CALCULUS TEST SOLUTIONS
STANFORD MATH TOURNAMENT
FEBRUARY 22, 2003

1. Given $f(x) = \frac{1}{x}$, find $f^{(100)}(x)$.

   Solution: $\mathbf{\frac{100!}{x^{101}}}$**.** Simply note the pattern: $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$.

2. A windup penguin moves along the x-axis with acceleration given by $a(t) = 2t - 2$ units per second. At $t = 1$ second, the penguin is moving left with a speed of 4 units per second. What is the total distance the penguin travels in the first four seconds ($t = 0$ to $t = 4$)?

   Solution: $\mathbf{\frac{34}{3}}$**.** The distance travelled is given by $\int_0^4 |v(t)|dt$ where $|v(t)|$ is the speed function. $v(t) = \int a(t)dt = t^2 - 2t + C$. We can evaluate C with the information that $v(1) = -4$. $C = -3$. The distance covered is then

$$\begin{aligned}\int_0^4 |t^2 - 2t - 3|dt &= \int_3^4 (t^2 - 2t - 3)dt - \int_0^3 (t^2 - 2t - 3)dt \\ &= \left[\frac{t^3}{3} - t^2 - 3t\right]_3^4 - \left[\frac{t^3}{3} - t^2 - 3t\right]_0^3 \\ &= \frac{7}{3} + 9 \\ &= \frac{34}{3}.\end{aligned}$$

3. Let $f(x) = (x-1)(x-2)(x-3)^2(x-4)(x-5)(x-6)$. Find $f''(3) - f'(3) + f(3)$.

   Solution: $\mathbf{-24}$**.** The only nonzero term in $f''(x)$ after substituting in $x = 3$ is the term in the product rule where we have differentiated the $(x-3)$ terms twice. Thus $f''(x) = 2(3-1)(3-2)(3-4)(3-5)(3-6) = -24$. Note every term in $f'(x)$ involves $(x-3)$ and thus $f'(3) = f(3) = 0$. Thus, $f''(3) - f'(3) + f(3) = -24$.

4. Evaluate
$$\int_{-2003\pi}^{2003\pi} (\cos^2 x - \pi x^2) \sin x \, \mathrm{d}x.$$

   Solution: **0.** Observing that $(\cos^2 x - \pi x^2)$ is even and $\sin x$ is odd, we have that the integrand is odd. The integral is therefore zero.

5. Consider a function $f : \mathbf{R} \to \mathbf{R}$. Given that $f(0) = 0$, $\lim_{h\to 0} \frac{f(h)}{h} = 7$, and $f(x+y) = f(x) + f(y) + 3xy$ for all $x, y \in \mathbf{R}$, what is $f(7)$?

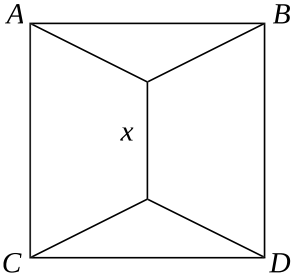
   Solution: $\mathbf{122.5 = \frac{245}{2}}$**.** We know that $f(x+h) = f(x) + f(h) + 3xh$, so

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{f(x) + f(h) + 3xh - f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(h)}{h} + 3x \\ &= 3x + 7\end{aligned}$$

By definition, this limit is $f'(x)$, so we know $f'(x) = 3x + 7$, and thus $f(x) = 1.5x^2 + 7x + k$ for some constant k. But we also know that $f(0) = 0$, so we must have $k = 0$, and therefore $f(7) = 122.5$.

6. Four cities–Athens, Berlin, Cupertino, and Denver–lie at the corners of a square with sides 100 miles long. They want to build a road system that links all 4 cities such that one can drive from one city to any other city (although possibly not directly.

   Building roads costs \$1 million per mile. They have determined that the cheapest system will be of the form



   where the roads are all line segments inside the square. The two triangles in the figure are isosceles. How much does the cheapest possible road system cost? (A radical in your answer is okay.)

