Geometry Test Solutions
Stanford Math Tournament
February 22, 2003

1. $ABCD$ is a square with sides of length 1. Suppose that a point E is placed somewhere on the edge CD. Let M be the maximum possible area of $\triangle ABE$, and let m be the minimum possible area of $\triangle ABE$. What is m/M?

   Solution: 1. Consider AB to be the base of the triangle. Then the base has length 1, and regardless of where E is placed, the height is 1 as well. Hence, the area of $\triangle ABE$ is $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$, so $m = M = \frac{1}{2}$. Thus, $m/M = 1$.

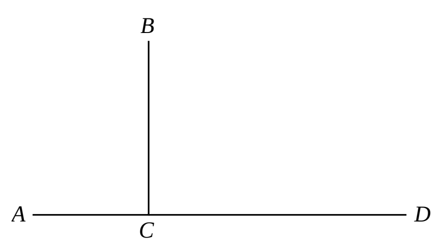
2. Points A, B, and C are such so that AC and BC define unique lines parallel to each other. If $A = (1, 2)$ and $B = (5, 6)$, and $AC = 3BC$, then what are the possible locations for C?

   Solution: $\{(\mathbf{4}, \mathbf{5}), (\mathbf{7}, \mathbf{8})\}$. Since AC and BC define unique lines, A, B, and C must be distinct. Also, since AC is parallel to BC, the points must be collinear. Draw the line through A and B. If C is on the opposite side of B than A, then $AC = AB + BC = 3BC$, which implies that $BC = \frac{1}{2}AB$. If $m = B - A = (4, 4)$, then $C = B + \frac{1}{2}m = (7, 8)$.

   On the other hand, if C is between A and B, then $AC = 3BC$ implies that C is three fourths of the way from A to B. Thus, $C = A + \frac{3}{4}m = (4, 5)$.

   Finally, C cannot be on the opposite side of A because then $BC > AC$ which contradicts $3BC = AC$. Thus, the two possibilities for C that we have already found are the only two points it can be.

3. In the diagram below, $\overline{BC} \perp \overline{AD}$, $AC = 6$, $CD = \frac{27}{2}$, and $m\angle BAC = m\angle CBD$. Find the length of BD.



   Solution: $\frac{\mathbf{9}}{\mathbf{2}}\sqrt{\mathbf{13}}$. First, notice that $\triangle ABC \sim \triangle BDC$ by AA since both are right triangles and $m\angle BAC = m\angle CBD$. Hence, there exists a constant k such that $BC = k \cdot AC$ and $CD = k \cdot BC$. Plugging the first equation into the second, we find that $CD = k^2 \cdot AC$, so

$$k^2 = \frac{CD}{AC} = \frac{27}{2} \cdot \frac{1}{6} = \frac{9}{4},$$

   and thus $k = \frac{3}{2}$.

   Now, we find that $BC = \frac{3}{2} \cdot 6 = 9$. Then, $AC^2 + BC^2 = AB^2$ by the Pythagorean theorem, so

$$AB^2 = 36 + 81 = 117,$$

and thus $AB = \sqrt{117} = 3\sqrt{13}$. And finally, again using the similarity of $\triangle ABC$ and $\triangle BDC$,

$$CD = \frac{3}{2}AB = \frac{9}{2}\sqrt{13}.$$

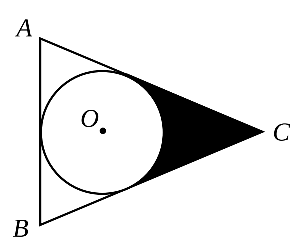
4. Patty and Selma are racing through a park. The park has two concentric circular paths joined by two radial paths, one of which hits the outer circle at the point where they enter the park. The exit is at the intersection of the other radial path and the outer circle. Patty follows the radial path to the inner circle, walks around the short way to the other radial path and down it to the exit. Selma just walks the short way around the outer circle to the exit. They move at the same rate and meet up at the exit at the same time. What is the smaller angle made by the two radial paths?

