Team Test Solutions
Stanford Math Tournament
February 22, 2003

1. What is the ratio of the area of an equilateral triangle to the area of the largest rectangle that can be inscribed inside the triangle?

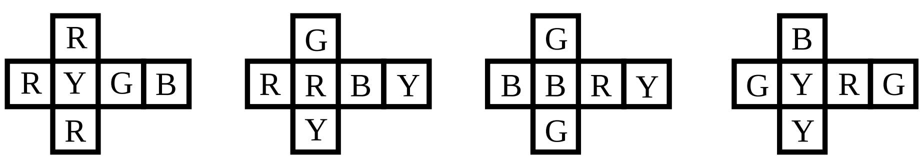
   Solution: 2 or 2 : 1 or 2 to 1. Let ABC be an equilateral triangle of side length s. Place triangle ABC on a coordinate grid so that $A = (-\frac{s}{2}, 0), B = (\frac{s}{2}, 0)$, and $C = (0, \frac{s\sqrt{3}}{2})$. Clearly one edge of the rectangle must lie on an edge of the triangle, so assume this is AB. Label the vertex of the rectangle on BC (x, y). Note that the equation of BC is $y = -\sqrt{3}x + \frac{s\sqrt{3}}{2}$. The area of the rectangle is $2xy = -2\sqrt{3}x^2 + s\sqrt{3}x = -2\sqrt{3}(x - \frac{s}{4})^2 + 2\sqrt{3}(\frac{s}{4})^2$. Maximum area occurs when $x = \frac{s}{4}$ and the area is then $\frac{s^2\sqrt{3}}{8}$. The area of the triangle is $\frac{s^2\sqrt{3}}{4}$, and thus the ratio is 2.

2. Define $P(x) = x^{12} + 12x^{11} + 66x^{10} + 220x^9 + 495x^8 + 792x^7 + 924x^6 + 792x^5 - 159505x^4 + 220x^3 + 66x^2 + 12x + 1$. Find $\frac{P(19)}{20^4}$.

   Solution: $\mathbf{25,599,869,679}$. The near symmetry of the polynomial suggests we look at binomial coefficients. Using Pascal's triangle, we can quickly note that $P(x) = (x+1)^{12} - 160000x^4 = (x+1)^{12} - 20^4x^4$. Therefore,

$$\frac{P(19)}{20^4} = \frac{20^{12} - 20^4 \cdot 19^4}{20^4} = 20^8 - 19^4 = 25,599,869,679.$$

3. Four flattened colored cubes are shown below, which are folded into cubes with the letters facing outward. Each of the cubes' faces has been colored red (R), blue (B), green (G) or yellow (Y). The cubes are stacked on top of each other in numerical order with cube #1 on bottom. The goal of the puzzle is to find an orientation for each cube so that on each of the four visible sides of the stack all four colors appear. Find any solution, and for each side of the stack, list the colors from bottom to top. List the sides in either clockwise or counter-clockwise order.



Cube 1 Cube 2 Cube 3 Cube 4

   Solution: $(R, B, Y, G), (B, Y, R, G), (G, R, B, Y), (Y, G, B, R)$. (Note that the 4-tuples ((R, B, Y, G), etc.) must be in this order, but the listing can start with any of them.) There are many approaches but perhaps the fastest is a graph theoretical approach. Draw a graph on 4 vertices where the edges are labelled. An edge is drawn between two vertices if they appear on opposite sides of a cube. The edge is labelled with the number of the cube it comes from. Then what we are looking for is two subgraphs that are each regular of degree 2 with four edges, one of each number. The two subgraphs must be disjoint. One subgraphs tells us which pairs of colors align on the left and right side. The other would give us the front and back. We can easily find such a solution and in this case it is unique. Starting at an arbitrary side and going clockwise the solution is (R, B, Y, G), (B, Y, R, G), (G, R, B, Y), (Y, G, B, R).

4. When evaluated, the sum $\sum_{k=1}^{2002}[k \cdot k!]$ is a number that ends with a long series of 9's. How many 9's are at the end of the number?

