General Test Solutions
Stanford Math Tournament
February 22, 2003

1. What is the probability that when you pick a person at random from a phonebook, he or she will have an above-average number of arms?

   A. Zero B. Close to zero C. Half D. Close to one E. One

   Solution: D. Once we observe that the average number of arms is slightly less than two, we immediately see that D is the correct answer.

2. Bobo needs to rent a truck. He could rent from Company Alpha for \$250 per week plus 10 cents per mile or he could rent from Company Beta for \$150 a week plus 25 cents per mile driven. At what weekly milage are the rental fees of the two companies equal?

   Solution: $\mathbf{\frac{2000}{3}}$. Let m be the number of miles, then we want $250 + .1m = 150 + .25m$. For small m, this will not be true. We solve the equality and find that $m = \frac{2000}{3}$ miles.

3. You have 12 red socks and 5 blue socks in your drawer. You take two socks at random. What is the probability of having matching socks?

   Solution: 19/34. The probability of having two red socks and two blue socks are $\frac{12}{17} \cdot \frac{11}{16}$ and $\frac{5}{17} \cdot \frac{4}{16}$ respectively, totaling $\frac{19}{34}$.

4. Six athletes, F, G, H, M, N, and O, swim in two separate 100-meter events, numbered 1 and 2. Each athlete finishes both events, and there are no ties. The same athlete who finishes second in event 1 finished fifth in event 2. M finished sixth in event 1. F finishes in one of the last three places in both events. If M finishes fourth in event 2, in which place must F finish in event 2?

   Solution: Sixth (last). F finishes in fourth, fifth or sixth place in event 2. F cannot finish fifth as the athlete that does so also scores second in event 1. Thus if M finishes fourth, F must finish sixth.

5. Suppose that $a * b = a^2 + ab + 3b + 1$. List all numbers a such that there is no b for which $a * b = 2$.

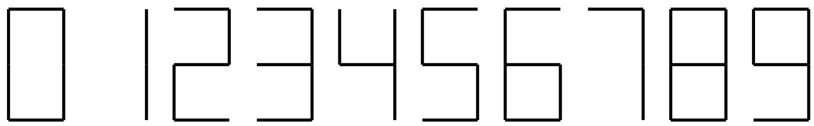
   Solution: $-\mathbf{3}$. If $a * b = 2$, then

$$a^2 + ab + 3b + 1 = 2$$
$$b(a + 3) = 1 - a^2.$$

   And from this we see that if $a \neq -3$, then $b = (1 - a^2)/(a + 3)$ satisfies $a * b$. However, if $a = -3$, then for any b,

$$a * b = a^2 + (a + 3)b + 1 = a^2 + 1 = 10.$$

6. Suppose you are writing numbers in the following font:



How many integers from 0 to 2003 inclusive have the property that if you rotate them 180°, you get the same number back (e.g., 956)? (Note: despite the asymmetrical spacing, if 1 is rotated, we still get 1 back.)

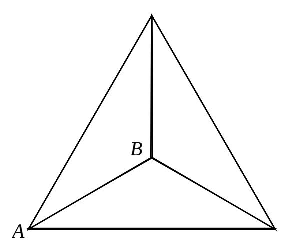
Solution: 49. There are 5 single-digit rotationally-symmetric numbers (0, 1, 2, 5, and 8) and 6 two-digit ones (11, 22, ... , 88, 69, and 96). Since three-digit rotationally-symmetric numbers are composed of a two-digit and a one-digit rotationally-symmetric number, there are $5 \times 6 = 30$. Four-digit rotationally-symmetric numbers are of the form $1AB1$ or 2002 since numbers greater than 2003 are not included. Noting that AB is rotationally-symmetric but can also be 00, we see that there are $6 + 1 = 7$ rotationally-symmetric numbers of the form $1AB1$, so there are 8 four-digit rotationally-symmetric numbers up to 2003. In total, there are therefore $5+6+30+8 = 49$ rotationally-symmetric numbers from 0 to 2003.

7. Ani is taking a true/false test with 6 questions. She gets 1 point for each question she gets right, loses one point for each question she gets wrong, and 0 points for each question she skips. In how many ways can she get a score of 5 on the exam?

