Stochastic mixing model with power law decay of variance

Sergei Fedotov,[1] Matthias Ihme,[2] and Heinz Pitsch[2]
[1]*School of Mathematics, The University of Manchester, Manchester M60 1QD United Kingdom*
[2]*Center for Turbulence Research, Stanford University, Stanford, California 94305-3035, USA*
(Received 27 January 2004; published 14 January 2005)

A stochastic mixing model based on the law of large numbers is presented that describes the decay of the variance of a conserved scalar in decaying turbulence as a power law, $\sigma_c^2 \propto t^{-\alpha}$. A general Lagrangian mixing process is modeled by a stochastic difference equation where the mixing frequency and the ambient concentration are random processes. The mixing parameter λ is introduced as a coefficient in the mixing frequency in order to account for initial length-scale ratio of the velocity and scalar field and other physical dependencies. We derive a nonlinear integral equation for the probability density function (pdf) of a conserved scalar that describes the relaxation of an arbitrary initial distribution to a δ-function. Numerical studies of this equation are conducted, and it is shown that λ has a distinct influence on the decay rate of the scalar. Results obtained from the model for the evolution of the pdf are in a good agreement with direct numerical simulation (DNS) data.

DOI: 10.1103/PhysRevE.71.016310 PACS number(s): 47.10.+g, 05.40.−a

I. INTRODUCTION

The mixing of a conserved scalar $c=c(t,\mathbf{x})$, advected by a turbulent flow, is a problem of both fundamental and practical interest [1,2]. One of the basic characteristics of the mixing process is the rate at which the scalar variance $\sigma_c^2(t) = \langle (c-\mu)^2 \rangle$ decays with time. Here μ is the mean value, and the angular brackets $\langle \cdot \rangle$ denote an averaging procedure. One of the simplest and widely used mixing models is the interaction by exchange with the mean (IEM) model [3,4]. In this model, the scalar relaxes toward its mean μ according to the simple equation

$$\frac{dc}{dt} = -\frac{1}{\tau}(c-\mu). \tag{1}$$

Other mixing models are based on mapping closure [5], the coalescence-dispersion model [6], the Langevin model [7], the Fokker-Planck models [8], the stochastic mixing frequency model [9], integral models [10], and the Euclidean minimum spanning tree model (EMST) [11]. Reviews of these can be found in [1,2].

There are two different laws governing the decay rate of a passive scalar: (a) the exponential law [12]

$$\sigma_c^2(t) \propto \exp\{-t/\tau\}, \tag{2}$$

with the characteristic time τ which is appropriate for stationary turbulence and (b) the power law [13,14]

$$\sigma_c^2(t) \propto t^{-\alpha}, \tag{3}$$

without any characteristic time scale that is typical for decaying turbulence.

Most theoretical models introduce a characteristic time scale and assume implicitly or explicitly the exponential decay rate. The main purpose of this paper is to study the mixing process following the power law (3). Experiment results show that the decay of the variance strongly depends on the initial ratio of the velocity and scalar length scales and that there is no universal decay exponent [15]. It is well known that the decay exponent α depends on the low wavenumber part of the scalar and velocity spectrum, $E_c(k,t)$ and $E(k,t)$ [13]. One can expand both spectra into a Taylor series: $E_c(k,t)=2\pi k^2[C_0+C_2k^2+\mathcal{O}(k^4)]$ and $E(k,t)=2\pi k^2[B_0+B_2k^2+\mathcal{O}(k^4)]$ [14]. For high Reynolds and Peclet numbers, the scalar variance can be considered as a function of C_0, B_0 and t alone and dimensional arguments lead to

$$\sigma_c^2(t) \propto C_0 B_0^{-3/5} t^{-6/5} \tag{4}$$

with $\alpha=1.2$. For C_0, B_0 equal to zero, the scaling argument results in different approximate decay laws with α equals to $6/7, 10/7$, or 2 [13].

Here we present a simple stochastic mixing model based on the law of large numbers (LLN) [16]. The reason why the LLN is involved in our formulation of the mixing problem is that the random conserved scalar $c=c(t,\mathbf{x}(t))$ appears to behave as a sample mean. It converges to the mean value μ, while the variance $\sigma_c^2(t)$ decays approximately as t^{-1}. Since the variance of the scalar decays faster than a sample mean (typically α is greater than unity), we will introduce some nonlinear modifications into the corresponding probability density function (PDF) equation [see Eq. (35)]. The main idea is to develop a robust model that is independent from restrictive assumptions regarding stochastic properties of the mixing process. Here we exploit the similarity of the behavior of a scalar c to that of the sample mean

$$c_n = \frac{1}{n}\sum_{k=1}^{n} \zeta_k, \tag{5}$$

where $\zeta_1, \ldots, \zeta_n$ is a sequence of mutually independent random variables, each having a mean μ and standard deviation σ_ξ^2. The LLN tells us that the random sum c_n tends to the mean value μ with probability one, while the variance $\sigma_{c_n}^2 = \langle (c_n-\mu)^2 \rangle$ decays as σ_ξ^2/n. Then, the probability density function of c_n tends to a δ-function, $\delta(c-\mu)$, as $n \to \infty$ independent of the initial shape of the PDF $p(0,c)$. In the present paper, where we consider decaying isotropic turbulence with

the constant scalar mean μ, the discrete increment n can be understood as the time variable.