   Solution: 499. The given sum is equivalent to $\sum_{k=1}^{2002}[(k+1)! - k!]$ which is equal to $2003! - 1!$. Now, clearly 2003! will end with a lot of zeros and when we subtract the 1 we get a series of 9's. We need to see how many factors of 10 divide 2003!. We have many more 2's than 5's, so 5's are the limiting factor. There are $\lfloor \frac{2003}{5} \rfloor + \lfloor \frac{2003}{5^2} \rfloor + \lfloor \frac{2003}{5^3} \rfloor + \lfloor \frac{2003}{5^4} \rfloor = 400 + 80 + 16 + 3 = 499$ 5's we can take out of 2003!, so 499 factors of 10 divide 2003!, so the number ends with 499 9's.

5. Find the positive integer n that maximizes the expression $\frac{200003^n}{(n!)^2}$.

   Solution: 447. Let $a_n = \frac{200003^n}{(n!)^2}$. If a_n is the maximum, then $a_{n-1} \leq a_n$ and $a_n \geq a_{n+1}$. The first inequality implies $\frac{200003^{n-1}}{((n-1)!)^2} \leq \frac{200003^n}{(n!)^2}$ which is equivalent to $n^2 \leq 200003$. The second inequality implies similarly that $(n+1)^2 \geq 200003$. The only n for which these two inequalities are true is $n = 447$.

6. Find $11^3 + 12^3 + ... + 100^3$.

   Hint: Develop a formula for $\sum_{k=1}^{n} k^3$.

   Solution: 25499475. Look at $\sum_{k=1}^{n} k^3$ for the first few values of k:

$$\begin{aligned} 1^3 &= 1 \\ 1^3 + 2^3 &= 9 \\ 1^3 + 2^3 + 3^3 &= 36 \\ 1^3 + 2^3 + 3^3 + 4^3 &= 100. \end{aligned}$$

   The fact that the right side in each of these is a perfect square looks pretty suspicious, and after looking at *which* perfect squares we're getting (and maybe trying a few more sums), we are led to guess that

$$\sum_{k=1}^{n} k^3 = (1 + 2 + \cdots + n)^2 = \left(\frac{n(n+1)}{2}\right)^2.$$

   And in fact, an inductive proof quickly shows us that this formula is correct. With this in place, we can easily find the original sum since

$$\begin{aligned} 11^3 + 12^3 + \cdots + 100^3 &= \sum_{k=1}^{100} k^3 - \sum_{k=1}^{10} k^3 \\ &= \left(\frac{100 \cdot 101}{2}\right)^2 - \left(\frac{10 \cdot 11}{2}\right)^2 \\ &= 5050^2 - 55^2 \\ &= 25499475 \end{aligned}$$

7. Six fair 6-sided dice are rolled. What is the probability that the sum of the values on the top faces of the dice is divisible by 7?

   Solution: $\mathbf{\frac{1111}{7776} = \frac{1111}{6^5}}$**.** Consider the more general problem of rolling n dice. Let P_n be the probability that the sum is divisible by 7. Suppose we know the sum of $n-1$ of the n dice. If this sum is divisible by 7, then the sum of all n cannot be. If this sum is not divisible by 7, then there is exactly one number on the n^{th} die that will work. Thus, P_n is the probability the first $n-1$ are not divisible by 7 multiplied by the probability of rolling what we need on the n^{th} die, which gives us $P_n = (1 - P_{n-1})\frac{1}{6}$. Since $P_1 = 0$ and $P_2 = \frac{1}{6}$, we get $P_3 = \frac{5}{36}, P_4 = \frac{31}{216}, P_5 = \frac{185}{1296}$, and finally, $P_6 = \frac{1111}{7776}$.

8. Several students take a quiz which has five questions, and each one is worth a point. They are unsure as to how many points they received, but all of them have a reasonable idea about their scores. Below is a table of what each person thinks is the probability that he or she got each score. Assuming their probabilities are correct, what is the probability that the sum of their scores is exactly 20?