Solution: 6. If Ani answers a single question wrong, then she can get at most 4 points, so we know that she can't miss any. From there, we see that the only way she can score 5 is if she answers 5 right and skips one. There are 6 choices for the one she skips, so there are 6 ways she can score a 5 on the exam.

8. An ant is allowed to walk along the edges of a tetrahedron. She starts at vertex A and wants to go to vertex B. She can use any *edge* at most once, and her journey ends once she gets to B. How many different paths can she take?

Solution: 7. To better visualize this, imagine we are looking down at the tetrahedron from above. (Note that it doesn't matter which vertices A and B are.)



There is 1 path using one side, 2 paths using 2 sides, 2 paths using 3 sides, and 2 paths using 4 sides. (The only restriction was that she can't use any *edge* more than once, so it's okay to pass through A again.) This gives us 7 ways total.

9. What is the perimeter of an isosceles right triangle that has an area of 1 square inch?

Solution: $2+2\sqrt{2}$. Let ℓ be the length of a leg, and let h be the length of the hypotenuse. We can orient the triangle such that the base and height are both ℓ, so $\frac{1}{2}\ell^2 = 1$, which implies that $\ell = \sqrt{2}$. Then $h = \sqrt{\ell^2+\ell^2} = \sqrt{4} = 2$. Hence, the perimeter is $2+2\sqrt{2}$.

10. A "Multiplication Day" is defined as a date for which the product of the number of the month and the number of the day equals the last two digits of the year. How many Multiplication Days occur between January 1, 1995 and December 31, 2003?

Solution: 13. The factors of 95 are 1, 5, 19, and 95 making 5/19/95 the only solution in 1995. The useful factors of 96 are 4, 6, 8, 12, 16, and 24, giving rise to the solutions 4/24/96, 6/16/96, 8/12/96, and 12/18/96. 97 is prime so 1997 has no Multiplication Days. The only useful factors of 98 are 7 and 14 giving 7/14/98. 1999 has two days, 9/11/99 and 11/9/99. 2000 has no solutions. 1/1/2001 is the only day in 2001. 2002 however has the two solutions 1/2/02 and 2/1/02. Lastly, 2003 has two days, 1/3/03 and 3/1/03. This gives us a total of 13 days.

11. What are the dimensions of a rectangular tract of land if its perimeter is 40 and its area 96?

Solution: $8, 12$. Let l, w be the length and width, respectively. Then, $2l + 2w = 40$ and $lw = 96$. Thus, $l(20-l) = 96$, which implies $l^2 - 20l + 96 = 0$. This factors as $(l-8)(l-12) = 0$ giving $l = 8$ or $l = 12$. Whichever value l takes, w takes the other.

12. Evaluate $2^{2^{2^2}}/2^{2^2}$.

Solution: 4096. This is simply $2^{2^{2^2}-2^2} = 2^{16-4} = 2^{12} = 4096$.

13. A mixture of 12 liters of Chemical A, 16 liters of Chemical B and 26 liters of Chemical C is required to kill the evil Wiffle bug. Commercial spray X contains 1, 2 and 2 parts respectively of these chemicals. Commercial spray Y contains only Chemical C and commercial spray Z contains only Chemicals A and B in equal amounts. How much of each spray (X, Y, Z) is needed to get the desired mixture?

Solution: $(4, 18, 8)$. Let x, y, z be the number of liters of Chemicals X, Y and Z, respectively that we use. Then, we need $x + z = 12$, $2x + z = 16$ and $2x + y = 26$. Subtracting the first equation from the second yields $x = 4$. Hence we can find $y = 18$ and $z = 8$.

14. A card is chosen randomly from a regular 52-card deck. What is the probability that it is red and is a six or less?

Solution: $\frac{5}{26}$. The possible cards that are six or less are sixes, fives, fours, threes and twos. There are two red suits, giving 10 desirable possibilities out of the 52-card deck. Thus, the probability is $\frac{10}{52} = \frac{5}{26}$.

15. On a twelve-hour digital clock displaying hours and minutes only, for how many minutes during each day is the sum of the digits 12?

Solution: 126. If the hours is 1 or 10, we need the sum of the other digits to be 11. There are 4 possibilities: 29, 38, 47 and 56. If the hour is 2 or 11, we have 5 possibilities, and hours 3 and 12 give us 6 possibilities each. If the hour is 4, 5, 6, or 7, we get possibilities, while 8 gives us 5 possibilities, and 9 gives us 4. Altogether, this gives us 63 different possibilities.. Each readout occurs twice each day, so the answer is 128.

