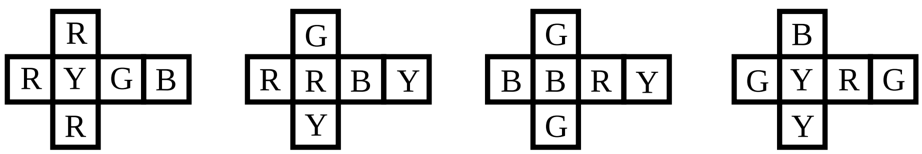
Team Test
Stanford Math Tournament
February 22, 2003

1. What is the ratio of the area of an equilateral triangle to the area of the largest rectangle that can be inscribed inside the triangle?

2. Define $P(x) = x^{12} + 12x^{11} + 66x^{10} + 220x^9 + 495x^8 + 792x^7 + 924x^6 + 792x^5 - 159505x^4 + 220x^3 + 66x^2 + 12x + 1$. Find $\frac{P(19)}{20^4}$.

3. Four flattened colored cubes are shown below, which are folded into cubes with the letters facing outward. Each of the cubes' faces has been colored red (R), blue (B), green (G) or yellow (Y). The cubes are stacked on top of each other in numerical order with cube #1 on bottom. The goal of the puzzle is to find an orientation for each cube so that on each of the four visible sides of the stack all four colors appear. Find any solution, and for each side of the stack, list the colors from bottom to top. List the sides in either clockwise or counter-clockwise order.




Cube 1 Cube 2 Cube 3 Cube 4

4. When evaluated, the sum $\sum_{k=1}^{2002} [k \cdot k!]$ is a number that ends with a long series of 9's. How many 9's are at the end of the number?

5. Find the positive integer $n$ that maximizes the expression $\frac{200003^n}{(n!)^2}$.

6. Find $11^3 + 12^3 + ... + 100^3$.

   *Hint:* Develop a formula for $\sum_{k=1}^{n} k^3$.

7. Six fair 6-sided dice are rolled. What is the probability that the sum of the values on the top faces of the dice is divisible by 7?

8. Several students take a quiz which has five questions, and each one is worth a point. They are unsure as to how many points they received, but all of them have a reasonable idea about their scores. Below is a table of what each person thinks is the probability that he or she got each score. Assuming their probabilities are correct, what is the probability that the sum of their scores is exactly 20?

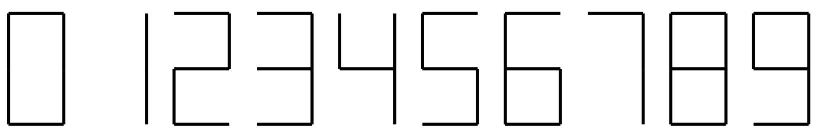
| Score / Student | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Allison | 0 | 0 | .25 | .5 | .25 | 0 |
| Barbara | 0 | .5 | .5 | 0 | 0 | 0 |
| Christi | 0 | 0 | 0 | 0 | 0 | 1 |
| David | 0 | 0 | 0 | 0 | .5 | .5 |
| Ed | 0 | .25 | .5 | .25 | 0 | 0 |
| Fred | .25 | .5 | .25 | 0 | 0 | 0 |
| Gary | 0 | 0 | 0 | .25 | .5 | .25 |

9. Let $F_n$ be the number of ways of completely covering an $3 \times n$ chessboard with $n$ $3 \times 1$ dominoes. For example, there are two ways of tiling a $3 \times 3$ chessboard with three $3 \times 1$ dominoes (all horizontal or all vertical). What is $F_{14}$?

10. Two players (Kate and Adam) are playing a variant of Nim. There are 11 sticks in front of the players and they take turns each removing either one or any prime number of sticks. The player who is forced to take the last stick loses. The problem with the game is that if player one (Kate) plays perfectly, she will always win. Give all the starting moves, if any, that lead to a sure win for Kate (assuming each player plays perfectly).

11. Define $f(x,y) = x^2 - y^2$ and $g(x,y) = 2xy$. Find all $(x,y)$ such that $(f(x,y))^2 - (g(x,y))^2 = \frac{1}{2}$ and $f(x,y) \cdot g(x,y) = \frac{\sqrt{3}}{4}$.
*Hint:* Consider $z = x + iy$, where $i = \sqrt{-1}$.

12. The numerals on digital clocks are made up of seven line segments, as shown below:



(The two vertical segments for 1 are on the right side.) When various combinations of them light up different numbers are shown. When a digit on the clock changes, some segments turn on and others turn off. For example, when a 4 changes into a 5 two segments turn on and one segment turns off, for a total of 3 changes. In the usual ordering $1, 2, 3, \ldots, 0$ there are a sum total of 32 segment changes (including the wrapping around from 0 back to 1). If we can put the digits in any order, what is the fewest total segment changes possible? (As above, include the change from the last digit back to the first.)

13. How many solutions are there to $(\cos 10x)(\cos 9x) = \frac{1}{2}$ for $x \in [0, 2\pi]$?

14. Find $\binom{2003}{0} + \binom{2003}{4} + \binom{2003}{8} + \cdots$.
*Hint:* Consider $(1 \pm i)$ and $(1 \pm 1)$.

15. Alice and Bob are playing a game that depends on $N$ and $M$, both positive integers. They start with a bag of $N$ marbles, and take turns removing at least one and up to $M$ marbles. Alice moves first, and the person who takes the last marble wins. If $N$ is chosen randomly between 97 and 2003 inclusive, and $M$ is chosen randomly between 1 and 10, what is the probability that Bob will win, assuming optimal play by both parties?