Student \ Score	0	1	2	3	4	5
Allison	0	0	.25	.5	.25	0
Barbara	0	.5	.5	0	0	0
Christi	0	0	0	0	0	1
David	0	0	0	0	.5	.5
Ed	0	.25	.5	.25	0	0
Fred	.25	.5	.25	0	0	0
Gary	0	0	0	.25	.5	.25

   Solution: $\mathbf{\frac{105}{512}}$**.** Together the students answered at least $2+1+5+4+1+0+3 = 16$ questions correctly. In addition, all students have a number of questions of which they are are completely uncertain whether they answered them correctly; Christi has 0, Barbara and David have 1, Alison, Ed, Fred, and Gary have 2. And based on the probabilities in the table, we can see that each student gives a 50% chance that he/she got each such question correct. There are 10 such questions, and the probability that the students' scores totalled 20 is the probability that exactly 4 of these 10 questions were answered correctly. There are $\binom{10}{4}$ ways of selecting the four problems that are answered correctly, and each of these possibilities has probability $\left(\frac{1}{2}\right)^{10}$ of occurring. Thus, the probability that this happens is $\binom{10}{4} \cdot \left(\frac{1}{2}\right)^{10} = \frac{210}{1024} = \frac{105}{512}$.

9. Let F_n be the number of ways of completely covering an $3 \times n$ chessboard with n 3×1 dominoes. For example, there are two ways of tiling a 3×3 chessboard with three 3×1 dominoes (all horizontal or all vertical). What is F_{14}?

   Solution: 129. Consider a $3 \times n$ chessboard. A corner square can be covered in two ways, and a moment's thought reveals that covering the corner with a domino parallel to the side of length 3 yields F_{n-1} arrangements, whereas covering the corner with a domino parallel to the side of length n yields F_{n-3} arrangements. We therefore have the recurrence relation $F_n = F_{n-1} + F_{n-3}$ and the initial conditions $F_1 = F_2 = 1$, $F_3 = 2$. F_{14} is easily computed to be 129.

10. Two players (Kate and Adam) are playing a variant of Nim. There are 11 sticks in front of the players and they take turns each removing either one or any prime

number of sticks. The player who is forced to take the last stick loses. The problem with the game is that if player one (Kate) plays perfectly, she will always win. Give all the starting moves, if any, that lead to a sure win for Kate (assuming each player plays perfectly).

Solution: {2}. The possible moves are $\{1, 2, 3, 5, 7, 11\}$. If Kate takes 11 sticks, then she has taken the last stick and lost. If she takes 7 sticks, then Adam can take 3 sticks leaving Kate to take the last remaining stick. If Kate takes 5 or 3 sticks, then Adam will take 5 or 7 sticks, respectively, leaving Kate the last stick. Thus Kate's only possible winning starting choices are 1 or 2. If the first move is 1, then Adam can take 5 sticks, leaving 5. Now, if Kate takes 1, 2, or 3 sticks, Adam can take 3, 2, or 1 respectively, always leaving Kate the last stick.

Thus, Kate should always take 2 sticks initially, and we will show this lets her win. If Adam takes 7, 5, 3 or 1 sticks after Kate takes 2 sticks, then Kate can take 1, 3, 5, or 7 respectively, leaving Adam one stick. If Adam takes 2 sticks after Kate took 2 sticks, then Kate should take 2 sticks again leaving 5 sticks. Then Adam must take 1, 2 or 3 sticks. Kate should take 3, 2 or 1 stick respectively always leaving one last stick for Adam.

From this, we see that 2 is the only starting move that gives Kate a guaranteed win. Hence, the answer is 2.

11. Define $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$. Find all (x, y) such that $(f(x, y))^2 - (g(x, y))^2 = \frac{1}{2}$ and $f(x, y) \cdot g(x, y) = \frac{\sqrt{3}}{4}$.
Hint: Consider $z = x + iy$, where $i = \sqrt{-1}$.