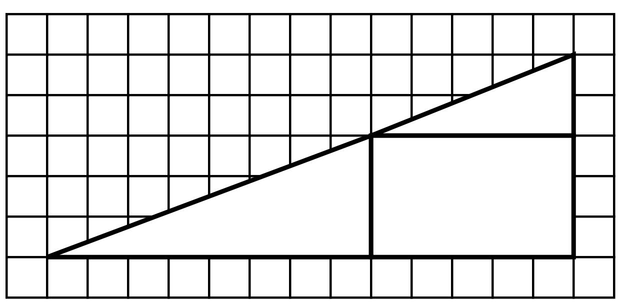
16. Homer has three kids, Bart, Lisa and Maggie. In 2003, Lisa's age is four times Maggie's age plus two. Bart goes into a time machine and goes back 11 years; then his age is three less than twice Lisa's age (in 1992). Bart then goes back to the present and Maggie takes a 15 year trip at light-speed such that she doesn't age but everyone else does. Upon Maggie's return, Bart finds he's seven times as old as Maggie. What is Maggie's age?

    Solution: 6. The scenario can be translated into three equations: $L = 4M + 2$, $B = 2(L-11)-3$, and $B+15 = 7M$. Solving these yields the solution $M = 6, L = 26$, $B = 27$. Thus Maggie is 6 years old.

17. A tropical island has eight towns, N, O, P, R, S, T, U, and W. Tourists must fly into P and then take roads to the other towns. There are roads joining N to O, N to P, N to R, O to U, P to S, P to T, S to T and T to W. A tourist travelling from O to W and visiting as few towns as possible must visit how many different towns between O and W?

    Solution: 3. O only connects to N and U while W only connects to T. Thus, there are more than two towns between them. As N and T both connect to P, there are a minimum of three towns between towns O and W.

18. Each square in the grid below has side length 1. What is the area of the enclosed region?



    Solution: 32. The area of the lower left triangle is $\frac{1}{2} \cdot 3 \cdot 8 = 12$, while the area of the upper right triangle is $\frac{1}{2} \cdot 2 \cdot 5 = 5$, and the area of the rectangle is $3 \cdot 5 = 15$. Summing these gives us 32.

    Notice that the entire enclosed region is *not* a triangle. The hypotenuse of the lower left triangle has a slope of 3/8, while the hypotenuse of the upper right triangle has a slope of 2/5.

19. A rancher has 100 meters of fencing to enclose a rectangular corral whose area is $350\,\text{m}^2$. He also wants to use some of the fencing to divide the corral into two equal

sized areas, and he wants this inner fence to be as short as possible. Find the length of the inner fence.

Solution: 10. Let x be the width and y be the length of the corral. If the inner fence is as short as possible, then it is simply a straight line of length y. Then $2x + 3y = 100$ and $xy = 350$. We can solve one for x and substitute in to yield the quadratic equations $3y^2 - 100y + 700 = 0$. This factors as $(3y - 70)(y - 10) = 0$. Thus $y = 10$ or $\frac{70}{3}$. The shorter is 10.

20. Two fair 5-sided dice are rolled. The faces are labelled 1, 2, 3, 4, and 5. What is the probability their sum is odd?

Solution: $\mathbf{\frac{12}{25}}$. To get an odd sum, we must get an even number with one die and an odd number with the other. There are 2 even numbers and 3 odd numbers, so there are $2 \cdot 3$ possible even/odd pairs, and each can occur in two ways (by reversing the roles of the dice). Thus, there are $2 \cdot 6 = 12$ ways to obtain an odd sum out of the $5^2 = 25$ possible results. Hence, the probability of getting an odd sum is $\frac{12}{25}$.

21. What is the smallest positive number k such that there are real numbers a and b satisfying $a + b = k$ and $ab = k$?

Solution: 4. a and b are the roots of the polynomial $x^2 - kx + k$. (There are various ways to show this. One is to multiply out $(x - a)(x - b)$, another is to substitute $b = k - a$ into the equation $ab = k$.) Using the quadratic equation, this polynomial has roots

$$x = \frac{k \pm \sqrt{k^2 - 4k}}{2}.$$

These roots are real if and only if $k^2 - 4k \geq 0$. And given that $k > 0$, we find that this only holds if $k \geq 4$.

