Advanced Topics Test
Stanford Math Tournament
February 22, 2003

1. Two coins are found in a fountain. One is a fair coin and the other has "heads" on both sides. One coin is chosen randomly and flipped 5 times. All 5 times it lands "heads" face up. What is the probability that the fair coin was chosen?

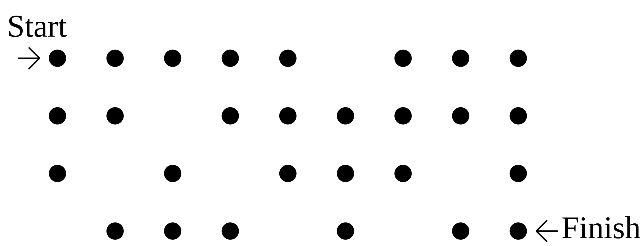
2. Let $\diamondsuit$ be a binary operator on positive real numbers that satisfies the following two rules: $(x \cdot y^2)\diamondsuit y = x(y\diamondsuit 1)$ and $(x\diamondsuit 1)\diamondsuit x = 1$. Given $1\diamondsuit 1 = 1$, find $23\diamondsuit 87$.

3. Sammy and Bevo each choose a real number at random between 1 and 10, inclusive. What is the probability that they differ by more than 4?

4. Let $a(x)$, $b(x)$, $c(x)$,and $d(x)$ be polynomials, none of which have roots of multiplicity greater than 1. Suppose that $\log_2 a(x) = \log_2 b(x) + \log_2 c(x) - \log_2 d(x)$. Furthermore, $b(x) = 0$ has exactly two solutions, $x = -7$ and $x = -11$; $c(x) = 0$ has exactly two solutions, $x = -5$ and $x = -8$; and $d(x) = 0$ has exactly two solutions, $x = -5$ and $x = -11$. Also, $\log_2 a(-6) = 2$. Find $\log_2 a(1)$.

5. There are 1,000 points equally spaced on a circle of radius 10. A point randomly; call it A. Pick another distinct point; call it B. Repeat this process until you have six distinct points A, B, C, D, E and F. What is the probability that the triangles ADC and BEF do not intersect each other?

6. In the diagram below, you can move from one dot to another adjacent dot by moving right, down, diagonally down to the right or diagonally up to the right, and two dots are adjacent if they are within one row and/or one column of each other. How many distinct paths are there from the starting dot to the ending dot?



7. All thirteen spades in a deck of cards are shuffled uniformly and dealt in a line. Let S be a statement about the order of the thirteen cards and $P(S)$ be the probability that S is true. For example, suppose S is "The five appears before the nine", then $P(S) = \frac{1}{2}$. How many of the values $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{50}$ can $P(S)$ not attain?

8. Euler and his wife (Katharina Gsell) throw a dinner party and invite four other married couples. Once everyone arrives, various people shake hands. Note that no person shakes hands with himself and no married couple shakes hands with each other. Euler asks his wife and everyone else at the party how many people's hands they have shaken and is shocked to find that every answer he receives is different (note that Euler doesn't consider the number of handshakes in which he participated). How many handshakes did Euler's wife participate in?

9. A number of otherwise identical cards are marked either red or blue on one side. In total, there are a red cards and b blue cards. The cards are shuffled and arranged in a stack face-down with their colors not showing. You can flip up to $a+b-1$ cards from the top of the pile over to look at their colors, but eventually, you must stick a card to your forehead *before looking at its color*. What are the minimum and maximum probabilities P_{min} and P_{max} with which you can pick a red card? You can use any strategy you wish. Give your answer in the form $(P_{\text{min}}, P_{\text{max}})$.

10. Evaluate $2 \cdot 3^{-1} - 4 \cdot 3^{-2} + 6 \cdot 3^{-3} - 8 \cdot 3^{-4} + \cdots$.