The main result of this paper is the derivation of the time-discrete nonlinear integral equation for the PDF $p(t,c)$

$$p(t+1,c) = \int_0^1 \int_{-1}^{\frac{1+t}{\lambda}-1} \frac{t+1}{1+t-\lambda(1+\varepsilon)} \times p\left(t, \frac{(t+1)(c-\mu)-\lambda(1+\varepsilon)(z-\mu)}{1+t-\lambda(1+\varepsilon)}\right) \times p(t,z)\psi(t,\varepsilon)d\varepsilon dz \quad (6)$$

for $t=1,2,3,\ldots$. Here $\psi(t,\varepsilon)$ is the PDF for the mixing frequency and λ is the mixing parameter (see below). The main property of this equation is that it describes the relaxation of $p(t,c)$ to a δ-function

$$p(t,c) \to \delta(c-\mu) \text{ as } t \to \infty. \quad (7)$$

The decay of the variance, $\sigma_c^2(t)=\langle (c(t)-\mu)^2\rangle$, is of the form $t^{-\alpha}$ for large t. The case $\alpha=1$ corresponds to the law of large numbers.

The remainder of this paper is organized as follows. In Sec. II we derive the integral equation from a stochastic difference equation describing the evolution of the PDF of a passive scalar in time. In Sec. III we solve the nonlinear integral equation numerically and analyze the influence of the different parameters on the decay rate. The evolution of the PDF is compared to data obtained from a direct numerical simulation (DNS).

II. STOCHASTIC MIXING MODEL

The evolution of a passive scalar $c=c(t,\mathbf{x})$ is governed by the stochastic PDE

$$\frac{\partial c}{\partial t} + \mathbf{v}(t,\mathbf{x}) \cdot \nabla c = D\nabla^2 c, \quad (8)$$

where $\mathbf{v}(t,\mathbf{x})$ is the random velocity field and D is the molecular diffusivity. The classical problem is to derive a closed equation for the Euler one-point PDF $p=p(c;t,\mathbf{x})=\langle \delta(c-c(t,\mathbf{x}))\rangle$. A detailed discussion of this, still unsolved, problem can be found in [2]. In the present paper we consider the mixing problem in the Lagrangian framework by introducing a passive scalar $c(t)=c(t,\mathbf{x}(t))$ of a particle volume moving with the velocity $\mathbf{v}(t,\mathbf{x}(t))$ (see Fig. 1).

A. Stochastic difference equation

The equation for the scalar $c(t)=c(t,\mathbf{x}(t))$ of the particle moving with the random velocity $\mathbf{v}(t,\mathbf{x}(t))$ can be approximated by the stochastic equation

$$\frac{dc}{dt} = -\gamma(t)(c-\tilde{c}(t)), \quad (9)$$

where the mixing frequency $\gamma(t)$ and the ambient concentration $\tilde{c}(t)$ are random processes. For the constant values of γ and $\tilde{c}$, we obtain the IEM model (1) with the exponential decay

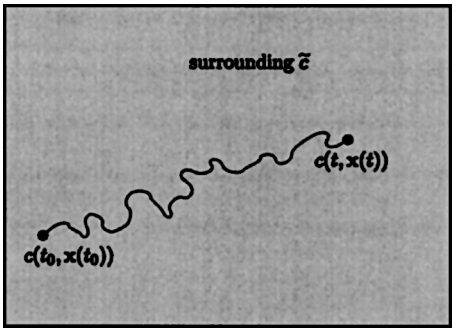

FIG. 1. Mixing problem along Lagrangian path way.

$$\frac{c(t)-\tilde{c}}{c(0)-\tilde{c}} = \exp\{-\gamma t\}. \quad (10)$$

The key feature of the present model is that $\tilde{c}(t)$ is a random process and functional of $c(t)$ itself. The crucial assumption that relates Eq. (9) to the law of large numbers is that the mean value $\langle\gamma(t)\rangle$ behaves as t^{-1} for large t. One can show that for a statistically homogeneous Gaussian scalar field $c=c(t,\mathbf{x})$, the mean value $\langle\gamma(t)\rangle$ is proportional to the ratio χ/σ_c^2, where χ is the mean-scalar dissipation rate, $\chi=2D\langle(\nabla c)^2\rangle$ (see p. 551 in Ref. [2]). It follows from the classical equation for the variance decay [2]

$$\frac{d\sigma_c^2}{dt} = -\chi, \quad (11)$$

that if the scalar variance $\sigma_c^2(t) \propto t^{-\alpha}$, then the mean scalar dissipation χ decays as $t^{-(1+\alpha)}$ and, therefore,

$$\langle\gamma(t)\rangle \propto t^{-1}. \quad (12)$$

Of course, these results are specific to a homogeneous Gaussian field, and it would be very difficult to validate them in a general case. Therefore, the above formula should be considered as the main assumption of the paper.