22. How many three digit numbers satisfy the conditions that there is no repetition in the digits, the number must contain a 5, and is less than 800? (Note that a 0 in the leading digit is not allowed.)

Solution: 168. The leading digit must be 1, 2, 3, 4, 5, 6 or 7. One of the three digits must be a 5. First assume that the first digit is 5. That leaves $9 \cdot 8$ choices for the last two digits, since repetition is not allowed. Now, if the second digit is 5, we have 6 possibilities for the first digit, which leaves 8 choices for the third, giving us $6 \cdot 8 = 48$ possibilities. Similarly, if the third digit is 5, there are 6 choices for the first and 8 for the second, giving us another 48 possibilities. Therefore, there are $9 \cdot 8 + 2 \cdot 48 = 168$ different numbers.

23. A merchant plans to sell two models of CD players at costs of \$250 and \$400. The \$250 model yields a profit of \$45 and the \$400 model yields one of \$50 per unit. The merchant estimates a monthly demand of 250 total units. For security purposes, he doesn't want to have more than \$70,000 in inventory (based on selling price) at once. Find the number of \$250 CD players he should stock to maximize profit.

Solution: 200. Let x be the number of \$250 CD players in stock and y be the number of \$400 players. To maximize profit, he should sell as many units as possible, so $x + y = 250$. Meanwhile, due to the inventory limit, $250x + 400y \leq 70000$ due to the inventory limit. From the first equation, we obtain $y = 250 - x$, and plugging this into the inequality yields

$$\begin{aligned} 250x + 400(250 - x) &\leq 70,000 \\ -150x &\leq -30,000 \\ x &\geq 200. \end{aligned}$$

Since the merchant makes the most profit off of the \$400 model, he should sell as many of those as possible, and consequently, as few of the \$250 model as he can. Since $x \geq 200$, he should minimize x, and therefore stock 200 of the \$250 model.

24. How many distinct permutations are there of the group of letters A, A, G, E, E, E, M?

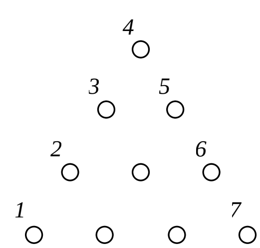
    Solution: 420. There are 7 letters hence 7! orderings. However, the two A's are indistinguishable, hence swapping them leaves us with the same permutation. We have thus overcounted by a factor of 2. Likewise, there are $3! = 6$ rearrangements of the three E's, thus the answer is $\frac{7!}{2!3!} = 420$ permutations.

25. A 1000 liter tank contains 50 liters of a 25% brine solution. If you have plenty of liters of a 75% brine solution, how many liters of the 75% solution do you need to add to the tank to get a 30% brine solution?

    Solution: $\frac{50}{9} = 5\frac{5}{9}$. Initially, $.25 \cdot 50 = 12.5$ liters of the solution is pure brine. Suppose that we add x liters of the 75% solution to the tank. This adds another $.75x$ liters of pure brine, and we want the ratio of pure brine to the total volume of the solution to be .3. Therefore,

$$\begin{aligned} \frac{12.5 + .75x}{50 + x} &= .3 \\ 12.5 + .75x &= 15 + .3x \\ .45x &= 2.5 \\ x &= \frac{50}{9}. \end{aligned}$$

26. Peter is going bowling. On each roll he always hits one of pins 1 through 7, as numbered below, with equal likelihood.



    When he hits pin x, then he knocks over all the pins in the equilateral triangle with x as its topmost point.

Now, suppose Peter bowls a full frame (one roll if he knocks all the pins down with his first roll, otherwise two consecutive rolls with no replacement of the pins in between). If he would hit pin x on his second roll, but has already knocked it down on his first roll, then he doesn't hit any pins at all. What is the probability that Peter knocks down exactly 6 pins in a frame?

Solution: $\mathbf{\frac{12}{49}}$**.** The four remaining pins can either be in a straight line down one of the sides, or in a diamond with pin 4 as its topmost point. If Peter hits pin x on his first roll and pin y on his second, we find that the following sequences knock over 6 pins:

$$(1,3),(2,3),(2,6),(3,1),(3,2),(3,3),(5,5),(5,6),(5,7),(6,5),(6,2),(7,5).$$

Thus, there are 12 ways for Peter to knock over 6 pins. There are 49 ways for him to bowl a frame, so his chance of knocking over 6 pins is $\frac{12}{49}$.

