Advanced Topics Test Solutions
Stanford Math Tournament
February 22, 2003

1. Two coins are found in a fountain. One is a fair coin and the other has "heads" on both sides. One coin is chosen randomly and flipped 5 times. All 5 times it lands "heads" face up. What is the probability that the fair coin was chosen?

   Solution: $\mathbf{\frac{1}{33}}$**.** The probability that the coin is fair is $\frac{A}{A+B}$, where A is the number of ways to obtain 5 consecutive heads with the fair coin, and B is the number of ways with the unfair coin. $A = 1$ and $B = 2^5 = 32$, so the answer is $\frac{1}{33}$.

2. Let $\diamondsuit$ be a binary operator on positive real numbers that satisfies the following two rules: $(x \cdot y^2)\diamondsuit y = x(y\diamondsuit 1)$ and $(x\diamondsuit 1)\diamondsuit x = 1$. Given $1\diamondsuit 1 = 1$, find $23\diamondsuit 87$.

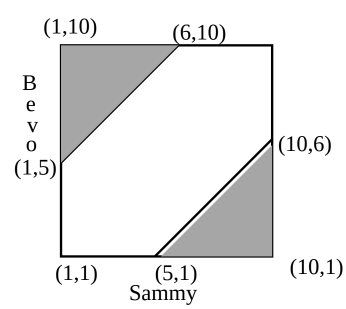
   Solution: $\mathbf{\frac{23}{87}}$**.** Take $y = 1$. Then $(x \cdot 1^2)\diamondsuit 1 = x(1\diamondsuit 1) = x$ by the first rule. Thus $x\diamondsuit 1 = x$. Now, by the second rule,

$$\begin{aligned} x\diamondsuit y &= (\frac{x}{y^2} \cdot y^2)\diamondsuit y \\ &= \frac{x}{y^2}(y\diamondsuit 1) \\ &= \frac{x}{y^2} \cdot y \\ &= \frac{x}{y}. \end{aligned}$$

   Thus $23\diamondsuit 87 = \frac{23}{87}$.

3. Sammy and Bevo each choose a real number at random between 1 and 10, inclusive. What is the probability that they differ by more than 4?

   Solution: $\mathbf{\frac{25}{81}}$**.** Call Sammy's number s and Bevo's number b. They differ by more than 4 if $b - s > 4$ or $s - b > 4$. All possible choices can be represented by the square shown below. The shaded region is where they differ by more than 4 and it has area $\frac{25}{2} + \frac{25}{2} = 25$. The probability is then $\frac{25}{9^2} = \frac{25}{81}$.



4. Let $a(x)$, $b(x)$, $c(x)$,and $d(x)$ be polynomials, none of which have roots of multiplicity greater than 1. Suppose that $\log_2 a(x) = \log_2 b(x) + \log_2 c(x) - \log_2 d(x)$. Furthermore, $b(x) = 0$ has exactly two solutions, $x = -7$ and $x = -11$; $c(x) = 0$ has exactly two

solutions, $x = -5$ and $x = -8$; and $d(x) = 0$ has exactly two solutions, $x = -5$ and $x = -11$. Also, $\log_2 a(-6) = 2$. Find $\log_2 a(1)$.

Solution: $4 + 2\log_2 3$ or $4 + \log_2 9$ or $\log_2 144$. From their roots, we know that $b(x) = c_1(x+7)(x+11)$, $c(x) = c_2(x+5)(x+8)$, and $d(x) = c_3(x+5)(x+11)$ for some c_1, c_2, c_3. Since

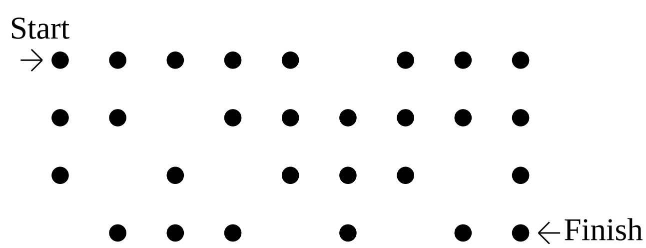
$$\log_2 a(x) = \log_2 \frac{b(x)c(x)}{d(x)} = \log_2 \frac{c_1c_2(x+7)(x+11)(x+5)(x+8)}{c_3(x+11)(x+5)},$$

we can see that $\log_2 a(x) = \log_2(c_4(x+7)(x+8))$, where $c_4 = \frac{c_1c_2}{c_3}$. So, $a(x) = c_4(x+7)(x+8)$. Since $a(-6) = 4$, then we see that $c_4 = 2$, so $a(x) = 2(x+7)(x+8)$, and $a(1) = 2 \cdot 8 \cdot 9 = 144$. So, $\log_2 a(1) = \log_2 144 = 4 + \log_2 9 = 4 + 2\log_2 3$.

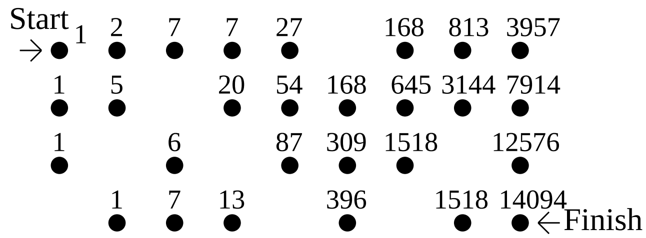
5. There are 1,000 points equally spaced on a circle of radius 10. A point randomly; call it A. Pick another distinct point; call it B. Repeat this process until you have six distinct points A, B, C, D, E and F. What is the probability that the triangles ADC and BEF do not intersect each other?