   Solution: $\$100(1+\sqrt{3})$ million. This problem is a little easier if we let $y = 50 - \frac{1}{2}x$ (the height of either of the triangles in the diagram) and look at the length of the roads in terms of y. The total length $L(y)$ of the roads is given by $L(y) = 100 - 2y + 4(\sqrt{y^2 + 50^2})$. Therefore,

$$\begin{aligned} L'(y) &= -2 + 4(\frac{1}{2} \cdot 2y \cdot \sqrt{y^2 + 2500}) \\ &= \frac{4y}{\sqrt{y^2 + 2500}} - 2. \end{aligned}$$

   The critical points of $L(y)$ occur when $L'(y) = 0$, or

$$\begin{aligned} \frac{4y}{\sqrt{y^2 + 2500}} &= 2 \\ 2y &= \sqrt{y^2 + 2500} \\ 4y^2 &= y^2 + 2500 \\ 3y^2 &= 2500 \\ y &= \frac{50}{\sqrt{3}}. \end{aligned}$$

   Therefore,

$$\begin{aligned} L(50/\sqrt{3}) &= 100 - 2 \cdot \frac{50}{\sqrt{3}} + 4(\sqrt{2500/3 + 2500}) \\ &= 100 - \frac{100}{\sqrt{3}} + 4 \cdot \frac{100}{\sqrt{3}} \\ &= 100(1 + \sqrt{3}) \end{aligned}$$

is either a maximum or a minimum. It is straightforward to check that this is a minimum, and therefore the cheapest possible system is $\$100(1+\sqrt{3})$ million.

7. A group of college students gets together to write math questions while eating cake. Teena eats at a rate of $c(t)$ pieces of cake per hour and writes at a rate of $q(t)$ questions per hour. The faster Teena eats cake, the less she wants to eat, so $c'(t) = -c(t)$. However, as she eats, she gets sleepy and writes questions more slowly, so $q'(t) = -3c(t)$.

   When the question writing session begins (at $t = 0$), Teena is writing 7 questions per hour and eating 2 pieces of cake per hour. But if she ever writes questions more slowly than 2 per hour, she gets kicked out. The question writing session lasts for 24 hours and thus ends at $t = 24$. Does Teena get kicked out of the question writing session before it ends? And if so, when?

   Solution: Yes, after $\ln 6$ hours. Given that $c'(t) = -c(t)$, we can solve for $c(t)$ by separation of variables:

$$\begin{aligned} \frac{c'}{c} &= -1 \\ \ln|c| &= -t + k_1 \\ c &= e^{-t+k_1} \text{ (since } c \text{ is always positive)} \\ c &= k_2 e^{-t} \end{aligned}$$

   Plugging in $t = 0$, we find that $k_2 = c(0) = 2$, and therefore $c(t) = 2e^{-t}$.

   Now, given that $q'(t) = -3c(t)$, we find that

$$\begin{aligned} q' &= -6e^{-t} \\ q &= 6e^{-t} + k_3. \end{aligned}$$

   Plugging in $t = 0$, we see that $k_3 = q(0) - 6 = 1$, so $q(t) = 6e^{-t} + 1$. This function does indeed decrease below 2 as t increases, and this occurs when

$$\begin{aligned} 6e^{-t} + 1 &= 2 \\ e^{-t} &= \frac{1}{6} \\ -t &= \ln\frac{1}{6} \\ t &= \ln 6. \end{aligned}$$

   Hence, Teena gets kicked out after $\ln 6$ hours.

8. An hourglass (a shape composed of two cones of the same size joined at the apex oriented so that the hourglass sits on one of the cones' base) has a maximal radius of 12 inches and a total height of 10 inches. The sand inside occupies one-fourth of the total volume of the hourglass, and it takes one minute for all the sand to drop from the top cone to the bottom one (the hourglass is a one minute timer). Assuming the sand falls at a constant rate, what is the rate of change in the height of the sand in the bottom cone 15 seconds before the timer runs out? Assume the surface of the sand in the bottom cone is always flat.