27. Write $\frac{1}{x^2+3x+2}$ as a sum of fractions, each of which has a denominator of the form $ax+b$.

Solution: $\mathbf{\frac{1}{x+1}-\frac{1}{x+2}}$**.** The denominators must be the linear factors of the polynomial x^2+3x+2, which are $x+1$ and $x+2$. Thus, we want to find A and B such that

$$\begin{aligned}\frac{1}{x^2+3x+2} &= \frac{A}{x+1}+\frac{B}{x+2}\\ \frac{1}{x^2+3x+2} &= \frac{A(x+2)+B(x+1)}{x^2+3x+2}\\ \frac{1}{x^2+3x+2} &= \frac{(A+B)x+(2A+B)}{x^2+3x+2}.\end{aligned}$$

Setting the numerators equal, we obtain the system of equations

$$\begin{aligned}A+B&=0\\ 2A+B&=1.\end{aligned}$$

Solving this, we find that $A=1$ and $B=-1$, so

$$\frac{1}{x^2+3x+2}=\frac{1}{x+1}-\frac{1}{x+2}.$$

28. Thirteen pirates are trying to divide up their booty. When everyone is given the same number, one is left over. In the ensuing fight, two pirates fall overboard. The remaining pirates again try to divide the coins evenly amongst themselves and find there is one coin left over. After another fight, four pirates are shot out of cannons. The remaining pirates again find that when they divide up the coins, one is leftover. They give up and donate all the coins to the Rice Math Club.

Afterwards, Greg counts the loot and finds out that the number of coins is the smallest odd number greater than 1 that would give the pirates all these problems. How many coins are there?

Solution: 2003. Let N be the number of coins. Since one coin is leftover when the loot is divided between 13 pirates, $N-1$ must be divisible by 13. Similarly, we find that $N-1$ must be divisible by 11 and 7 as well.

Since 7, 11, and 13 are all prime numbers, and $N > 1$, we see that $N-1$ must be a multiple of $7 \cdot 11 \cdot 13 = 1001$. However, $N-1$ cannot be 1001 since this would give $N = 1002$, which is even. Therefore, $N-1$ must be 2002, so $N = 2003$.

29. Find the number of diagonals in a regular 17-sided polygon.

    Solution: 119. Pick one of the 17 vertices. It has diagonals with 14 other vertices (not counting the vertex nor its neighbors). However, each diagonal is then counted twice, once for each end. This gives $\frac{17 \cdot 14}{2} = 119$ diagonals.

30. Certificates of deposit (CDs) pay 10% annually and municipal bonds pay 8% interest annually. Over the year, an investor wants to invest one-fourth of his money into stocks, and stocks grow by 12% each year. He wants a combined return of 10% on his total investment of \$500,000. How much is put into each type of investment? Give (CD:bonds:stocks).

    Solution: $(\mathbf{250,000 : 125,000 : 125,000})$. Let c, b, and s be the amount invested in CDs, bonds and stocks, respectively. He invests one-fourth of his money in stocks, so $s = \frac{1}{4} \cdot 500,000 = 125,000$. The remaining \$375,000 is divided between CDs and bonds, so $c + b = 375000$, and thus $c = 375,000 - b$. The total return he wants is 10%, so

$$\begin{aligned} .1c + .08b + .12s &= .1 \cdot 500,000 \\ .1 \cdot (375,000 - b) + .08b + .12 \cdot 125,000 &= 50,000 \\ 37,500 - .1b + .08b + 15,000 &= 50,000 \\ .02b &= 2,500 \\ b &= 125,000. \end{aligned}$$

    It then follows that $c = 250,000$, and the ordered triple $(c : b : s)$ is $(250,000 : 125,000 : 125,000)$.

31. A box has a face with area 8, a face with area 15, and a face with area 10. Find the volume of the box.

    Solution: $\mathbf{20\sqrt{3}}$. Let the length, width and height be l, w, and h respectively. Then $wl = 15$, $wh = 8$, and $lh = 10$. Multiplying these three equations together yields $l^2w^2h^2 = 8 \cdot 15 \cdot 10 = 1200$. Thus, volume is $lwh = \sqrt{1200} = 20\sqrt{3}$.

