

$$|z^m(j) - x_o(j)| \leq 1.5k\mu|x_o(k)| \quad \forall j.$$

*If  $k\mu < \frac{1}{3.1}$ , then at stage  $m+s$  and for every  $j$  we will have,*

$$|z^{m+s}(j) - x_o(j)| \leq 1.5(k\mu)^{s+1}|x_o(k)|.$$

Since the proof of this lemma is very similar to the proof of Lemma IV.2, it is omitted.

*Outline of the proof of Theorem II.3:* The proof is an induction that combines the above lemmas. Suppose that  $x_o(1), x_o(2), \dots, x_o(r)$  are already in the active set. According to Lemma IV.2 all these terms will remain in the active set, and according to Lemma IV.3 after  $\ell_r$  steps  $x_o(r+1)$  will also get into the active set. In one more step, the error on each element gets smaller than  $1.5k\mu|x_o(r+1)|$ , and everything can be repeated. Lemma IV.4 provides the first step of the induction. Finally when all the elements are in the active set lemma IV.5 tells us that the error goes to zero exponentially fast. ■

Since the proof of the convergence of IST is very similar to IHT we do not repeat it here. You may refer to [16] for more details.

## V. PROOF OF CONVERGENCE FOR THE IST ALGORITHM

As mentioned before the main ideas of the proof of the IST algorithm are very similar to those of the IHT. We will mention the proof in detail but will try to emphasize more on the differences. The following lemma helps us find some bounds on the error of the algorithm at each step.

**Lemma V.1.** *Suppose that  $x_o(1), x_o(2), \dots, x_o(r)$ ,  $r \leq k$ , are in the active set at the  $m^{\text{th}}$  step. Also assume that*

$$|x^m(j) - x_o(j)| \leq 4k\mu|x_o(r)|, \quad \forall j \in I^m,$$

*and  $k\mu < \frac{1}{4.1}$ . Then at stage  $m+s$ ,  $\forall i \in I^{m+s}$  we have the following upper bound for  $|x^{m+s}(i) - x_o(i)|$ ,*

$$|x_o(r+1)| (2k\mu + \dots + (2k\mu)^s) + 2(2k\mu)^{s+1}|x_o(r)|.$$

*Moreover,  $x_o(1), x_o(2), \dots, x_o(r)$  remain in the active set.*

<sup>2</sup>This result holds even if  $k\mu < \frac{1}{2}$ . For the sake of consistency with the other parts of the proof we state it in this way

*Proof:* As before, this can be proved by induction. We assume that at step  $m+s$  the upper bound holds and  $x_o(1), x_o(2), \dots, x_o(r)$  are in the active set and we prove the same things for  $m+s+1$ . Similar to what we saw before,

$$\begin{aligned}
& |z^{m+s+1}(i) - x_o(i)| \\
& \leq \sum_{j \in I^{m+s} \setminus \{i\}} |\langle \phi_i, \phi_j \rangle w^{m+s}(j)| + \sum_{j \in \{1, 2, \dots, k\} \setminus I^{m+s} \cup \{i\}} |\langle \phi_i, \phi_j \rangle w^{m+s}(j)|, \\
& \stackrel{1}{=} \sum_{j \in I^{m+s} \setminus \{i\}} |\langle \phi_i, \phi_j \rangle w^{m+s}(j)| + \sum_{j \in \{r+1, \dots, k\} \setminus I^{m+s} \cup \{i\}} |\langle \phi_i, \phi_j \rangle w^{m+s}(j)|, \\
& \stackrel{2}{\leq} (k-1)\mu(2k\mu|x_o(r+1)| + \dots + (2k\mu)^s|x_o(r+1)| \\
& \quad + 2(2k\mu)^{s+1}|x_o(r)|) + k\mu|x_o(r+1)| := \alpha_s.
\end{aligned}$$

Equality (1) is using the assumption that the first  $r$  elements are in the active set at stage  $m+s$ . Inequality (2) is also due to the assumptions of the induction and the fact that  $w^{m+s}(j) = x_o(j) - x^{m+s}(j)$ .