The appearance of the power-law decay in the deterministic case can be understood if we assume that the mixing frequency γ is $\lambda(t_0+t)^{-1}$. Then

$$\frac{dc}{dt} = -\frac{\lambda}{t_0+t}(c-\tilde{c}). \quad (13)$$

The solution to this simplified equation takes the form of the power-law decay

$$\frac{c(t)-\tilde{c}}{c(0)-\tilde{c}} = \left(\frac{t_0}{t_0+t}\right)^{\lambda}. \quad (14)$$

Since our aim here is to relate the mixing problem to the law of large numbers for the sample mean (5), it is more convenient to rewrite Eq. (9) as a stochastic difference equation for the concentration c_n

$$c_{n+1}-c_n = -\gamma_n(c_n-\tilde{c}_n), \quad 0 \leqslant c_n \leqslant 1, \quad n=1,2,3,\ldots, \quad (15)$$

where γ_n and $\tilde{c}_n$ are the mixing frequency and the random ambient concentration at the time n. We assume that they are

sequences of mutually independent random variables with the densities

$$\omega(n,\gamma)=\frac{d}{d\gamma}\mathbf{P}\{\gamma_n<\gamma\},\quad \phi(n,\tilde{c})=\frac{d}{d\tilde{c}}\mathbf{P}\{\tilde{c}_n<\tilde{c}\} \quad (16)$$

and the first moments

$$\langle\gamma_n\rangle=\int_0^1\gamma\omega(n,\gamma)d\gamma=\frac{\lambda}{1+n},\quad \langle\tilde{c}_n\rangle=\int_0^1\tilde{c}\phi(n,\tilde{c})d\tilde{c}=\mu. \quad (17)$$

In what follows we assume that the mean concentration μ is constant. The mean mixing frequency $\langle\gamma_n\rangle$ has been chosen in such a way that it is equal to $\lambda/2$ at time $n=1$. The parameter λ can be regarded as a measure of the mixing frequency. Since $0\leqslant c_n\leqslant 1$, γ_n obeys the inequality $0\leqslant\gamma_n\leqslant 1$. By introducing the deviations from the mean μ for the concentration of the particle and its surrounding, respectively,

$$u_n=c_n-\mu,\quad \xi_n=\tilde{c}_n-\mu, \quad (18)$$

we can rewrite Eq. (15) as

$$u_{n+1}=u_n-\gamma_n(u_n-\xi_n),\quad -\mu\leqslant u_n\leqslant 1-\mu,\quad n=1,2,3,\ldots, \quad (19)$$

where ξ_n is a sequence of zero mean, independent random variables with the density $\phi(n,\xi)$ and $-\mu\leqslant\xi\leqslant 1-\mu$.

B. Law of large numbers and forward Kolmogorov equation

To illustrate the connection between the mixing problem and the law of large numbers, consider the case when the sequence ξ_n is statistically stationary; that is, $\phi(\xi)$ is independent of n and the mixing frequency γ_n is a deterministic sequence of the form

$$\gamma_n=\frac{1}{1+n} \quad (20)$$

with $\lambda=1$. Let us write Eq. (19) in a slightly different form

$$u_{n+1}=u_n-\frac{1}{1+n}(u_n-\xi_{n+1}). \quad (21)$$

If we assume $u_1=\xi_1$, it follows from Eq. (21) that $u_2=(\xi_1+\xi_2)/2, u_3=(\xi_1+\xi_2+\xi_3)/3$, and so on. Therefore, the solution of Eq. (21) can be written as a sample mean

$$u_n=\frac{1}{n}\sum_{k=1}^{n}\xi_k, \quad (22)$$

which tends to the zero mean as $n\to\infty$ while the variance $\langle u_n^2\rangle\propto n^{-1}$ [17].

The advantage of having Eq. (19) is that we can easily derive an equation for the probability density function

$$p(n,u)=\frac{d}{du}\mathbf{P}\{u_n<u\},\quad n=1,2,3,\ldots. \quad (23)$$

It follows from Eqs. (19) and (20) that u_n is a discrete Markov process and the PDF for this process satisfies the forward Kolmogorov equation [16]

$$p(n+1,u)=\frac{n+1}{n}\int_{-\mu}^{1-\mu}p\left(n,\frac{(n+1)u-\xi}{n}\right)\varphi(\xi)d\xi,$$
$$n=1,2,3\ldots. \quad (24)$$

Here we used the inverse equation

$$u_n=\frac{(n+1)u_{n+1}-\xi_n}{n}. \quad (25)$$

The main properties of the solution of the Kolmogorov equation (24) with the arbitrary initial condition $p(1,u)$ are

$$p(n,u)\to\delta(u)\ \text{as}\ n\to\infty \quad (26)$$

and

$$\int_{-\mu}^{1-\mu}u^2p(n,u)du\to n^{-1}\ \text{as}\ n\to\infty. \quad (27)$$

C. General case: Random mixing frequency and nonlinear PDF equation

To account for the entire spectrum of time scales, we now assume that the mixing frequency γ_n is a stochastic variable. It is convenient to write

$$\gamma_n=\frac{\lambda}{1+n}(1+\varepsilon_n), \quad (28)$$

where ε_n is the sequence of zero mean, independent random variables. Since $0\leqslant\gamma_n\leqslant 1$, it follows from Eq. (28) that

$$-1\leqslant\varepsilon_n\leqslant\frac{1+n}{\lambda}-1. \quad (29)$$

The decay of the variance of the mixing frequency $\sigma_\gamma^2(n)$ can be determined as

$$\sigma_\gamma^2(n)=\langle(\gamma_n-\langle\gamma_n\rangle)^2\rangle=\frac{\lambda^2\langle\varepsilon_n^2\rangle}{(1+n)^2}. \quad (30)$$