Solution: $\frac{3}{10} = .3$. Two triangles on a circle intersect if and only if the vertices are interspersed. By this, we mean the ordering $ABEDCF$ is interspersed since the letters ADC don't occur together, but $ABEFDC$ is not because we can rotate the ordering to get an equivalent ordering $DCABEF$, in which the letters of ADC and BEF both occur together. To avoid over-counting always start viewing the possible distributions at point A and move clockwise. There are clearly 5! total ways to order the points. Note that each has equal probability of being chosen. The 6 orientations that are not interspersed are $ACD(BEF)$, $ADC(BEF)$, $AC(BEF)D$, $AD(BEF)C$, $A(BEF)CD$, and $A(BEF)DC$. Note that in each orientation, the triangle BEF can occur in any of 6 possible orderings as well. This gives us 36 total orientations where the vertices do not intersect and thus the probability is $\frac{36}{5!} = \frac{3}{10}$.

6. In the diagram below, you can move from one dot to another adjacent dot by moving right, down, diagonally down to the right or diagonally up to the right, and two dots are adjacent if they are within one row and/or one column of each other. How many distinct paths are there from the starting dot to the ending dot?



Solution: 14094. We can count the number of ways to get to any point recursively. You must start at the starting dot and from each point you can come from above it, left of it, above and left diagonally or below and left diagonally provided that these directions are possible. When you consider a new dot D, look at the number of ways you can get to each dot from which it can be reached; the sum of these is the number of ways of reaching D. Once we look at the final dot, we find that there are 14094 ways of getting there.



7. All thirteen spades in a deck of cards are shuffled uniformly and dealt in a line. Let S be a statement about the order of the thirteen cards and $P(S)$ be the probability that S is true. For example, suppose S is "The five appears before the nine", then $P(S) = \frac{1}{2}$. How many of the values $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{50}$ can $P(S)$ not attain?

   Solution: 13. Notice that $P(S)$ must be of the form $P(S) = \frac{N(S)}{13!}$ with $N(S)$ an integer, since the number of ways to arrange 13 cards in a row is 13!. So for $P(S) = \frac{1}{n}$, n must satisfy $n \cdot N(S) = 13!$, that is, n divides 13!. The numbers from 1 to 50 that do not satisfy this are 17, 19, 23, 29, 31, 34, 37, 38, 41, 43, 46, 47, and 49, so thirteen values are impossible.

   We now show that the other values for n are possible. Since n divides 13!, let $N(S) = \frac{13!}{n}$. Define S to be a listing of $N(S)$ possible orderings of the thirteen cards (where S is true if one of them is the correct ordering). Each ordering can occur in only one way, so $P(S) = \frac{N(S)}{13!} = \frac{1}{n}$ as desired. Thus, only the thirteen values listed above are impossible to attain.

8. Euler and his wife (Katharina Gsell) throw a dinner party and invite four other married couples. Once everyone arrives, various people shake hands. Note that no person shakes hands with himself and no married couple shakes hands with each other. Euler asks his wife and everyone else at the party how many people's hands they have shaken and is shocked to find that every answer he receives is different (note that Euler doesn't consider the number of handshakes in which he participated). How many handshakes did Euler's wife participate in?

   Solution: 4. Euler receives 9 different answers to his question. However, each person shakes hands with at most 8 people. Thus the answers Euler gets are the numbers 0,1,2,. . . ,8. Clearly the person who shook hands with no people must be married to the person who shook everyone else's hands. (The person who shook 8 hands clearly shook hands with everyone except himself and his spouse. If someone shook no hands, then she had better be the aforementioned spouse). Likewise, one can reason that the people who shook 1 person's hands and 7 people's hands must be married. This continues until we find that the person who shook four people's hands must be Euler's wife.

9. A number of otherwise identical cards are marked either red or blue on one side. In total, there are a red cards and b blue cards. The cards are shuffled and arranged in a stack face-down with their colors not showing. You can flip up to $a+b-1$ cards from the top of the pile over to look at their colors, but eventually, you must stick a card to your forehead *before looking at its color.* What are the minimum and maximum probabilities $P_{\min}$ and $P_{\max}$ with which you can pick a red card? You can use any strategy you wish. Give your answer in the form $(P_{\min}, P_{\max})$.

Solution: $\mathbf{\frac{a}{a+b}}$. Imagine that when you flip over a card to see its color, a dealer deals from the top of the deck, but when you decide to stick a card on your forehead, he deals it from the bottom. We can see that the probability of getting a red card is always $\frac{a}{a+b}$, regardless of your strategy.

10. Evaluate $2 \cdot 3^{-1} - 4 \cdot 3^{-2} + 6 \cdot 3^{-3} - 8 \cdot 3^{-4} + \cdots$.

Solution: $\mathbf{\frac{3}{8} = .375}$. Call the answer s. Then

$$\begin{aligned} s &= (-2)\sum_{n=1}^{\infty} n\left(-\frac{1}{3}\right)^n \\ &= (-2)\sum_{n=1}^{\infty}\sum_{j=1}^{n}\left(-\frac{1}{3}\right)^n \\ &= (-2)\sum_{j=1}^{\infty}\sum_{n=j}^{\infty}\left(-\frac{1}{3}\right)^n \\ &= (-2)\sum_{j=1}^{\infty}\frac{\left(-\frac{1}{3}\right)^j}{1-\left(-\frac{1}{3}\right)} \\ &= \left(-\frac{3}{2}\right)\sum_{j=1}^{\infty}\left(-\frac{1}{3}\right)^j \\ &= \left(-\frac{3}{2}\right)\frac{\left(-\frac{1}{3}\right)}{1-\left(-\frac{1}{3}\right)} \\ &= \frac{3}{8} \end{aligned}$$