At least one of the largest  $k+1$  coefficients of  $z$ , corresponds to an element whose index is not in  $\{1, 2, \dots, k\}$ , and the magnitude of this coefficient is less than  $\alpha_s$ . Therefore the threshold value is less than or equal to  $\alpha_s$ . Applying the soft thresholding to  $z$  will at most add  $\alpha_s$  to the distance of  $z^{s+1}(i)$  and  $x_o(i)$ , and this completes the proof of the upper bound. The main thing that should be checked is whether the first  $r$  elements will remain in the active set or not. For  $i \in \{1, 2, \dots, r\}$  we have,

$$\begin{aligned}
|z^{m+s+1}(i)| & \geq |x_o(i)| - |z^{m+s+1}(i) - x_o(i)|, \\
& \geq |x_o(i)| - k\mu|x_o(r)|(1 + 2k\mu + \dots + (2k\mu)^{s+1}) \\
& \quad - 2k\mu(2k\mu)^{s+1}|x_o(r)| \geq |x_o(i)| - \frac{|x_o(r)|}{2.05} \\
& \geq |x_o(r)| - \frac{|x_o(r)|}{2.05}. \tag{10}
\end{aligned}$$

If the sequence in the above expression is multiplied by 2, the result will be a sequence in the form of the sequences mentioned in lemma IV.1 for  $\alpha = 2k\mu$ ,  $\beta = 2$  and the last equality is based on that lemma.

If  $i \notin \{1, 2, \dots, k\}$ ,

$$\begin{aligned}
|z^{m+s+1}(i)| & \leq k\mu|x_o(r)|(1 + 2k\mu + \dots + (2k\mu)^{s+1}) \\
& \quad + 2k\mu(2k\mu)^{s+1}|x_o(r)| \leq \frac{|x_o(r)|}{2.05}.
\end{aligned}$$

Since  $\min_{\{i: i \leq r\}} |z^{m+s+1}(i)| > \max_{\{i: i > k\}} |z^{m+s+1}(i)|$ , the first  $r$  elements remain in the active set. The base of the induction is also clear since it is the same as the assumptions of the lemma.  $\blacksquare$

**Lemma V.2.** Suppose that  $k \leq \frac{\mu^{-1}}{4.1}$ , and  $x_o(1), x_o(2), \dots, x_o(r)$ ,  $r \leq k$ , are in the active set at the  $m^{\text{th}}$  step. Also, assume that  $\frac{|x_o(r)|}{|x_o(r+1)|} \leq 2^{\ell_r - 5}$ . If

$$|x^m(j) - x_o(j)| \leq 4k\mu|x_o(r)|, \quad \forall j \in I^m,$$

then after  $\ell_r$  steps  $x_o(r+1)$  will get into the active set, and

$$|x^{m+\ell_r+1}(j) - x_o(j)| \leq 4k\mu|x_o(r+1)|, \quad \forall j \in I^{m+\ell_r+1}.$$

*Proof:* As before we try to find a bound for the error at time  $m + \ell_r$ . For  $i \in \{1, 2, \dots, k\}$ ,

$$\begin{aligned}
|z^{m+\ell_r}(i) - x_o(i)| & \leq \frac{1}{2}|x_o(r+1)|(2k\mu + \dots + (2k\mu)^{\ell_r}) \\
& \quad + (2k\mu)^{\ell_r+1}|x_o(r)| \leq \frac{|x_o(r+1)|}{2.1} + \frac{|x_o(r+1)|}{64}
\end{aligned}$$

and therefore for  $i = r+1$ ,

$$\begin{aligned}
|z^{m+\ell_r}(r+1)| & \geq |x_o(r+1)| - |z^{m+\ell_r}(i) - x_o(i)| \geq \\
& |x_o(r+1)| - \frac{|x_o(r+1)|}{2.1} - \frac{|x_o(r+1)|}{64} \tag{11}
\end{aligned}$$

Since  $|z^{m+\ell_r}(r+1)| > \max_{\{i: k < i\}} |z^{m+\ell_r}(i)|$ , the  $r+1^{\text{th}}$  element will get into the active set at this stage. On the other hand for any  $i \in I^{m+\ell_r}$  we have  $|x^{m+\ell_r}(i) - x_o(i)| \leq x_o(r+1)$ . For the next stage of the algorithm we will have at most  $2k$  non-zero

$x^{m+\ell_r}(i) - x_o(i)$  and absolute value of each of them is less than  $|x_o(r+1)|$ . Therefore  $|z^{m+\ell_r+1}(i) - x_o(i)| \leq 2k\mu|x_o(r+1)|$  and after thresholding we have,  $|x^{m+\ell_r+1}(i) - x_o(i)| \leq 4k\mu|x_o(r+1)|$  for  $i \in I^{m+\ell_r+1}$ .

