Algebra Test Solutions
Stanford Math Tournament
February 22, 2003

1. Using the digits 2, 0, 0, and 3 only, you are allowed to form integers of the form

$$a,\ a^b,\ a^{b^c},\ \text{or}\ a^{b^{c^d}}.$$

For example, 3200 and 300^2 are two possibilities. What is the *second* largest such number you can make?

Solution: $\mathbf{2^{300}}$. $3^{200} = 9^{100} > 2^{300} = 8^{100}$. All other possibilities are easily seen to be smaller.

2. Suppose that $a * b = a^2 + ab + 3b + 1$. List all numbers a such that there is no b for which $a * b = 2$.

Solution: $-\mathbf{3}$. If $a * b = 2$, then

$$\begin{aligned} a^2 + ab + 3b + 1 &= 2 \\ b(a+3) &= 1 - a^2. \end{aligned}$$

And from this we see that if $a \neq -3$, then $b = (1-a^2)/(a+3)$ satisfies $a * b$. However, if $a = -3$, then for any b,

$$a * b = a^2 + (a+3)b + 1 = a^2 + 1 = 10.$$

3. What is the smallest positive number k such that there are real numbers a and b satisfying $a + b = k$ and $ab = k$.

Solution: 4. a and b are the roots of the polynomial $x^2 - kx + k$. (There are various ways to show this. One is to multiply out $(x-a)(x-b)$, another is to substitute $b = k - a$ into the equation $ab = k$.) Using the quadratic equation, this polynomial has roots

$$x = \frac{k \pm \sqrt{k^2 - 4k}}{2}.$$

These roots are real if and only if $k^2 - 4k \geq 0$. And given that $k > 0$, we find that this only holds if $k \geq 4$.

4. Harry, Hermione, and Ron go to Diagon Alley to buy chocolate frogs. If Harry and Hermione each spend one-fourth of their own money, they would spend 3 galleons in total. If Harry and Ron each spend one-fifth of their own money, they would spend 2 galleons in total. Everyone has a whole number of galleons, and the number of galleons between the three of them is a multiple of 7. How many galleons *could* Harry have? List all possibilities.

Solution: $\{\mathbf{1, 8}\}$. Let A, B, and C be the number of galleons Harry, Hermione, and Ron have, respectively. Then the given information provides us with the following conditions:

$$\begin{aligned} A+B &= 12 \\ A+C &= 10 \\ qA+B+C &= 7D, \text{ where } D \text{ is an integer} \end{aligned}$$

By adding the first two and subtracting the third, we get $A = 22 - 7D$. Since A, B, and C must be nonnegative, we have $A = 1$ or 8.

5. In the following equation, x and y are digits and the subscripts are number bases. $(11xy)_7 = (310x)_5$. Find (x, y).

   Solution: $\mathbf{(1,2)}$. Written in base 10, the equation $(11xy)_7 = (310x)_5$ becomes

$$\begin{aligned} 7^3 + 7^2 + x \cdot 7^1 + y &= 3 \cdot 5^3 + 5^2 + x \\ 392 + 7x + y &= 400 + x \\ 6x + y &= 8. \end{aligned}$$

   Since x is a digit in a base 5 number, we must have $0 \leq x \leq 4$, and since y is a digit in a base 7 number, we must have $0 \leq y \leq 6$. The only solution that satisfies these constraints is $x = 1$, $y = 2$.

6. Assume the polynomial $p(x) = x^8 + 86x^6 - 87x^4 + 212x^2 + 4$ has no complex roots. How many negative real roots does it have?

   Solution: 4. Since $p(x)$ has no complex roots, all 8 of its roots are real. Note however that the polynomial is even, i.e. symmetric about the y-axis. Thus, $P(x)$ has an equal number of positive and negative roots, meaning that it has $\frac{8}{2} = 4$ negative real roots.

7. Let r_1, r_2, and r_3 be the solutions of the equation $x^3 - 2x^2 + 4x + 10 = 0$. Compute $(r_1+2)(r_2+2)(r_3+2)$.

   Solution: 14. Since r_1, r_2, and r_3 are solutions, we know that

$$(x - r_1)(x - r_2)(x - r_3) = x^3 - 2x^2 + 4x + 10.$$

   Plugging in $x = -2$ gives us

$$\begin{aligned} (-2 - r_1)(-2 - r_2)(-2 - r_3) &= (-2)^3 - 2(-2)^2 + 4(-2) + 10 \\ (-1)(r_1+2)(r_2+2)(r_3+2) &= -14 \\ (r_1+2)(r_2+2)(r_3+2) &= 14. \end{aligned}$$

8. Taking positive square roots, evaluate

$$\sqrt{72 + \sqrt{72 + \sqrt{72 + \cdots}}}.$$

   Solution: 9. Let x equal the given expression. Then x satisfies $x = \sqrt{72 + x}$, and hence $x^2 - x - 72 = 0$. From this, we easily see that $x = -8$ or 9, and since we are taking positive square roots, it is clear that $x = 9$ is the only possibility.

9. Solve for x:

$$\log_2 \log_4 x + \log_4 \log_2 x = 2.$$

Solution: $\mathbf{x = 16}$**.** First, we want to know the relationship between $\log_2 a$ and $\log_4 a$ for any positive number a. Let $b = \log_4 a$. Then $a = 4^b = 2^{2b}$, which implies that $\log_2 a = 2b$. Therefore, $\log_2 a = 2 \log_4 a$.

Now, let $y = \log_4 x = 2 \log_2 x$. Substituting this into the equation for x, we find that

$$\log_2 y + \log_4 2y = 2.$$

Noting that $\log_2 y = 2 \log_4 y = \log_4 y^2$, we then obtain

$$\begin{aligned} \log_4 y^2 + \log_4 2y &= 2 \\ \log_4 2y \cdot y^2 &= 2 \\ 2y^3 &= 4^2 \\ y &= 2. \end{aligned}$$

And finally, $x = 4^y = 16$.

10. Let $b > 0$ and $c > 0$. Suppose that the sequence $x_1, x_2, x_3, \ldots$ is defined by

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 1 \\ x_{n+2} &= bx_{n+1} + cx_n, \quad n \geq 0. \end{aligned}$$

The ratio x_{n+1}/x_n approaches a finite number R as n goes to infinity. What is R?

Solution: $\mathbf{\frac{b+\sqrt{b^2+4c}}{2}}$**.** For each positive integer n, let $r_n = x_{n+1}/x_n$. Taking the recurrence relation for x_{n+2} and dividing by x_{n+1}, we get

$$\begin{aligned} \frac{x_{n+2}}{x_{n+1}} &= b + c\frac{x_n}{x_{n+1}} \\ r_{n+1} &= b + cr_n^{-1}, \end{aligned}$$

which is a recurrence relation for r_n.

Based on the given information, we know that the sequence $r_1, r_2, r_3, \ldots$ approaches R as n goes to ∞. Then R must be a fixed point of the recursion for r_n; that is, it must satisfy

$$R = b + cR^{-1}.$$

(If R *didn't* satisfy this relation, then whenever the sequence r_n got very close to R, it would immediately jump away from R.) Therefore

$$R^2 - bR - c = 0.$$

Using the quadratic equation, we find that

$$R = \frac{b \pm \sqrt{b^2 + 4c}}{2}.$$

One can easily see that x_n is positive for all n, and therefore r_n must be positive for all n as well. Therefore, R must be positive, too, meaning that the only possibility is

$$R = \frac{b + \sqrt{b^2 + 4c}}{2}.$$