When both ε_n and ξ_n are random variables, we have to specify the joint probability density for ε and ξ. If we denote it by $\mathbf{\Psi}(n,\varepsilon,\xi)$, then the forward Kolmogorov equation for $p(n,u)$ takes the form

$$p(n+1,u)=\int_{-\mu}^{1-\mu}\int_{-1}^{\frac{1+n}{\lambda}-1}\frac{n+1}{1+n-\lambda(1+\varepsilon)}$$
$$\times p\left(n,\frac{(n+1)u-\lambda(1+\varepsilon)\xi}{1+n-\lambda(1+\varepsilon)}\right)$$
$$\times\mathbf{\Psi}(n,\varepsilon,\xi)d\varepsilon d\xi. \quad (31)$$

This equation follows from the stochastic difference equation for the Markov jump process:

$$u_{n+1}=u_n-\frac{\lambda}{1+n}(1+\varepsilon_n)(u_n-\xi_n)\quad -\mu\leqslant u_n\leqslant 1-\mu \quad (32)$$

and its inverse

$$u_n = \frac{(n+1)u_{n+1} - \lambda(1+\varepsilon)\xi_n}{1+n-\lambda(1+\varepsilon)}. \tag{33}$$

Since γ_n is determined by the random scalar dissipation rate, it is natural to assume that γ_n and ξ_n are independent. Moreover, the PDF for ξ_n must be related to $p(n,u)$. The simplest choice would be to assume $p(n,\xi)$, which gives

$$\Psi(n,\varepsilon,\xi) = p(n,\xi)\psi(n,\varepsilon). \tag{34}$$

In this case the generalized nonlinear integral equation for $p(n,u)$ can be written as

$$p(n+1,u) = \int_{-\mu}^{1-\mu} \int_{-1}^{\frac{1+n}{\lambda}-1} \frac{n+1}{1+n-\lambda(1+\varepsilon)} \times p\left(n, \frac{(n+1)u - \lambda(1+\varepsilon)\xi}{1+n-\lambda(1+\varepsilon)}\right) \times p(n,\xi)\psi(n,\varepsilon)d\varepsilon d\xi \tag{35}$$

for $n=1,2,3,\ldots$. It follows from Eq. (30) that the variance of the mixing frequency $\sigma_\gamma^2(n)$ decays as n^{-2} if the variance $\sigma_\varepsilon^2=\langle\varepsilon_n^2\rangle$ is constant.

We expect that the variance of a passive scalar $\sigma_c^2(n) = \langle(c_n-\mu)^2\rangle = \langle u_n^2\rangle$ behaves as $n^{-\alpha}$ for large n. Taking the analytical solution of the simplified model (13) into account, we anticipate a strong dependence of α on the mixing parameter λ. The trivial case $\lambda=0$ describes the motion of an inert particle in its surrounding and Eq. (19) has the simple solution $u_n=u_1$ for all n. However, if λ is close to its upper limit, the Lagrangian particle volume experiences strong interactions with its surrounding. Intuitively, it can be expected that the mixing parameter λ has a distinct influence on the decay of σ_u^2 [see solution (14) to simplified equation].

III. NUMERICAL RESULTS

The integral equation (35) describes the relaxation of an arbitrary initial PDF of a conserved scalar c to a δ-distribution for large n. Because of the nonlinearity in p, this equation needs to be solved numerically. In this section, we analyze the behavior of the solution of Eq. (35) for different values of the parameter λ, initial conditions for $p(n,u)$, and the density function $\psi(n,\varepsilon)$ for the stochastic variable ε_n. In particular, we study the dependence of the decay exponent α on λ and $\psi(n,\varepsilon)$.

Let us first discuss some numerical issues. In order to advance the density $p(n+1,u)$ in time, Eq. (35) must be integrated over the ξ and ε space, which results in the overall complexity $\mathcal{O}(N_u \times N_\xi \times N_\varepsilon)$ for each time step. Here, N_ζ denotes the number of grid points in the ζ direction. This makes the computation rather time-consuming. However, a computer code can be easily parallelized along the u coordinate. The initial density of u relaxes to a δ-distribution with steep gradients around $u=0$. An adaptive grid using the equidistribution principle is employed in u and ξ space in order to resolve the shape of the PDF properly. In the equidistribution principle [18], an initially given number of grid points is distributed in such a way that the condition $\int_{u_i}^{u_{i+1}} w(u)du = (N_u-1)^{-1}\int_{-\mu}^{1-\mu} w(u)du$ is satisfied. The weight function $w(u)$ is a positive function measuring the variation of the solution.

TABLE I. Fixed parameters used for the numerical simulation.

Parameter	Description	Value
nEnd	number of time steps	100 000
μ	mean value	0.5
N_u	dimension of u	101
N_ξ	dimension of ξ	101
N_ε	dimension of ε	101

A. Parameters and initial conditions

From experiments and DNS studies it is known that the density distribution of a passive scalar can be reasonably approximated by a β distribution [1,2]. Recall that the β distribution has nonzero probability in the interval $[u_{\min}, u_{\max}]$ and has the form

$$\beta(u;\langle u\rangle,\sigma_u^2) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(u_{\max}-u_{\min})^{1-a-b}(u-u_{\min})^{a-1} \times (u_{\max}-u)^{b-1}. \tag{36}$$

It is fully described by two parameters, which are functions of $\langle u\rangle$ and σ_u^2

$$a = \frac{\langle u\rangle - u_{\min}}{u_{\max}-u_{\min}}\left[\frac{(\langle u\rangle - u_{\min})(u_{\max}-\langle u\rangle)}{\sigma_u^2} - 1\right], \tag{37}$$

$$b = \frac{u_{\max} - \langle u\rangle}{u_{\max}-u_{\min}}\left[\frac{(\langle u\rangle - u_{\min})(u_{\max}-\langle u\rangle)}{\sigma_u^2} - 1\right]. \tag{38}$$

The β distribution adopts a wide range of shapes: for the maximum variance $\sigma_u^2=(\langle u\rangle-u_{\min})(u_{\max}-\langle u\rangle)$, a double

TABLE II. Combinations of initial distributions used for the parameter study: $\beta(\sigma_\zeta^2)$ (β distribution with prescribed variance σ_ζ^2) $\delta(\zeta)$ (Dirac δ-function for ζ).

