

ALGEBRA TEST
STANFORD MATH TOURNAMENT
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1. Using the digits 2, 0, 0, and 3 only, you are allowed to form integers of the form

$$a, a^b, a^{b^c}, \text{ or } a^{b^{c^d}}.$$

For example, 3200 and 300^2 are two possibilities. What is the *second* largest such number you can make?

2. Suppose that $a * b = a^2 + ab + 3b + 1$. List all numbers a such that there is no b for which $a * b = 2$.
3. What is the smallest positive number k such that there are real numbers a and b satisfying $a + b = k$ and $ab = k$.
4. Harry, Hermione, and Ron go to Diagon Alley to buy chocolate frogs. If Harry and Hermione each spend one-fourth of their own money, they would spend 3 galleons in total. If Harry and Ron each spend one-fifth of their own money, they would spend 2 galleons in total. Everyone has a whole number of galleons, and the number of galleons between the three of them is a multiple of 7. How many galleons *could* Harry have? List all possibilities.
5. In the following equation, x and y are digits and the subscripts are number bases. $(11xy)_7 = (310x)_5$. Find (x, y) .
6. Assume the polynomial $p(x) = x^8 + 86x^6 - 87x^4 + 212x^2 + 4$ has no complex roots. How many negative real roots does it have?
7. Let r_1, r_2 , and r_3 be the solutions of the equation $x^3 - 2x^2 + 4x + 10 = 0$. Compute $(r_1 + 2)(r_2 + 2)(r_3 + 2)$.
8. Taking positive square roots, evaluate

$$\sqrt{\sqrt{72 + \sqrt{72 + \sqrt{72 + \dots}}}}$$

9. Solve for x :

$$\log_2 \log_4 x + \log_4 \log_2 x = 2.$$

10. Let $b > 0$ and $c > 0$. Suppose that the sequence x_1, x_2, x_3, \dots is defined by

$$\begin{aligned}x_0 &= 1 \\x_1 &= 1 \\x_{n+2} &= bx_{n+1} + cx_n, \quad n \geq 0.\end{aligned}$$

The ratio x_{n+1}/x_n approaches a finite number R as n goes to infinity. What is R ?