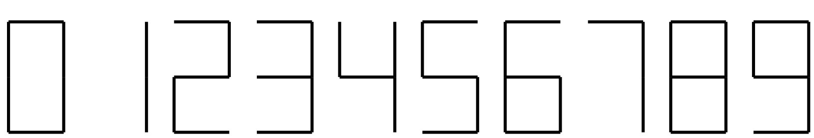
Solution: $\mathbf{(\frac{\sqrt{2}}{4}(1+\sqrt{3}), \frac{\sqrt{2}}{4}(-1+\sqrt{3})), (\frac{\sqrt{2}}{4}(1-\sqrt{3}), \frac{\sqrt{2}}{4}(1+\sqrt{3})), (\frac{\sqrt{2}}{4}(-1-\sqrt{3}), \frac{\sqrt{2}}{4}(1-\sqrt{3}))}$, **and** $\mathbf{(\frac{\sqrt{2}}{4}(-1+\sqrt{3}), \frac{\sqrt{2}}{4}(-1-\sqrt{3}))}$. First, notice that if we set $z = x + iy$ where $i = \sqrt{-1}$, then $z^2 = f(x, y) + ig(x, y)$. Applying the same reasoning again, we find that $z^4 = f^2 - g^2 + 2ifg$. Now, suppose we find z such that

$$z^4 = f^2 - g^2 + 2ifg = \frac{1}{2} + i\frac{\sqrt{3}}{2}. \qquad (*)$$

This equation holds if and only if the real and imaginary parts are equal, or equivalently, $f^2 - g^2 = \frac{1}{2}$ and $fg = \frac{\sqrt{3}}{4}$. Hence, retrieving the x and y that gave us z, we get a solution to the problem, and all solutions may be found in this manner.

Notice that the complex number on the right side of $(*)$ can be drawn in the complex plane as a vector with magnitude 1 and angle $\frac{\pi}{3}$ with the x-axis. Thus $z^4 = e^{i\frac{\pi}{3}} = \cos(\frac{\pi}{3} + 2\pi k) + i\sin(\frac{\pi}{3} + 2\pi k)$. We can see then that $z = \cos(\frac{\pi}{12} + \frac{\pi k}{2}) + i\sin(\frac{\pi}{12} + \frac{\pi k}{2})$, where $k = 0, 1, 2$ or 3. We can solve for these sine and cosine values exactly using the fact that $\sin\frac{\pi}{12} = \sin(\frac{\pi}{3} - \frac{\pi}{4}) = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$ and $\cos\frac{\pi}{12} = \frac{\sqrt{2}}{4}(1+\sqrt{3})$. All four points are $(\frac{\sqrt{2}}{4}(1+\sqrt{3}), \frac{\sqrt{2}}{4}(-1+\sqrt{3})), (\frac{\sqrt{2}}{4}(1-\sqrt{3}), \frac{\sqrt{2}}{4}(1+\sqrt{3})), (\frac{\sqrt{2}}{4}(-1-\sqrt{3}), \frac{\sqrt{2}}{4}(1-\sqrt{3}))$, and $(\frac{\sqrt{2}}{4}(-1+\sqrt{3}), \frac{\sqrt{2}}{4}(-1-\sqrt{3}))$.

12. The numerals on digital clocks are made up of seven line segments, as shown below:



(The two vertical segments for 1 are on the right side.) When various combinations of them light up different numbers are shown. When a digit on the clock changes, some segments turn on and others turn off. For example, when a 4 changes into a 5 two segments turn on and one segment turns off, for a total of 3 changes. In the usual ordering $1, 2, 3, \ldots, 0$ there are a sum total of 32 segment changes (including the wrapping around from 0 back to 1). If we can put the digits in any order, what is the fewest total segment changes possible? (As above, include the change from the last digit back to the first.)

Solution: 16. First, since we are summing the segment changes in a complete cycle, the total number of changes must be even. (Any segment that is turned on must eventually be turned off, and any segment that is turned off must eventually be turned on.)

Now, we look for the digit transitions that involve only one change. After some thought, we find that these are $0 \leftrightarrow 8$, $1 \leftrightarrow 7$, $4 \leftrightarrow 9$, $5 \leftrightarrow 6$, and $6 \leftrightarrow 8$, so at most 5 of the ten digit transitions can involve only one segment change. The other 5 must involve at least 2 segment changes each, giving us a lower bound of 15 total changes. However, this is odd, and the total number of changes must be even, so there must be at least 16 changes altogether. And indeed, the ordering 3, 7, 1, 4, 9, 5, 6, 0, 8, 2 has 16 total changes.