Run	$p(1,u)$	$\psi(1,\varepsilon)=\psi(\varepsilon)$
BB1	$\beta(0.20)$	$\beta(0.500)$
BB2	$\beta(0.20)$	$\beta(0.250)$
BB3	$\beta(0.20)$	$\beta(0.125)$
BB4	$\beta(0.15)$	$\beta(0.500)$
BB5	$\beta(0.15)$	$\beta(0.250)$
BB6	$\beta(0.15)$	$\beta(0.125)$
BB7	$\beta(0.05)$	$\beta(0.500)$
BB8	$\beta(0.05)$	$\beta(0.250)$
BB9	$\beta(0.05)$	$\beta(0.125)$
BD1	$\beta(0.20)$	$\delta(\varepsilon)$
BD2	$\beta(0.15)$	$\delta(\varepsilon)$
BD3	$\beta(0.05)$	$\delta(\varepsilon)$

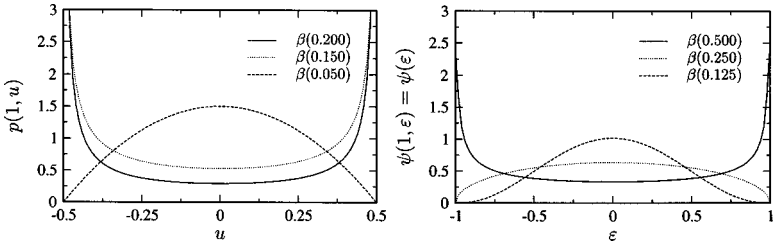

FIG. 2. Initial conditions for $p(1,u)$ and $\psi(1,\varepsilon)=\psi(\varepsilon)$; $\lambda=1.0$.

δ-distribution at $u_{\min}$ and $u_{\max}$ is approached and for small σ_u^2 a Gaussian distribution around $\langle u\rangle$ is obtained. Throughout the following simulations, this distribution is used as initial condition for $p(n,u)$.

To solve Eq. (35), we have to provide an appropriate density function for the random variable ε. We assume for simplicity that ε has a stationary distribution. A reasonable choice would be to assume that ε is distributed following Eq. (36). A particular choice for $\psi(\varepsilon)$ is a δ-distribution that corresponds to the deterministic mixing frequency (20). In what follows, we use both distributions. The constant parameters for all simulations of Eq. (35) are summarized in Table I.

B. Dependence of the decay exponent α on the initial distribution $p(1,u)$ and $\psi(\varepsilon)$

In the first numerical experiment, the dependence of the decay exponent α on the initial distribution $p(1,u)$ and $\psi(\varepsilon)$ is investigated. Therefore, prescribed distributions with different initial variance for $\sigma_u^2(1)$ and σ_ε^2 are used (see Table II and Fig. 2).

For this simulation λ is chosen to be unity and ε is in the range $-1\leqslant\varepsilon\leqslant(1+n)/\lambda-1$. The maximum possible value for ε is then found for $n=1$.

The decay rate of σ_u^2 and the decay exponent α are evaluated and plotted in Fig. 3. In the left column, the four different combinations with initial variance $\sigma_u^2(1)=0.20$ are plotted, the middle column shows the decay rate for the cases with $\sigma_u^2(1)=0.15$. In the right column, the evolution of $p(n,u)$ is shown for the monomodal initial β distribution with $\sigma_u^2(1)=0.05$.

From the bottom row of Fig. 3 it can be seen that $p(n,u)$ experiences strong dynamics over the initial interval, say for $n\leqslant 10$. Over this interval, the variance does not follow the power-law decay. The influence of the initial distribution of ε on the evolution of $p(n,u)$ and, consequently, on $\sigma_u^2(n)$ vanish for large n. This is most pronounced for the cases BB7–BB9 and BD3 (right column). Here the decay exponents for all of these cases collapse to a single curve, independent of $\psi(\varepsilon)$. From this simulation, one can conclude that the initially assumed distribution for ε has only marginal influence on the decay exponent α for large n. During the simulation, the value for α never reaches a steady state. The final convergence rate for $n=100\,000$ is $|d\alpha/dn|<1.0\times10^{-6}$ so that α is in the range $1.50\leqslant\alpha\leqslant1.65$ for all cases.

C. Dynamic behavior of the PDF $p(n,u)$

We will now discuss the influence of $\psi(\varepsilon)$ on the dynamics of $p(n,u)$. Here the mixing parameter $\lambda=0.75$ is used. In order to obtain insight into the transient behavior, we choose a double δ-distribution and a bimodal β distribution as initial condition for $p(n,u)$, namely, $p(1,u)=[\delta(\mu+u)+\delta(1-\mu-u)]/2$ and $p(1,u)=\beta(0.15)$. For clarity, the different combinations used here are summarized in Table III.