Solution: $\frac{\sqrt[3]{5}}{90}$ inches per sec. The total volume of the hourglass is 480π and thus the volume of the sand is 120π. The sand falls at a constant rate, so after $3/4$ of a minute, $3/4$ of the total volume of sand, or $90\pi\text{in}^3$, has fallen into the bottom cone.

Let h be the height of the empty part of the bottom cone (which is also a cone), and let r be the radius of the top surface of the sand. A triangular cross-section of this empty cone is similar to a triangular cross-section of the original cone, so $\frac{h}{5} = \frac{r}{12}$, and therefore $r = \frac{12}{5}h$. Hence, the volume V of this cone, in terms of h, is

$$V = \frac{1}{3}\pi\left(\frac{12}{5}h\right)^2 h = \frac{48}{25}\pi h^3.$$

This quantity equals the volume of the entire bottom cone minus the volume of the sand, which leads us to the equation

$$\begin{aligned} V &= 240\pi - 90\pi \\ \frac{48}{25}\pi h^3 &= 150\pi \\ h &= \frac{5}{2}\sqrt[3]{5}. \end{aligned}$$

Next, since the bottom cone is filling at a rate of $120\pi\frac{\text{in}^3}{\text{min}} = 2\pi\frac{\text{in}^3}{\text{sec}}$, we find that $dV/dt = -2\pi$. Therefore,

$$\begin{aligned} \frac{144}{25}\pi h^2\frac{dh}{dt} &= -2\pi \\ \frac{dh}{dt} &= -\frac{50}{144h^2}. \end{aligned}$$

Plugging in the value of h we obtained earlier gives us $dh/dt = -\sqrt[3]{5}/90$. And since the height of the sand increases at the same rate as h decreases, we conclude that the height of the sand is increasing at $-\sqrt[3]{5}/90\frac{\text{in}}{\text{sec}}$.

9. Let $s > 0$. Consider straight lines drawn from $(0, r)$ to $(s - r, 0)$ for all $0 \le r \le s$. We define $f(x)$ as the maximum y-value that any of these lines take on at x. Find $f(x)$ for $0 \le x \le s$.

   Solution: $(\sqrt{\mathbf{s}} - \sqrt{\mathbf{x}})^2$. Consider a fixed x_0 with $0 \le x_0 \le s$. Consider a line going from $(0, r)$ to $(s - r, 0)$ (where $0 \le s - r \le x_0$). The slope of this line is $r/(r - s)$, and it's y-intercept is r, so the equation of the line is

   $$y = \frac{r}{r - s}x_0 + r.$$

   Hence,

   $$f(x_0) = \max_{0 \le r \le s - x_0}\left(\frac{r}{r - s}x_0 + r\right).$$

   To evaluate this, we let $g(r) = \frac{r}{r-s}x_0 + r$ and find the maximum of this function by setting the derivative equal to 0:

   $$\begin{aligned} g'(r) &= \frac{(r - s) - r}{(r - s)^2}x_0 + 1 \\ &= 1 - \frac{s}{(s - r)^2}x_0. \end{aligned}$$

Setting this equal to 0 yields

$$\begin{aligned}\frac{s}{(s-r)^2}x_0 &= 1\\ sx_0 &= (s-r)^2\\ \pm\sqrt{sx_0} &= s-r\\ r &= s \pm \sqrt{sx_0}.\end{aligned}$$