   Solution: 2 radians. Let R be the radius of the outer circle, and let r be the radius of the inner circle. Let *theta* be the measure of the arc that Selma walks. Then the distance Selma walks is $R\theta$, and the distance Patty walks is $2(R - r) + r\theta$. Setting them equal, we find that

$$R\theta = 2R - 2r + r\theta$$
$$R(\theta - 2) = r(\theta - 2).$$

   Since $R > r$ by assumption, this is only possible if $\theta = 2$. This is smaller than the other angle formed by the segments $(2\pi - 2)$, so this is the angle we are looking for.

5. Circle O is inscribed in $\triangle ABC$ and has radius 1. $\triangle ABC$ is isosceles with $AC = BC$. Suppose that $AB = 2\sqrt{3}$. Find the area of the shaded region.



   Solution: $\mathbf{\sqrt{3} - \frac{\pi}{3}}$**.** Let D be the point at which circle O intersects $\overline{AB}$, and let E be the point at which it intersects $\overline{AC}$. Then $\triangle ADC$ and $\triangle CEO$ are both right triangles, and since $m\angle ECO = m\angle ACD$, we see that $\triangle ADC \sim \triangle CEO$ as well. Therefore, there exists a constant k such that

$$AD = kEO, \quad CD = kCE, \quad AC = kCO.$$

   Since $\triangle ABC$ is isosceles with $AC = BC$, D must be the midpoint of AB, and hence $AD = \sqrt{3}$. Meanwhile, EO is a radius of circle O, so $EO = 1$. Thus, the equation $AD = kEO$ implies that $k = \sqrt{3}$.

   Next, since O is inscribed in $\triangle ABC$, the segment AO bisects $\angle DAE$, from which we quickly see that $\triangle ADO$ and $\triangle AEO$ are congruent. Therefore, $AE = AD = \sqrt{3}$.

Now, let $x = CO$, and let $y = CE$. Then $AC = y + AE = y + \sqrt{3}$, and $CD = x + DO = x + 1$ since DO is a radius of O. The equations $CD = \sqrt{3}CE$ and $AC = \sqrt{3}CO$ then yield

$$x + 1 = \sqrt{3}y$$
$$\sqrt{3}x = y + \sqrt{3}.$$

Solving these equations gives us $x = 2$ and $y = \sqrt{3}$.

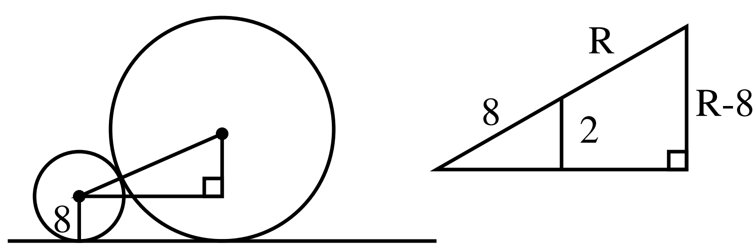
Since $y = \sqrt{3}$, we see that $AC = 2\sqrt{3} = AB$. Thus, all three sides of $\triangle ABC$ are equal, so it is equilateral. Hence, the area of the shaded region is just one third the area of $\triangle ABC$ minus the area of O, and this is $\sqrt{3} - \frac{\pi}{3}$.

6. In trapezoid $ABCD$ with $AB \parallel CD$, $AB = 20$, $CD = 3$, $\angle ABC = 32°$ and $\angle BAD = 58°$. Compute the distance from the midpoint of AB to the midpoint of CD.

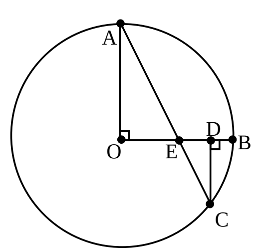
   Solution: $\frac{17}{2}$. Extend AD and BC to their intersection and call that E. Note that $\angle AEB$ is a right angle. The quantity we want is the difference between the medians to the hypotenuses of triangles ABE and CBE. Since the median to the hypotenuse of a right triangle is half the length of the hypotenuse, the answer is $\frac{1}{2}AB - \frac{1}{2}CD = \frac{17}{2}$.

7. Two spherical balls lie on the ground touching. If one ball has a radius of 8 inches and the point of contact is 10 inches above the ground, what is the radius of the other ball?