At $n=2$ the initial distribution for the case Dyn1 splits up into four peaks at positions $u=\lambda(\xi-\gamma)/2+\gamma$ with $\gamma=\{-\mu,1-\mu\}$ and $\xi=\{-\mu,1-\mu\}$. In the succeeding time, those peaks move toward the center $u=0$ while the tails flatten. It is clear from the Fig. 4 that the transitional behavior of the PDF for the case Dyn1 is fairly unrealistic. The main reason for this is the assumption $\psi(\varepsilon)=\delta(\varepsilon)$ (i.e., the absence of

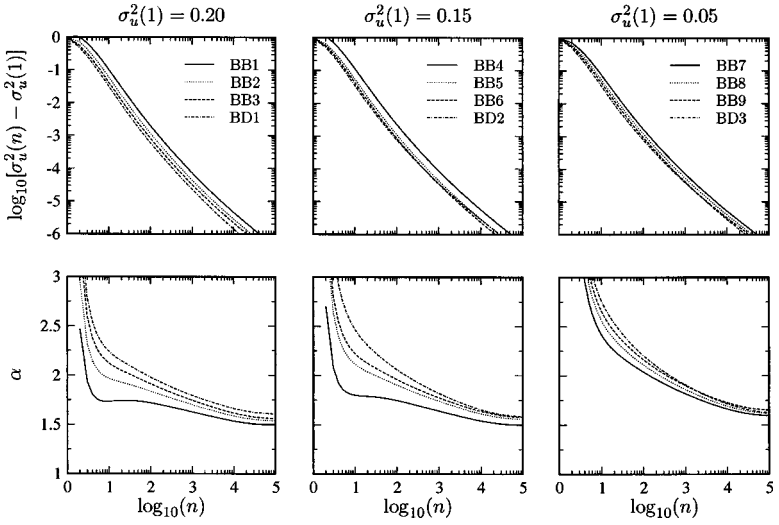

FIG. 3. Evolution of σ_u^2 and decay exponent α for the initial β distribution in u.

TABLE III. Combinations used to study dynamic behavior of $p(n,u)$ for $\lambda=0.75$.

Run	$p(1,u)$	$\psi(\varepsilon)$
Dyn1	$[\delta(\mu+u)+\delta(1-\mu-u)]/2$	$\delta(\varepsilon)$
Dyn2	$\beta(0.15)$	$\beta(0.50)$

random fluctuations in the exchange parameter γ_n). An entirely different dynamic behavior is obtained for the case Dyn2. The initially smooth density function $p(1,u)$ transits rapidly to a unimodal distribution and converges to the δ-PDF.

We can summarize that the prescribed distribution of ε influences the transitional dynamics of $p(n,u)$. However, its effect on the decay exponent α for large n is insignificant.

D. Mixing parameter λ

From experiments and DNS studies it is well known that the decay rate strongly depends on the initial ratio of the velocity and scalar length scales ℓ_u and ℓ_c [15]. The free parameter λ is introduced in the mixing frequency (28) in order to link the present model to experiment data. It was concluded previously that the presumed form of $\psi(\varepsilon)$ has only insignificant influence on the decay rate. However, the dynamic behavior of $p(n,u)$ is greatly influenced by the stochastic variable ε. In anticipation of the overall weak influence of $\psi(\varepsilon)$ on the decay exponent we arbitrarily chose $\psi(\varepsilon)=\beta(0.20)$, whereas $\psi(\varepsilon)$ has nonzero probability only in the interval $-1\leqslant\varepsilon\leqslant 2/\lambda-1$. The parameters used for the different test cases are summarized in Table IV.

Experiments on turbulent mixing show that the variance σ_u^2 decreases faster with increasing ratio ℓ_u/ℓ_c, until about $\ell_u/\ell_c\sim 5$ [15]. For ratios that are greater than 5, the dependence is negligible. This tendency can be resembled by changing the mixing parameter λ. Figure 5 shows the distinct dependence of the mixing parameter on the decay of σ_u^2. The power-law exponent α is approximately 1.55 for $\lambda=1$ (LA1) and decreases to about 0.62 for $\lambda=0.25$ (LA4), which corresponds to a "slow" mixing process.

E. Comparison with DNS data

In this section, the evolution of the PDF $p(t,u)$ is compared to data obtained from a direct numerical simulation (DNS) [19,20]. The three-dimensional DNS is conducted for decaying, isotropic turbulence in a periodic box. After the turbulence is fully developed, which is indicated by an increase of the characteristic length scale of the velocity field, the passive scalar is initialized in discrete "blobs." This is expressed by a double δ-function for the PDF of u with peaks at its minimum and maximum value.

TABLE IV. Combinations used to study the influence of the mixing parameter λ on the evolution of the variance.