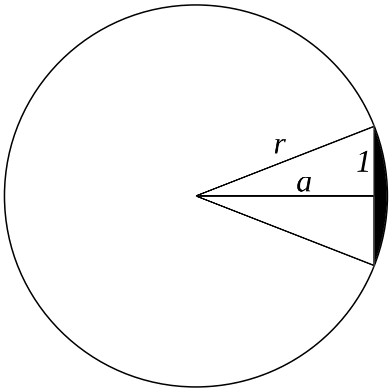
But $s+\sqrt{sx_0} > s$, which is not allowed, so $r = s - \sqrt{sx_0}$ must be the maximum, which we can check by finding the second derivative. So we find that

$$\begin{aligned}f(x_0) &= g(s-\sqrt{sx_0})\\ &= \frac{s-\sqrt{sx_0}}{s-\sqrt{sx_0}-s}x_0 + (s-\sqrt{sx_0})\\ &= \frac{s-\sqrt{sx_0}}{-\sqrt{sx_0}}x_0 + s - \sqrt{sx_0}\\ &= s - 2\sqrt{sx_0} + x_0\\ &= (\sqrt{s}-\sqrt{x_0})^2.\end{aligned}$$

Since x_0 was arbitrary (within the given domain), we conclude that $f(x) = (\sqrt{s}-\sqrt{x})^2$ in general.

10. Sammy the Owl is making a one-eyed Jack'o'Lantern. He buys a perfectly spherical pumpkin of radius 1 foot. He first carves out the inside so that the inner radius is 10 inches. Then, using a cylindrical drill bit of radius 1, he drills a circular hole for the eye. The hole is perfectly straight, with the tip of the drill bit aimed radially inward. What is the total volume in cubic inches of pumpkin that he has removed?

Solution: $\boldsymbol{\pi(\frac{5456}{3} + 198\sqrt{11} - \frac{286\sqrt{143}}{3})}$ **in^3.** First, Sammy removes $\frac{4}{3}\pi(10)^3 = \frac{4000\pi}{3}$ cubic inches from the center of the pumpkin. Next his drill removes a volume that is almost a cylinder, but is curved on the ends. If we take a cross section, then the curved regions on either end look like the shaded region in the diagram below:



Let's figure out the volume of such a curved region for any r. First, notice that by the Pythagorean theorem, $a = \sqrt{r^2-1}$. Next, assume that the pumpkin is centered at the origin, with the hole centered on the positive x-axis. The curved region is the solid of revolution formed by rotating the graph of $y = \sqrt{r^2 - x^2}$, $a \le x \le r$, around the x-axis. Hence, the volume of the region, as a function of r, is

$$V(r) = \int_a^r \pi y^2 \, dx = \int_a^r \pi (r^2 - x^2) \, dx.$$

Evaluating the integral, we find that

$$\begin{aligned} V(r) &= \pi \left[r^2 x - \frac{1}{3} x^3 \right]_a^r \\ &= \pi \left(\frac{2}{3} r^3 - a r^2 + \frac{a^3}{3} \right) \\ &= \pi \left(\frac{2}{3} r^3 - a \left(r^2 - \frac{1}{3} a^2 \right) \right) \\ &= \pi \left(\frac{2}{3} r^3 - \sqrt{r^2 - 1} \left(\frac{2}{3} r^2 + \frac{1}{3} \right) \right). \end{aligned}$$

Now, we return to the hole that Sammy has drilled in the pumpkin. The cylinder we are using to approximate it has radius 1, is centered on the x-axis, and extends for $\sqrt{99} \le x \le \sqrt{143}$ (looking at the a values for $r = 10$ and $r = 12$), so it has volume $\pi(\sqrt{143} - \sqrt{99})$. It is missing a curved region of volume $V(10) = \pi(\frac{2000}{3} - 201\sqrt{11})$ on the inside, and it has an additional curved region of volume $V(12) = \pi(1152 - \frac{289\sqrt{143}}{3})$ on the outside. Hence, the total volume removed for the hole is

$$\pi \left(\sqrt{143} - \sqrt{99} + 1152 - \frac{289\sqrt{143}}{3} - \frac{2000}{3} + 201\sqrt{11} \right),$$

which simplifies to $\pi(\frac{1456}{3} + 198\sqrt{11} - \frac{286\sqrt{143}}{3})$. Finally, adding the original $\frac{4000\pi}{3}$ in^3 that was removed, we get a total volume of $\pi(\frac{5456}{3} + 198\sqrt{11} - \frac{286\sqrt{143}}{3})$ in^3 removed from the pumpkin.