The base of the induction is also clear from the assumptions of this lemma and the proof is complete. ■

For the IHT algorithm we proved that at the first step the first element will pass the threshold. Since the selection step of IST and IHT is exactly the same, we can claim that the same thing is true for IST, i.e. the largest magnitude coefficient will pass the threshold. Also, as we saw for IHT, the error was less than  $k\mu|x_o(1)|$ . Therefore, for the IST we have,  $|x^1(j) - x_o(j)| < 2k\mu|x_o(1)|$ . These bounds are even better than the bounds we need for V.1 and V.2 and V.3.

The following lemma will explain what happens when the algorithm detects all the non-zero elements.

**Lemma V.3.** *Suppose that  $x_o(1), \dots, x_o(k)$ , are in the active set at the  $m^{\text{th}}$  step. Also assume that,*

$$|x^m(j) - x_o(j)| \leq 4k\mu|x_o(k)|.$$

*If  $k\mu < \frac{1}{4.1}$ , at stage  $m+s$  all the elements remain in the active set and for every  $j$  we will have,*

$$|z^{m+s}(j) - x_o(j)| \leq 2(2k\mu)^{s+1}|x_o(k)|$$

The proof of this lemma is very similar to the other lemmas and is omitted.

*Outline of the proof of Theorem II.4:* The proof is a simple induction by combining the above lemmas. Suppose that  $x_o(1), x_o(2), \dots, x_o(r)$  are already in the active set. According to Lemma V.1 all these terms will remain in the active set, and according to Lemma V.2 after  $\ell_r$  steps  $x_o(r+1)$  will also get into the active set. In one more step, the error on each element gets smaller than  $4k\mu|x_o(r+1)|$ , and everything can be repeated. Although we have not mentioned the first step of the induction it is not difficult to see that step is also true and it is very similar to the first step of IHT. Finally when all the elements are in the active set lemma V.3 tells us that the error goes to zero exponentially fast. ■

## VI. DISCUSSION AND COMPARISON WITH OTHER WORK

There is a huge amount of work on iterative thresholding algorithms, and we cannot mention all of them here; The interested reader is referred to [3]. Most of these papers are dealing with a fixed threshold that does not depend on iteration. In that case, there are rigorous results that give sufficient conditions for the IST algorithm to converge to the solution of  $\mathcal{P}_1$  [12], and for the IHT algorithm to a local minimum of  $\mathcal{P}_0$  [17]. The idea of choosing iteration dependent thresholds is also not new, and some simple variations were introduced in [11]. The  $k$  largest element thresholding policy was first introduced in [13] and was first used for IHT in [15]. It was also shown that if the  $\Phi$  matrix satisfies restricted isometry property (RIP) of order  $3k$ , the IHT converges to the sparsest solution. There are some basic differences in our approach. First, we are dealing with deterministic settings, and in these settings RIP conditions they have provided are much weaker than ours ( $k\mu < \frac{1}{3\sqrt{32}}$  compared to  $k\mu < \frac{1}{3.1}$ ). Under these more general conditions, as we observed, the performance of IHT is not as simple as what is mentioned in [15], and it may not recover  $x_o$  in just  $k$  steps. But it will finally recover the sparsest signal and we give bounds on the number of iterations it needs to converge. Secondly, as discussed in the last section, our approach was easily adapted to IST, and can be adapted to the other types of thresholds. Moreover, our method gives us an ordering among  $\ell_1$ , OMP, IHT and IST which may be useful for deciding on the choice of the algorithm. Finally there is another effort on analyzing the performance of IST by coherence that shows the possibility of success of such an algorithm at the first iteration [18]. But this result does not have any conclusion about the next iterations of IST in case it does not recover all the non-zero elements at the first step.

Also as it was mentioned in the paper the ITI algorithm is also close to CoSaMP, subspace pursuit and StOMP. But there are some main differences between ITI and these algorithms. In the ITI algorithm, coefficients can get into and out of the active set. This is different from StOMP in which once an element gets into the active set we force it to remain in the active set. Also, unlike CoSaMP and subspace pursuit ITI is not a two stage algorithm. The main idea that we analyzed this algorithm in this paper is the similarity of ITI to the iterative hard thresholding, that was mentioned in section IV.

## VII. CONCLUSION

In this paper, we analyzed iterative hard and soft thresholding, and proved that under certain conditions they work properly. These conditions are slightly weaker than their counterparts for  $\ell_1$  and OMP. But these algorithms are very simple to implement and much faster than both convex relaxation and greedy methods, and they are much more desirable for large scale problems.

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