Run	λ	$p(1,u)$	$\psi(\varepsilon)$	$[\varepsilon_{\min},\varepsilon_{\max}]$
LA1	1.00	$\beta(0.15)$	$\beta(0.20)$	$[-1,1]$
LA2	0.75	$\beta(0.15)$	$\beta(0.20)$	$[-1,5/3]$
LA3	0.50	$\beta(0.15)$	$\beta(0.20)$	$[-1,3]$
LA4	0.25	$\beta(0.15)$	$\beta(0.20)$	$[-1,7]$

It is well known that the numerical resolution for a DNS, expressed by the number of computational grid points N in each spatial direction, scales with the Reynolds number Re, $N\sim \mathrm{Re}^{\alpha_{\mathrm{DNS}}}$ with $\alpha_{\mathrm{DNS}}=\frac{3}{4}$ where the Reynolds number is based on the characteristic length scale and velocity scale of the largest eddies. This clearly shows that for a three-dimensional simulation even with todays computational resources DNS is restricted to low and moderate Reynolds number flows. In this DNS, the initial Reynolds number based on the Taylor length scale is $\mathrm{Re}_\lambda=33$. Here, Re is based on the velocity and length scale at the integral scales of the turbulence. The governing equations are solved using a pseudospectral code in a cubic domain with $N^3=128^3$ grid points. For isotropic decaying turbulence, the decay of the turbulent kinetic energy k follows a power law

$$k(t)\sim t^{-n} \tag{39}$$

and experiments suggest $n=1.3\pm0.2$. Due to the rapid decay of k, a DNS is performed only over a relatively short time period. In the present study, the DNS is performed over a nondimensional time interval $T=2.28t_{\mathrm{eddy}}$, where t_{eddy} is the initial large eddy turnover time, based on the integral length scale and rms velocity [19,20]. The present model describes the decay of the variance of a passive scalar for very long time and high Reynolds number, which might invalidate a comparison of the model with the DNS data. However, in the following, it is shown that the model is able to describe the initial decay of σ_u^2 accurately.

The integral equation (35) can be rewritten in the following form:

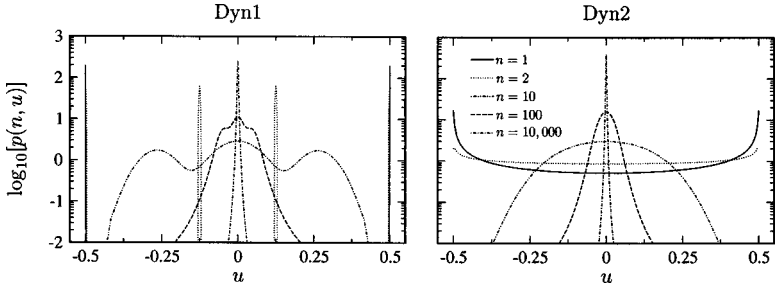

FIG. 4. Transitional behavior of $p(n,u)$ for different initial conditions and mixing parameter.

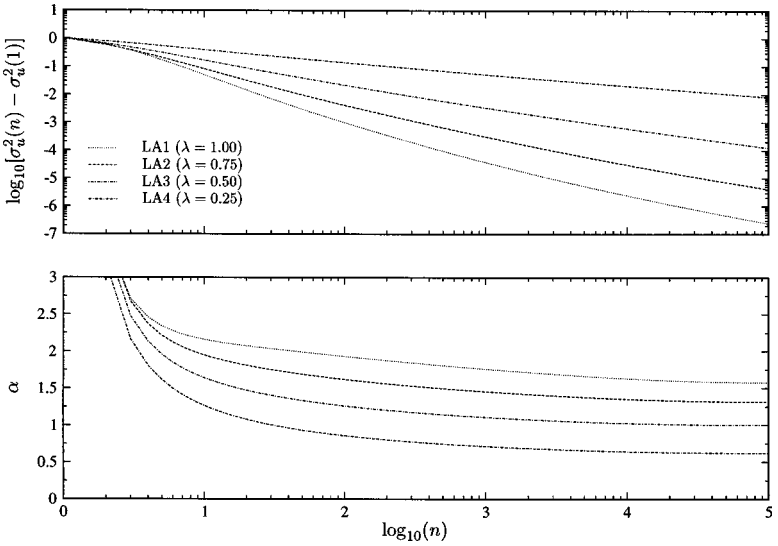

FIG. 5. Decay of σ_u^2 for different mixing intensities λ.

$$p(t+\Delta t,u) = \int_{-\mu}^{1-\mu} \int_{-1}^{\frac{1+t}{\lambda \Delta t}-1} \frac{1+t}{1+t-\Delta t\lambda(1+\varepsilon)} \times p\left(t, \frac{(1+t)u - \Delta t\lambda\xi(1+\varepsilon)}{1+t-\Delta t\lambda(1+\varepsilon)}\right) \times p(t,\xi)\psi(t,\varepsilon)d\varepsilon d\xi, \quad (40)$$

where the time t has been normalized with t_{eddy}, and the concentrations u and ξ are normalized by μ. Equations (35) and (40) are derived under the assumption that ξ_n is a sequence of independent random variables [see Eq. (19)]. This assumption restricts the time increment Δt to a lower bound, which must be larger than a turbulent correlation time. For the comparison of the model with the DNS we use $\Delta t = \frac{1}{2}$ and $p(\frac{1}{2},u) = p_{\text{DNS}}(\frac{1}{2},u) \approx \beta(0.178)$. The PDF for $\psi(\varepsilon)$ is $\beta(1.0)$ for $-1 \leqslant \varepsilon \leqslant (1+\Delta t)/(\lambda \Delta t) - 1$ with $\lambda = 0.85$. The number of grid points is $N_u = N_\varepsilon = N_\xi = 200$. An adaptive grid has been employed. The evolution of the PDF $p(t,u)$ for four different time increments is shown and compared to the DNS data in Fig. 6.