13. How many solutions are there to $(\cos 10x)(\cos 9x) = \frac{1}{2}$ for $x \in [0, 2\pi]$?

Solution: 18. Rewriting $(\cos 10x)(\cos 9x)$ as $\frac{1}{2}(\cos x + \cos 19x)$ we can visualize the graph as a cosine wave $(\cos 19x)$ with period $\frac{2\pi}{19}$, oscillating within an envelope set by $\frac{1}{2}(\cos x - 1) \leq y \leq \frac{1}{2}(\cos x + 1)$. Clearly, once the top of the envelope is below $\frac{1}{2}$ $(\cos x < 0)$, there are no solutions. Thus, we only need to consider the intervals $[0, \frac{\pi}{2}]$ and $[\frac{3\pi}{2}, 2\pi]$. The function is even around $x = \pi$, so we can just consider the first interval and double our answer. In this interval, $\cos 19x$ goes through 9.5 half-cycles, with each full half-cycle crossing $\frac{1}{2}$. This gives us 9 solutions (the last half of a half-cycle does not reach $\frac{1}{2}$) in $[0, \frac{\pi}{2}]$, for a total of 18 solutions.

14. Find $\binom{2003}{0} + \binom{2003}{4} + \binom{2003}{8} + \cdots$.

Hint: Consider $(1 \pm i)$ and (1 ± 1).

Solution: $\mathbf{2^{2001} - 2^{1000}}$. Using the binomial theorem, we can see that

$$2^{2003} = (1+1)^{2003} = \binom{2003}{0} + \binom{2003}{1} + \binom{2003}{2} + \binom{2003}{3} + \binom{2003}{4} + \cdots.$$

Note also that

$$(1+i)^{2003} = \binom{2003}{0} + i\binom{2003}{1} - \binom{2003}{2} - i\binom{2003}{3} + \binom{2003}{4} + \cdots.$$

We can also use the equations

$$0 = (1-1)^{2003} = \binom{2003}{0} - \binom{2003}{1} + \binom{2003}{2} - \binom{2003}{3} + \binom{2003}{4} + \cdots$$

and

$$(1-i)^{2003} = \binom{2003}{0} - i\binom{2003}{1} - \binom{2003}{2} + i\binom{2003}{3} + \binom{2003}{4} + \cdots.$$

If we call the sum we're looking for S, then adding these four equations yields

$$2^{2003} + (1+i)^{2003} + (1-i)^{2003} = 4\left[\binom{2003}{0} + \binom{2003}{4} + \binom{2003}{8} + \cdots\right] = 4S.$$

Now, notice that $(1+i)$ is a complex number with a magnitude of $\sqrt{2}$ and an angle of $\frac{\pi}{4}$. Therefore, $(1+i)^{2003} = (\sqrt{2}e^{i\frac{\pi}{4}})^{2003} = 2^{\frac{2003}{2}}e^{\frac{i2003\pi}{4}}$ which has magnitude $2^{\frac{2003}{2}}$ and angle $\frac{2003\pi}{4} = \frac{3\pi}{4}$. Thus $(1+i)^{2003} = 2^{\frac{2003}{2}}(\frac{-1+i}{\sqrt{2}}) = 2^{1001}(-1+i)$. Similarly, $(1-i)^{2003} = 2^{1001}(-1-i)$, so

$$4S = 2^{2003} + 2^{1001}(-1+i-1-i) = 2^{2003} - 2^{1002}.$$

Solving for S yields $S = 2^{2001} - 2^{1000}$.

15. Alice and Bob are playing a game that depends on N and M, both positive integers. They start with a bag of N marbles, and take turns removing at least one and up to M marbles. Alice moves first, and the person who takes the last marble wins. If N is chosen randomly between 97 and 2003 inclusive, and M is chosen randomly between 1 and 10, what is the probability that Bob will win, assuming optimal play by both parties?

    Solution: $\mathbf{\frac{385}{1907}}$. Bob wins if and only if N is a multiple of $M+1$, so the probability is calculated to be $\frac{1}{10}\cdot\frac{1}{1907}(953+635+476+381+317+273+238+212+191+174) = \frac{385}{1907}$.