32. Patty is picking peppermints off a tree. They come in two colors, red and white. She picks fewer than 30 total peppermints but at least one of each color. In addition, she always picks fewer white peppermints than five times the number of reds. How many different combinations of peppermints can she go home with?

    Solution: 346. Let w be the number of white and r be the number of red. Then $r + w < 30$ and and $w < 5r$. For $r = 1$ there are 4 choices for w. For $r = 2$ there are 9 choices, $r = 3$ gives 114 choices, $r = 4$ gives 19 choices, and $r = 5$ gives 24 choices. Therefore for $r \leq 5$, there are $4 + 9 + 14 + 19 + 24 = 70$ possible pairs (r, w).

When $6 \leq r < 28$, the restriction $w < 5r$ is overridden by $w + r < 30$. So for each such r, there are $29 - r$ possible choices for w. This gives

$$23 + 22 + 21 + \cdots + 2 + 1 = \frac{23 \cdot 24}{2} = 276$$

more possible pairs (r, w). Hence, there are $276 + 70 = 346$ possibilities in all.

33. We can express any four-digit number as $ABCD$, where A is the first digit, B is the second digit, etc. If any of the conditions below hold, we say that the number is "interesting":

- A, B, C, and D are all even,
- A, B, C, and D are all odd,
- $A > B > C > D$,
- $A < B < C < D$, or
- $A = B = C = D$.

How many "interesting" four-digit numbers are there?

Solution: 1445. First observe that we can discard the fifth condition as it is subsumed by the first two. The four conditions admit $4 \cdot 5 \cdot 5 \cdot 5 = 500$, $5 \cdot 5 \cdot 5 \cdot 5 = 625$, $\binom{10}{4} = 210$, and $\binom{9}{4} = 126$ numbers respectively. The number of four-digit numbers that satisfy at least two conditions is $3\binom{5}{4} + \binom{4}{4} = 16$. Noting that no three conditions can hold simultaneously, we know by inclusion-exclusion that the total number of four-digit numbers satisfying any of the conditions is $500 + 625 + 210 + 126 - 16 = 1445$.

34. Let O be an octagon with with vertices labelled $V_1, V_2, \ldots, V_8$ consecutively. Draw in all the diagonals of the octagon except for diagonals between V_1 and V_5, V_2 and V_6, V_3 and V_7, and V_4 and V_8. Now consider all triangles whose vertices are vertices of the octagon, and whose edges are the diagonals we have just drawn in. How many such triangles are there?

Solution: $\frac{8 \cdot 6 \cdot 4}{3!} = 32$. Let's pick the vertices of a triangle in order. We have 8 choices for the first vertex. The second vertex cannot be the first one we chose, nor can it be the one directly opposite that vertex on the octagon (since that diagonal wasn't drawn in). Hence, there are only 6 choices for the second vertex. The third vertex cannot be either of the first two vertices, nor can it be either of the two vertices opposite these two. Thus, there are 4 choices for the third vertex. Altogether, this gives us $8 \cdot 6 \cdot 4$ ways of choosing a triangle.

However, each unique triangle is counted $3!$ times since this is the number of ways of ordering its vertices. So in the end, there are $8 \cdot 6 \cdot 4/3! = 32$ triangles.

35. Let $i = \sqrt{-1}$. Evaluate

$$i + i^2 + i^3 + \cdots + i^{2003}.$$

Solution: -1. First, notice that $i^4 = 1$, so for any integer $k \geq 0$,

$$i^{4k} = (i^4)^k = 1, \qquad i^{4k+1} = i \cdot i^{4k} = i,$$
$$i^{4k+2} = i^2 \cdot i^{4k} = i^2 = -1, \quad i^{4k+3} = i^3 \cdot i^{4k} = i^3 = -i.$$

From this, we see that

$$i^{4k+1} + i^{4k+2} + i^{4k+3} + i^{4k+4} = i - 1 - i + 1 = 0.$$

We can organize the first 2000 terms in the sum into groups of four like this, so all these terms cancel to zero. This leaves us with

$$\begin{aligned} i^{2001} + i^{2002} + i^{2003} &= i^{4\cdot 500+1} + i^{4\cdot 500+2} + i^{4\cdot 500+3} \\ &= i - 1 - i \\ &= -1. \end{aligned}$$