In the beginning of the process, the distribution of the initially unmixed scalar is described by a double-δ function. Then, for early times, turbulent stirring and molecular mixing lead to large probabilities of weakly mixed regions with either very large or very small concentrations. This state is still described by a bimodal PDF as shown in Fig. 6. For later times ($t \approx 1.5$), the probabilities for very high and low values decrease in favor of a broad distribution around the mean value. For even later times ($t > 1.5$), the PDF assumes a unimodal distribution with a maximum at the mean value. The peaks close to the extreme values of u are slightly overpredicted for later times, but the general behavior of the PDF evolution is predicted quite accurately.

IV. CONCLUSION

In this paper we presented a stochastic model for the mixing of a passive scalar in decaying turbulence. The model is based on the law of large numbers and involves two stochastic processes: the random mixing frequency and the stochastic ambient concentration. The mean mixing frequency decays as t^{-1} and links the mixing model to the law of large numbers. We gave a heuristic derivation of the nonlinear integral equation for the scalar PDF (35) describing the relaxation of an arbitrary initial distribution to a δ-function. The sensitivity of the model to different initial conditions and parameters is numerically analyzed. It was shown that the distributions for the stochastic mixing frequency and the initial scalar PDF have only insignificant influence on the decay rate of the variance σ_c^2, which was described by a power law, $\sigma_c^2 \propto t^{-\alpha}$. This is to be expected because the initial distribution of the scalar is almost never measured in experiments, assuming implicitly that either the scalar is distributed

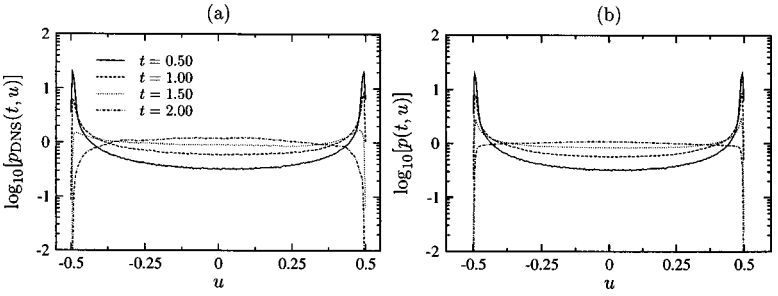

FIG. 6. Comparison of the PDFs with the results from DNS by Sripakagorn *et al.* [19,20] for $\lambda = 0.85$; (a) shapes of the PDFs extracted from the DNS at time $t = 0.5$, 1.0, 1.5, and 2.0, (b) shapes of PDFs obtained from the model for same time as in (a).

initially as a double δ-function or generally disregard the dependence of the decay of σ_c^2 on the initial probability density function. It was shown that with increasing values of the mixing parameter, the variance of the passive scalar decreases faster. The connection of our model to physical mixing processes is provided through the free parameter λ, which can account for initial length-scale dependencies ℓ_u/ℓ_c and other physical effects. We showed that the results obtained from the model for the evolution of the PDF are in a good agreement with DNS data.

[1] N. Peters, *Turbulent Combustion* (Cambridge University Press, Cambridge, England, 2000).

[2] S. B. Pope, *Turbulent Flows* (Cambridge University Press, Cambridge, England, 2000).

[3] J. Villermaux and J. C. Devillon, in *Proceedings of the Second International Symposium on Chemical Reaction Engineering* (1972), pp. 1–13.

[4] C. Dopazo and E. E. O'Brien, Acta Astronaut. **1**, 1239 (1974).

[5] H. Chen, S. Chen, and R. H. Kraichnan, Phys. Rev. Lett. **63**, 2657 (1989).

[6] R. L. Curl. AIChE J. **9**, 175 (1963).

[7] L. Valino and C. Dopazo, Phys. Fluids A **3**, 3034 (1991).

[8] R. O. Fox, Phys. Fluids A **4**, 1230 (1992).

[9] V. A. Sabelnikov and M. A. Gorokhovski, in Second International Symposium on Turbulence and Shear Flow Phenomena, Royal Institute of Technology (KTH), Stockholm, Sweden, June 27–29, 2001; O. Soulard, V. Sabelnikov, and M. Gorokhovski, Int. J. Heat Fluid Flow **25**, 875 (2004).

[10] V. A. Kaminskii, A. B. Rabinovich, A. Ya. Fedorov, and V. A. Frost, Teor. Osn. Khim. Tekhnol. **31**, 243 (1997).

[11] S. Subramaniam and S. B. Pope, Combust. Flame **115**, 487 (1998).

[12] D. R. Fereday, P. H. Haynes, A. Wonhas, and J. C. Vassilicos, Phys. Rev. E **65**, 035301(R) (2002); D. T. Son, *ibid.* **59**, 3811 (1999); E. Balkovsky and A. Fouxon, *ibid.* **60**, 4164 (1999).

[13] M. Lesieur, *Turbulence in Fluids* (Kluwer, Dordrecht, 1997).

[14] J. R. Chasnov, Phys. Fluids **6**, 1036 (1994).

[15] P. A. Durbin, Phys. Fluids **25**, 1328 (1982).

[16] W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1966).

[17] C. Knessl, B. J. Matkowsky, Z. Schuss, and C. Tier, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **46**, 1006 (1985).

[18] V. D. Liseikin, *Grid Generation Methods* (Springer, Berlin, 1999).

[19] P. Sripakagorn, G. Kosály, and H. Pitsch, CTR Annual Research Briefs, Stanford University/NASA Ames 117–128, (2000).

[20] P. Sripakagorn, S. Mitarai, G. Kosály, and H. Pitsch, J. Fluid Mech. **518**, 231 (2004).