   Solution: $\frac{40}{3}$. Since the point of contact is above the given ball's radius, then the other ball has a larger radius. Let R be the unknown radius. Draw a line from the center of one ball to the center of the other. The line drawn goes through the tangent point. We can make two similar right triangles as in the diagram below. Thus $\frac{R+8}{8} = \frac{R-8}{2}$, which implies $R = \frac{40}{3}$.



8. In circle O, $OA \perp OB$ and $OB \perp CD$. CD has length $\sqrt{3}$ and arc AC has length 6. $\angle AOC = 120°$. Find DB.



Solution: $2\sqrt{3}-3$. Adding in the segment OC, we get an isosceles triangle OAC. Notice that $OA = OC = r$, the radius of the circle. Then, the law of cosines gives us $36 = r^2 + r^2 - 2r^2\cos 120°$, which implies that $r = 2\sqrt{3}$.

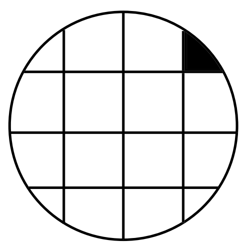
Now, observe that $\triangle AOE \sim \triangle CDE$. Thus,

$$\frac{AE}{EC} = \frac{AO}{CD}$$
$$\frac{6-EC}{EC} = \frac{2\sqrt{r}}{\sqrt{3}}.$$

Solving this equation, we find that $EC = 2$, and therefore $AE = 4$. Using the Pythagorean Theorem, we can then find $OE = 2$ and $ED = 1$. Thus

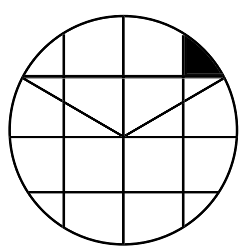
$$DB = 2\sqrt{3} - 2 - 1 = 2\sqrt{3} - 3.$$

9. A circular pizza of diameter 16 is cut so that two perpendicular diameters are each divided into 4 equal lengths. Find the area of the shaded corner piece.

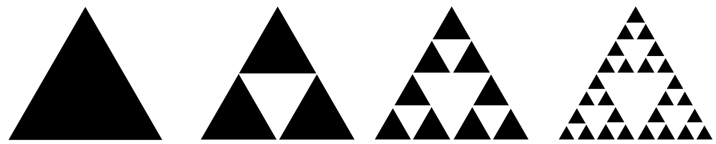


Solution: $\frac{16\pi}{3} + 16 - 16\sqrt{3}$. The entire circle has an area of 64π, and the four squares in the center have total area 64. Call the area of a side piece s and the area of the corner piece c. Then $8s + 4c = 64\pi - 64$, which simplifies to $2s + c = 16\pi - 16$.

Now consider the triangle in the diagram below. The triangle has a $120°$ central angle, and its area is $\frac{1}{2}\cdot 8\sqrt{3}\cdot 4 = 16\sqrt{3}$. The area of the entire sector is $\frac{64\pi}{3}$, so the area of the sector minus the area of the triangle is $\frac{64\pi}{3} - 16\sqrt{3}$. However, the area of this region is also $2s + 2c$, which implies that $2s + 2c = \frac{64\pi}{3} - 16\sqrt{3}$. Subtracting the first equation from this equation yields $c = \frac{16\pi}{3} + 16 - 16\sqrt{3}$.



10. Let the area of the first figure below, the solid equilateral triangle, be 1. Then suppose that smaller equilateral triangles are successively removed step by step in the manner depicted below. What is the perimeter of the shaded region of the nth figure?



Solution: $\mathbf{2 \cdot 3^{3/4} \cdot (\frac{3}{2})^{n-1}}$**.** Let s be the length of a side in the first figure. Each angle is $60°$, and the area of the triangle is 1, so

$$\begin{aligned} \frac{1}{2}s^2 \sin 60° &= 1 \\ s^2 &= \frac{4}{\sqrt{3}} \\ s &= \frac{2}{\sqrt[4]{3}}. \end{aligned}$$

The perimeter of this figure is then $3s = 2 \cdot 3^{3/4}$.

Now, notice that after each step, the perimeter of the shaded region increases by a factor of $3/2$. Thus, in the nth figure, the perimeter has increased by this factor $n-1$ times, so its perimeter is $2 \cdot 3^{3/4} \cdot (\frac{3}{2})^{n-1}$.